KYUNGPOOK Math. J. 56(2016), 273-293
http://dx.doi.org/10.5666/KMJ.2016.56.1.273
pISSN 1225-6951 eISSN 0454-8124

Biharmonic Hypersurfaces with Constant Scalar Curvature in E_s^5

Deepika
University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Sector-16C, Dwarka, New Delhi-110078, India
e-mail : sdeep2007@gmail.com

Ram Shankar Gupta*
Department of Mathematics, Central University of Jammu, Sainik colony, Jammu-180011, India
University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Sector-16C, Dwarka, New Delhi-110078, India
e-mail : ramshankar.gupta@gmail.com

A. Sharfuddin
Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi-110025, India
e-mail : sharfuddin_ahmad12@yahoo.com

Abstract. In this paper, we obtain that every biharmonic non-degenerate hypersurfaces in semi-Euclidean space E_s^5 with constant scalar curvature of diagonal shape operator has zero mean curvature.

1. Introduction

In 1964, Eells and Sampson [16] introduced the notion of poly-harmonic maps as a natural generalization of the well-known harmonic maps. Thus, while $\phi : (M, g) \to (N, h)$ harmonic maps between Riemannian manifolds are critical points of the energy functional $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$, the biharmonic maps are critical points of the bienergy functional $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$, where $\tau = trace\nabla d\phi$

* Corresponding Author.
Received February 4, 2015; revised July 26, 2015; accepted November 3, 2015.
2010 Mathematics Subject Classification: 53D12, 53C40, 53C42
Key words and phrases: Biharmonic submanifolds, Mean curvature vector, Chen's conjecture.

is the tension field of ϕ.

The study of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in the middle of 1980s. In particular, he proved that biharmonic surfaces in Euclidean 3-spaces are minimal. There are many non-existence results in Euclidean spaces developed by I. Dimitric in his doctoral thesis [14] and paper [15]. Based on these results, B. Y. Chen [7] in 1991 posed the following well-known conjecture: The only biharmonic submanifolds of Euclidean spaces are the minimal ones. Also, the conjecture was later proved for hypersurfaces in Euclidean 4-spaces [22] and for hypersurfaces with three distinct principal curvatures in E^5 [23]. Recently, it was proved that Chen's conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension [11] and for hypersurfaces in Euclidean spaces of arbitrary dimension with three distinct principal curvatures [20]. Also, it was shown that the conjecture is true for every biharmonic hypersurfaces in E^5 with zero scalar curvature [12]. The conjecture is a local problem and understanding local structure of biharmonic submanifolds to the point of minimality is a complex task. That may be the possible reason for the conjecture to be open till now in general so far. The global version of Chen's conjecture for biharmonic submanifolds in Euclidean space was studied in [17]. There exist lots of examples of proper biharmonic submanifolds in spheres (see, for instance [2-6, 18-19]).

In contrast to the submanifolds of Euclidean spaces, Chen's conjecture is not true always for submanifolds of pseudo-Euclidean spaces. For example, B. Y. Chen and S. Ishikawa [9, 10] obtained some examples of proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces E^4_s for s = 1, 2, 3, (see also [8]). But for hypersurfaces in pseudo-Euclidean spaces, it is reasonable that Chen's conjecture is also right. It was proved that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal [9, 10]. In [13], F. Defever et al. proved that the biharmonic conjecture is true for non-degenerate hypersurfaces of semi-Euclidean 4-spaces. A. Arvanitoyeorgos et al. [1] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal. Recently, it was proved that every biharmonic hypersurfaces with three distinct principal curvatures of diagonal shape operator in E^5_s must be minimal [21]. It led us to investigate biharmonic hypersurface in semi-Euclidean 5-spaces with four distinct principal curvatures.

In this paper, we study biharmonic non-degenerate hypersurfaces of constant scalar curvature in semi-Euclidean spaces E^5_s with diagonal shape operator.

2. Preliminaries

Let (M^4_r, g), $r = 0, 1, 2, 3, 4$, be a 4-dimensional hypersurface isometrically immersed in a 5-dimensional semi-Euclidean space $(E^5_s, \overline{g})$, $s = 0, 1, 2, 3, 4, 5$ and $g = \overline{g}_{|M^4_r}$. We denote by ξ unit normal vector to M^4_r with $\overline{g}(\xi, \xi) = \varepsilon$, where $\varepsilon = \pm 1$, according as M^4_r is pseudo-Riemannian or Riemannian hypersurface.

Let $\overline{\nabla}$ and ∇ denote linear connections on E^5_s and M^4_r, respectively. Then, the

Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \forall \ X,Y \in \Gamma(TM_r^4), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -A_\xi X, \tag{2.2}$$

where h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$\overline{g}(h(X,Y),\xi) = g(A_\xi X, Y). \tag{2.3}$$

The mean curvature is given by

$$\varepsilon H = \frac{1}{4}\text{trace}A. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X,Y)Z = g(AY,Z)AX - g(AX,Z)AY, \tag{2.5}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.6}$$

respectively, where R is the curvature tensor and

$$(\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y), \tag{2.7}$$

for all $X,Y,Z \in \Gamma(TM_r^4)$.

A biharmonic submanifold in a semi-Euclidean space is called proper biharmonic if it is not minimal. The necessary and sufficient condition for M_r^4 to be biharmonic in E_s^5 is

$$\triangle H + \varepsilon H \text{trace}A^2 = 0, \tag{2.8}$$

$$A(\text{grad}H) + 2\varepsilon H \text{grad}H = 0, \tag{2.9}$$

where H denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function f is given by [8]

$$\triangle f = -\sum_{i=1}^{4} \epsilon_i (e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal local tangent frame on M_r^4 and $g(e_i, e_i) = \epsilon_i$.

A vector X in E_s^5 is called spacelike, timelike or lightlike according as $\overline{g}(X,X) > 0$, $\overline{g}(X,X) < 0$ or $\overline{g}(X,X) = 0$, respectively. A non-degenerate hypersurface M_r^4 of E_s^5 is called Riemannian or pseudo-Riemannian according as the induced metric on M_r^4 from the indefinite metric on E_s^5 is definite or indefinite. A shape operator of pseudo-Riemannian hypersurfaces is not diagonalizable always unlike the Riemannian hypersurfaces.

3. Biharmonic Non-Degenerate Hypersurfaces of Constant Scalar Curvature in E^5_s

We have the following cases.

(a) *The case of four distinct principal curvatures*

In this section, we study biharmonic non-degenerate hypersurfaces Riemannian or pseudo-Riemannian M^4_r with diagonal shape operator. We also assume that mean curvature is not constant and grad$H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset U of M^4_r, with grad$_pH \neq 0$ for all $p \in U$. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2\varepsilon H$. The grad$H$ can be spacelike or timelike. Without losing generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2\varepsilon H & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.1}$$

The grad$H$ can be expressed as

$$\text{grad}H = \sum_{i=1}^{4} e_i(H)e_i. \tag{3.2}$$

As we have taken e_1 parallel to grad$H$, consequently

$$e_1(H) \neq 0, e_2(H) = 0, e_3(H) = 0, e_4(H) = 0. \tag{3.3}$$

We express

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \epsilon_k \omega^k_{ij} e_k, \qquad i, j = 1, 2, 3, 4. \tag{3.4}$$

Using compatibility conditions $\nabla_{e_k} g(e_i, e_i) = 0$ and $\nabla_{e_k} g(e_i, e_j) = 0$, we obtain

$$\omega^i_{ki} = 0, \qquad \omega^j_{ki} + \omega^i_{kj} = 0, \tag{3.5}$$

for $i \neq j$, and $i, j, k = 1, 2, 3, 4$.

From Codazzi equation (2.6), we have $g((\nabla_{e_i} A)e_j, e_j) = g((\nabla_{e_j} A)e_i, e_j)$ and then using (2.7), (3.1) and (3.4), we obtain

$$\epsilon_j e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega^j_{ji}, \tag{3.6}$$

and similarly, $g((\nabla_{e_i}A)e_j, e_k) = g((\nabla_{e_j}A)e_i, e_k)$ gives

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{3.7}$$

respectively, for distinct $i, j, k = 1, 2, 3, 4$.

Since $\lambda_1 = -2\varepsilon H$, from (3.3), we get

$$e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, e_4(\lambda_1) = 0. \tag{3.8}$$

Also, it is easy to show that

$$[e_i, e_j](\lambda_1) = 0, \qquad i, j = 2, 3, 4,$$

which gives

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{3.9}$$

for $i \neq j$ and $i, j = 2, 3, 4$.

Now, we show that $\lambda_j \neq \lambda_1, j = 2, 3, 4$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, then from (3.6), we find

$$\epsilon_j e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0, \tag{3.10}$$

which contradicts the first expression of (3.8).

Since M_r^4 has four distinct principal curvatures, from (2.4), we obtain that

$$\lambda_2 + \lambda_3 + \lambda_4 = 6\varepsilon H. \tag{3.11}$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.8) and (3.5), we find

$$\omega_{1i}^1 = 0, \qquad i = 1, 2, 3, 4. \tag{3.12}$$

Putting $k = 1, j \neq i$, and $i, j = 2, 3, 4$ in (3.7), and using (3.9), we get

$$\omega_{ij}^1 = \omega_{ji}^1 = \omega_{1i}^j = \omega_{i1}^j = 0, \qquad j \neq i, \text{and } i, j = 2, 3, 4. \tag{3.13}$$

Thus, we have the following:

Lemma 3.1. *Let $M_r^4, r = 0, 1, 2, 3, 4,$ be a biharmonic non-degenerate hypersurface of non-constant mean curvature with four distinct principal curvatures in semi-Euclidean space $E_s^5, s = 0, 1, 2, 3, 4, 5,$ having the shape operator given by* (3.1) *with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then, we obtain*

$$\nabla_{e_1} e_i = 0, \quad i = 1, 2, 3, 4, \tag{3.14}$$

$$\nabla_{e_i} e_1 = \omega_{i1}^i e_i \epsilon_i, \quad i = 2, 3, 4, \tag{3.15}$$

$$\nabla_{e_i} e_i = \sum_{i \neq j, j=1}^{4} \omega_{ii}^j e_j \epsilon_j, \quad i = 2, 3, 4, \tag{3.16}$$

$$\nabla_{e_i} e_j = \sum_{k \neq j, k=2}^{4} \omega_{ij}^k e_k \epsilon_k, \quad i, j = 2, 3, 4, and \;\; i \neq j, \tag{3.17}$$

where ω_{ij}^i *satisfy (3.5) and (3.6) for* $i, j = 1, 2, 3, 4$.

Evaluating Riemannian curvatures, using Lemma 3.1, Gauss equation and comparing the coefficients with respect to an orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we find the following:

•$X = e_1, Y = e_2, Z = e_1$,

$$e_1(\omega_{22}^1)\epsilon_2 - (\omega_{22}^1)^2 = -2\varepsilon\epsilon_1 H\lambda_2. \tag{3.18}$$

•$X = e_1, Y = e_3, Z = e_1$,

$$e_1(\omega_{33}^1)\epsilon_3 - (\omega_{33}^1)^2 = -2\varepsilon\epsilon_1 H\lambda_3. \tag{3.19}$$

•$X = e_1, Y = e_4, Z = e_1$,

$$e_1(\omega_{44}^1)\epsilon_4 - (\omega_{44}^1)^2 = -2\varepsilon\epsilon_1 H\lambda_4. \tag{3.20}$$

•$X = e_1, Y = e_2, Z = e_2$,

$$e_1(\omega_{22}^3)\epsilon_2 - \omega_{22}^3\omega_{22}^1 = 0. \tag{3.21}$$

$$e_1(\omega_{22}^4)\epsilon_2 - \omega_{22}^4\omega_{22}^1 = 0. \tag{3.22}$$

•$X = e_1, Y = e_3, Z = e_3$,

$$e_1(\omega_{33}^2)\epsilon_3 - \omega_{33}^2\omega_{33}^1 = 0. \tag{3.23}$$

$$e_1(\omega_{33}^4)\epsilon_3 - \omega_{33}^4\omega_{33}^1 = 0. \tag{3.24}$$

•$X = e_1, Y = e_4, Z = e_4$,

$$e_1(\omega_{44}^2)\epsilon_4 - \omega_{44}^2\omega_{44}^1 = 0. \tag{3.25}$$

$$e_1(\omega_{44}^3)\epsilon_4 - \omega_{44}^3\omega_{44}^1 = 0. \tag{3.26}$$

$\bullet X = e_2, Y = e_3, Z = e_2,$

$$e_2(\omega_{33}^2) + e_3(\omega_{22}^3) - \epsilon_1\omega_{22}^1\omega_{33}^1 - \epsilon_4\omega_{22}^4\omega_{33}^4 - \epsilon_2(\omega_{22}^3)^2 - \epsilon_3(\omega_{33}^2)^2 + (\omega_{32}^4\omega_{23}^4 - \omega_{34}^2\omega_{43}^2 - \omega_{42}^3\omega_{24}^3)\epsilon_4 = \epsilon_2\epsilon_3\lambda_2\lambda_3. \tag{3.27}$$

$$e_3(\omega_{22}^1) + \epsilon_3\omega_{22}^3\omega_{33}^1 - \epsilon_2\omega_{22}^3\omega_{22}^1 = 0. \tag{3.28}$$

$$e_3(\omega_{22}^4) + \epsilon_3\omega_{22}^3\omega_{33}^4 - \epsilon_2\omega_{22}^3\omega_{22}^4 = 0. \tag{3.29}$$

$\bullet X = e_2, Y = e_4, Z = e_2,$

$$e_2(\omega_{44}^2) + e_4(\omega_{22}^4) - \epsilon_1\omega_{22}^1\omega_{44}^1 - \epsilon_3\omega_{22}^3\omega_{44}^3 - \epsilon_2(\omega_{22}^4)^2 - \epsilon_4(\omega_{44}^2)^2 + (\omega_{42}^3\omega_{24}^3 - \omega_{32}^4\omega_{23}^4 - \omega_{34}^2\omega_{43}^2)\epsilon_3 = \epsilon_2\epsilon_4\lambda_2\lambda_4. \tag{3.30}$$

$$e_4(\omega_{22}^1) + \epsilon_4\omega_{22}^4\omega_{44}^1 - \epsilon_2\omega_{22}^4\omega_{22}^1 = 0. \tag{3.31}$$

$$e_4(\omega_{22}^3) + \epsilon_4\omega_{22}^4\omega_{44}^3 - \epsilon_2\omega_{22}^4\omega_{22}^3 = 0. \tag{3.32}$$

$\bullet X = e_3, Y = e_4, Z = e_3,$

$$e_3(\omega_{44}^3) + e_4(\omega_{33}^4) - \epsilon_1\omega_{33}^1\omega_{44}^1 - \epsilon_2\omega_{33}^2\omega_{44}^2 - \epsilon_3(\omega_{33}^4)^2 - \epsilon_4(\omega_{44}^3)^2 + (\omega_{34}^2\omega_{43}^2 - \omega_{32}^4\omega_{23}^4 - \omega_{42}^3\omega_{24}^3)\epsilon_2 = \epsilon_3\epsilon_4\lambda_3\lambda_4. \tag{3.33}$$

$$e_4(\omega_{33}^1) + \epsilon_4\omega_{33}^4\omega_{44}^1 - \epsilon_3\omega_{33}^4\omega_{33}^1 = 0. \tag{3.34}$$

$$e_4(\omega_{33}^2) + \epsilon_4\omega_{33}^4\omega_{44}^2 - \epsilon_3\omega_{33}^4\omega_{33}^2 = 0. \tag{3.35}$$

$\bullet X = e_2, Y = e_3, Z = e_3,$

$$e_2(\omega_{33}^1) + \epsilon_2\omega_{33}^2\omega_{22}^1 - \epsilon_3\omega_{33}^2\omega_{33}^1 = 0. \tag{3.36}$$

$$e_2(\omega_{33}^4) + \epsilon_2\omega_{33}^2\omega_{22}^4 - \epsilon_3\omega_{33}^2\omega_{33}^4 = 0. \tag{3.37}$$

$\bullet X = e_2, Y = e_4, Z = e_4,$

$$e_2(\omega_{44}^1) + \epsilon_2\omega_{44}^2\omega_{22}^1 - \epsilon_4\omega_{44}^2\omega_{44}^1 = 0. \tag{3.38}$$

$$e_2(\omega_{44}^3) + \epsilon_2\omega_{44}^2\omega_{22}^3 - \epsilon_4\omega_{44}^2\omega_{44}^3 = 0. \tag{3.39}$$

$\bullet X = e_3, Y = e_4, Z = e_4,$

$$e_3(\omega_{44}^1) + \epsilon_3\omega_{44}^3\omega_{33}^1 - \epsilon_4\omega_{44}^3\omega_{44}^1 = 0. \tag{3.40}$$

$$e_3(\omega_{44}^2) + \epsilon_3\omega_{44}^3\omega_{33}^2 - \epsilon_4\omega_{44}^3\omega_{44}^2 = 0. \tag{3.41}$$

From (3.1) and Gauss equation, the scalar curvature ρ is given by

$$\rho = 12H^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2. \tag{3.42}$$

Using (2.5), (2.8), (2.10), (3.1) and Lemma 3.1, we find

$$-\epsilon_1 e_1 e_1(H) + (\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1)\epsilon_1 e_1(H) + \varepsilon H(16H^2 - \rho) = 0. \tag{3.43}$$

From (3.3) and Lemma 3.1, we obtain

$$e_i e_1(H) = 0, \qquad i = 2, 3, 4. \tag{3.44}$$

For constant scalar curvature ρ, using (3.43) and (3.44), we get

$$e_i(\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1) = 0, \qquad i = 2, 3, 4. \tag{3.45}$$

Also, using Lemma 3.1, it is easy to see that

$$[e_1, e_i] = \epsilon_i\omega_{ii}^1 e_i, \qquad i = 2, 3, 4. \tag{3.46}$$

Now, we have

Lemma 3.2. *Let $M_r^4, r = 0,1,2,3,4,$ be a biharmonic non-degenerate hypersurface of constant scalar curvature with four distinct principal curvatures in semi-Euclidean space $E_s^5, s = 0,1,2,3,4,5,$ having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then, $e_i(\lambda_j) = 0$, for $i, j = 2, 3, 4,$ and $i \neq j$.*

Proof. Operating with e_2 on both sides of (3.42), (3.11) and using (3.6), we find

$$(\lambda_2 - \lambda_4)^2\omega_{44}^2\epsilon_4 + (\lambda_2 - \lambda_3)^2\omega_{33}^2\epsilon_3 = 0. \tag{3.47}$$

Differentiating (3.47) along e_1 and using (3.6), (3.23), (3.25), we get

$$[-2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4)\omega_{22}^1 + \epsilon_4(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_4)\omega_{44}^1]\omega_{44}^2\epsilon_4 + [-2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\omega_{22}^1 + \epsilon_3(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_3)\omega_{33}^1]\omega_{33}^2\epsilon_3 = 0.$$

and using (3.47) in the above equation, we get

$$\begin{aligned}[2\epsilon_2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 + \epsilon_4(2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3)\omega_{44}^1 \\ -\epsilon_3(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^1]\omega_{44}^2 = 0.\end{aligned} \tag{3.48}$$

Similarly, acting with e_1 and e_2 on (3.11), successively and using (3.6), (3.45), (3.36), (3.38) and (3.47), subsequently, we obtain

$$[\epsilon_2(\lambda_4 - \lambda_3)\omega_{22}^1 + \epsilon_4(\lambda_3 - \lambda_2)\omega_{44}^1 + \epsilon_3(\lambda_2 - \lambda_4)\omega_{33}^1]\omega_{44}^2 = 0. \tag{3.49}$$

Equations (3.48) and (3.49) show that either ω_{44}^2, or the expression between square brackets, has to vanish. We now prove that ω_{44}^2 has to be zero. In fact, if $\omega_{44}^2 \neq 0$, then the expressions between square brackets has to be zero:

$$2\epsilon_2(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)\omega_{22}^1+\epsilon_4(2\lambda_1+\lambda_2-3\lambda_4)(\lambda_2-\lambda_3)\omega_{44}^1 -\epsilon_3(2\lambda_1+\lambda_2-3\lambda_3)(\lambda_2-\lambda_4)\omega_{33}^1=0. \tag{3.50}$$

$$\epsilon_2(\lambda_4-\lambda_3)\omega_{22}^1+\epsilon_4(\lambda_3-\lambda_2)\omega_{44}^1+\epsilon_3(\lambda_2-\lambda_4)\omega_{33}^1=0. \tag{3.51}$$

Eliminating ω_{22}^1 from (3.50) and (3.51), we get

$$(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\epsilon_3\omega_{33}^1+\epsilon_4\omega_{44}^1)=0, \tag{3.52}$$

which shows that

$$\epsilon_3\omega_{33}^1+\epsilon_4\omega_{44}^1=0. \tag{3.53}$$

If (3.53) is true, then using it to eliminate ω_{33}^1, from (3.50) and (3.51), we find

$$2\epsilon_2(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)\omega_{22}^1+\epsilon_4[(2\lambda_1+\lambda_2-3\lambda_4)(\lambda_2-\lambda_3) +(2\lambda_1+\lambda_2-3\lambda_3)(\lambda_2-\lambda_4)]\omega_{44}^1=0. \tag{3.54}$$

$$\epsilon_2(\lambda_4-\lambda_3)\omega_{22}^1+\epsilon_4(\lambda_3+\lambda_4-2\lambda_2)\omega_{44}^1=0. \tag{3.55}$$

From (3.54) and (3.55), we obtain

$$(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)=0, \tag{3.56}$$

which is contradiction of the fact that principal curvatures are distinct. Therefore, $\omega_{44}^2=0$, which gives $\omega_{33}^2=0$ in view of (3.47). Consequently, $e_2(\lambda_3)=e_2(\lambda_4)=0$.

Now, we claim $e_3(\lambda_2)=e_3(\lambda_4)=0$. To prove this operating with e_3 on both sides of (3.42), (3.11) and using (3.6), we find

$$(\lambda_3-\lambda_4)^2\omega_{44}^3\epsilon_4+(\lambda_3-\lambda_2)^2\omega_{22}^3\epsilon_2=0. \tag{3.57}$$

Differentiating (3.57) along e_1 and using (3.6), (3.21) and (3.26), we get

$$[-2\epsilon_3(\lambda_1-\lambda_3)(\lambda_3-\lambda_4)\omega_{33}^1+\epsilon_4(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_4)\omega_{44}^1]\omega_{44}^3\epsilon_4+[-2\epsilon_3(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)\omega_{33}^1+\epsilon_2(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_2)\omega_{22}^1]\omega_{22}^3\epsilon_2=0.$$

Using (3.57) in the above equation, we get

$$[2\epsilon_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\omega_{33}^1+\epsilon_4(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_2)\omega_{44}^1 -\epsilon_2(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_4)\omega_{22}^1]\omega_{44}^3=0. \tag{3.58}$$

Similarly, acting with e_1 and e_3 on (3.11), successively and using (3.6), (3.45), (3.28), (3.40) and (3.57), subsequently we obtain

$$[\epsilon_3(\lambda_4-\lambda_2)\omega_{33}^1+\epsilon_4(\lambda_2-\lambda_3)\omega_{44}^1+\epsilon_2(\lambda_3-\lambda_4)\omega_{22}^1]\omega_{44}^3=0. \tag{3.59}$$

Equations (3.58) and (3.59) show that either ω_{44}^3, or the expression between square brackets, has to vanish. We now prove that ω_{44}^3, has to be zero. In fact, if $\omega_{44}^3\neq 0$, then the expressions between square brackets has to be zero:

$$\begin{aligned}2\epsilon_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\omega_{33}^1+\epsilon_4(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_2)\omega_{44}^1\\ -\epsilon_2(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_4)\omega_{22}^1=0.\end{aligned} \tag{3.60}$$

$$\epsilon_3(\lambda_4-\lambda_2)\omega_{33}^1+\epsilon_4(\lambda_2-\lambda_3)\omega_{44}^1+\epsilon_2(\lambda_3-\lambda_4)\omega_{22}^1=0. \tag{3.61}$$

Eliminating ω_{33}^1 from (3.60) and (3.61), we get

$$(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)(\epsilon_2\omega_{22}^1+\epsilon_4\omega_{44}^1)=0, \tag{3.62}$$

which shows that

$$\epsilon_2\omega_{22}^1+\epsilon_4\omega_{44}^1=0. \tag{3.63}$$

If (3.63) is true, then using it to eliminate ω_{22}^1, from (3.60) and (3.61), we find

$$\begin{aligned}2\epsilon_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\omega_{33}^1+\epsilon_4[(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_2)\\ +(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_4)]\omega_{44}^1=0.\end{aligned} \tag{3.64}$$

$$\epsilon_3(\lambda_4-\lambda_2)\omega_{33}^1+\epsilon_4(\lambda_2+\lambda_4-2\lambda_3)\omega_{44}^1=0. \tag{3.65}$$

From (3.64) and (3.65), we obtain

$$(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)=0, \tag{3.66}$$

which is a contradiction of the fact that principal curvatures are distinct. Therefore, $\omega_{44}^3=0$, which gives $\omega_{22}^3=0$ in view of (3.57). Consequently, $e_3(\lambda_2)=e_3(\lambda_4)=0$.

Now, we claim $e_4(\lambda_2)=e_4(\lambda_3)=0$. To prove this acting e_4 on both sides of (3.42), (3.11) and using (3.6), we find

$$(\lambda_4-\lambda_3)^2\omega_{33}^4\epsilon_3+(\lambda_4-\lambda_2)^2\omega_{22}^4\epsilon_2=0. \tag{3.67}$$

Differentiating (3.67) along e_1 and using (3.6), (3.22) and (3.24), we get

$$[-2\epsilon_4(\lambda_1-\lambda_4)(\lambda_4-\lambda_3)\omega_{44}^1+\epsilon_3(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_3)\omega_{33}^1]\omega_{33}^4\epsilon_3+[-2\epsilon_4(\lambda_1-\lambda_4)(\lambda_4-\lambda_2)\omega_{44}^1+\epsilon_2(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_2)\omega_{22}^1]\omega_{22}^4\epsilon_2=0$$

and using (3.67) in the above equation, we get

$$[2\epsilon_4(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)\omega_{44}^1+\epsilon_3(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_2)\omega_{33}^1 \\ -\epsilon_2(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_3)\omega_{22}^1]\omega_{33}^4=0. \tag{3.68}$$

Similarly, acting with e_1 and e_4 on (3.11), successively and using (3.6), (3.45), (3.31), (3.34) and (3.67), subsequently we obtain

$$[\epsilon_4(\lambda_3-\lambda_2)\omega_{44}^1+\epsilon_3(\lambda_2-\lambda_4)\omega_{33}^1+\epsilon_2(\lambda_4-\lambda_3)\omega_{22}^1]\omega_{33}^4=0. \tag{3.69}$$

Equations (3.68) and (3.69) show that either ω_{33}^4, or the expression between square brackets, has to vanish. We now prove that ω_{33}^4, has to be zero. In fact, if $\omega_{33}^4\neq 0$ then the expressions between square brackets has to be zero:

$$2\epsilon_4(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)\omega_{44}^1+\epsilon_3(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_2)\omega_{33}^1 \\ -\epsilon_2(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_3)\omega_{22}^1=0. \tag{3.70}$$

$$\epsilon_4(\lambda_3-\lambda_2)\omega_{44}^1+\epsilon_3(\lambda_2-\lambda_4)\omega_{33}^1+\epsilon_2(\lambda_4-\lambda_3)\omega_{22}^1=0. \tag{3.71}$$

Eliminating ω_{44}^1 from (3.70) and (3.71), we get

$$(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)(\epsilon_2\omega_{22}^1+\epsilon_3\omega_{33}^1)=0, \tag{3.72}$$

which shows that

$$\epsilon_2\omega_{22}^1+\epsilon_3\omega_{33}^1=0. \tag{3.73}$$

If (3.73) is true, then using it to eliminate ω_{22}^1, from (3.70) and (3.71), we find

$$2\epsilon_4(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)\omega_{44}^1+\epsilon_3[(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_2) \\ +(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_3)]\omega_{33}^1=0. \tag{3.74}$$

$$\epsilon_4(\lambda_3-\lambda_2)\omega_{44}^1+\epsilon_3(\lambda_2+\lambda_3-2\lambda_4)\omega_{33}^1=0. \tag{3.75}$$

From (3.74) and (3.75), we obtain

$$(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)=0, \tag{3.76}$$

which is a contradiction of the fact that principal curvatures are distinct. Therefore, $\omega_{33}^4=0$, which gives $\omega_{22}^4=0$ in view of (3.67). Consequently, $e_4(\lambda_2)=e_4(\lambda_3)=0$, which completes the proof. □

Now, we have:

Lemma 3.3. *Let $M_r^4, r=0,1,2,3,4,$ be a biharmonic non-degenerate hypersurface of constant scalar curvature with four distinct principal curvatures in semi-Euclidean space $E_s^5, s=0,1,2,3,4,5,$ having the shape operator given by* (3.1) *with respect to suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$. Then, we have*

$$(\lambda_3-\lambda_4)\epsilon_2\omega_{22}^1+(\lambda_4-\lambda_2)\epsilon_3\omega_{33}^1+(\lambda_2-\lambda_3)\epsilon_4\omega_{44}^1=0 \tag{3.77}$$

or

$$\omega^i_{kj} = 0, \tag{3.78}$$

for $i \neq j \neq k$, and $i, j, k = 2, 3, 4$.

Proof. Using Lemma 3.2 in (3.16) and (3.17), we get

$$\nabla_{e_2} e_3 = \omega^4_{23} e_4 \epsilon_4, \quad \text{and} \quad \nabla_{e_3} e_2 = \omega^4_{32} e_4 \epsilon_4,$$

respectively. Using it along with (2.5) and Lemma 3.1 to evaluate $g(R(e_1, e_2)e_3, e_4)$, and $g(R(e_1, e_3)e_2, e_4)$, we obtain

$$e_1(\omega^4_{23}) - \epsilon_2 \omega^1_{22} \omega^4_{23} = 0, \quad \text{and} \quad e_1(\omega^4_{32}) - \epsilon_3 \omega^1_{33} \omega^4_{32} = 0,$$

respectively.

Putting $j = 4, k = 2, i = 3$ in (3.7), we get

$$(\lambda_3 - \lambda_4)\omega^4_{23} = (\lambda_2 - \lambda_4)\omega^4_{32}. \tag{3.79}$$

Differentiating (3.79) with respect to e_1 and simplifying, we get

$$\omega^4_{32}(\epsilon_2 \omega^1_{22} - \epsilon_4 \omega^1_{44}) = \omega^4_{23}(\epsilon_3 \omega^1_{33} - \epsilon_4 \omega^1_{44}). \tag{3.80}$$

Now, (3.79) and (3.80) are homogeneous system of equations in two variables ω^4_{32} and ω^4_{23} having either non trivial solution or trivial solution. If it has trivial solution only, then, we have $\omega^4_{32} = 0$ and $\omega^4_{23} = 0$.

If it has non-trivial solution also, then the determinant formed by coefficients of ω^4_{32} and ω^4_{23} in (3.79) and (3.80) will be zero, i.e.,

$$(\lambda_3 - \lambda_4)\epsilon_2 \omega^1_{22} + (\lambda_4 - \lambda_2)\epsilon_3 \omega^1_{33} + (\lambda_2 - \lambda_3)\epsilon_4 \omega^1_{44} = 0.$$

Similarly, we can prove that either $\omega^3_{42} = \omega^3_{24} = \omega^2_{43} = \omega^2_{34} = 0$ or the determinant formed by their coefficients is zero. This completes the proof of the Lemma.□

Then we have the following two cases:

Case A: Let (3.78) holds, then using Lemma 3.2 in (3.27), (3.30) and (3.33), we find

$$-\epsilon_1 \omega^1_{22} \omega^1_{33} = \epsilon_2 \epsilon_3 \lambda_2 \lambda_3, \tag{3.81}$$

$$-\epsilon_1 \omega^1_{22} \omega^1_{44} = \epsilon_2 \epsilon_4 \lambda_2 \lambda_4, \tag{3.82}$$

and

$$-\epsilon_1 \omega^1_{33} \omega^1_{44} = \epsilon_3 \epsilon_4 \lambda_3 \lambda_4, \tag{3.83}$$

respectively.

Using (3.11), (3.18)∼(3.20) and (3.42), we get

$$\begin{aligned}&3e_1e_1(H)-\varepsilon\{\tfrac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1}\}\\ &-e_1(H)\{\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\}=-\epsilon_1H(24H^2-\rho).\end{aligned}\tag{3.84}$$

From (3.43) and (3.84), we find

$$\begin{aligned}&\varepsilon\{\tfrac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1}\}-2e_1(H)\{\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\\ &+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\}=\epsilon_1H[24H^2(1+2\varepsilon)+\rho(1+3\varepsilon)].\end{aligned}\tag{3.85}$$

Using (3.81)∼(3.83) in (3.85), we obtain

$$\begin{aligned}\epsilon_1e_1(H)\{\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\}&=H^3(6-18\varepsilon)\\ &-\tfrac{\rho H(1+8\varepsilon)}{4}+\tfrac{3\lambda_2\lambda_3\lambda_4}{4}.\end{aligned}\tag{3.86}$$

Using (3.43) and (3.86), we have

$$\epsilon_1e_1e_1(H)=H^3(6-2\varepsilon)-\frac{\rho H(1+12\varepsilon)}{4}+\frac{3\lambda_2\lambda_3\lambda_4}{4}.\tag{3.87}$$

On the other hand, using (3.6) and (3.81)∼(3.83), we have

$$\begin{aligned}e_1(\lambda_2\lambda_3\lambda_4)&=\lambda_2\lambda_3\lambda_4(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})\\ &-6\varepsilon\epsilon_1H(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1})(\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1})(\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}).\end{aligned}\tag{3.88}$$

Using (3.18)∼(3.20) and (3.81)∼(3.83), we find

$$12\varepsilon H\lambda_2\lambda_3\lambda_4=e_1(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\frac{e_1(\lambda_4)}{\lambda_4-\lambda_1}).\tag{3.89}$$

Also, from (3.81)∼(3.83), we obtain

$$(\lambda_2\lambda_3\lambda_4)^2=-\epsilon_1(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\frac{e_1(\lambda_4)}{\lambda_4-\lambda_1})^2.\tag{3.90}$$

Differentiating (3.90) along e_1, and using (3.88)∼(3.89), we get

$$e_1(\lambda_2\lambda_3\lambda_4)=2\lambda_2\lambda_3\lambda_4(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_4)}{\lambda_4-\lambda_1}).\tag{3.91}$$

Acting with e_1 on both sides of (3.86) and using (3.18)∼(3.20), (3.86), (3.87) and (3.91), we find

$$\begin{aligned}e_1(H)\{(6-54\varepsilon)H^2-\tfrac{(5+8\varepsilon)}{4}\rho\}&=\{\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\}\\ &\{H^3(12-20\varepsilon)-\tfrac{\rho H(2+20\varepsilon)}{4}\}.\end{aligned}\tag{3.92}$$

Differentiating again (3.92) along e_1 and using (3.18)∼(3.20), (3.81)∼(3.83), (3.42) and (3.87), we obtain

$$\epsilon_1\{H^3(6-2\varepsilon)-\tfrac{\rho H(1+12\varepsilon)}{4}+\tfrac{3\lambda_2\lambda_3\lambda_4}{4}\}\{(6-54\varepsilon)H^2-\tfrac{(5+8\varepsilon)}{4}\rho\}+2e_1^2(H)(6-54\varepsilon)H = \epsilon_1(\rho-24H^2)\{H^3(12-20\varepsilon)-\tfrac{\rho H(2+20\varepsilon)}{4}\}+\{\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\}\{3H^2(12-20\varepsilon)-\tfrac{\rho(2+20\varepsilon)}{4}\}e_1(H),$$

which on using (3.86), gives

$$\begin{aligned}\epsilon_1\{H^3(6-2\varepsilon)-\tfrac{\rho H(1+12\varepsilon)}{4}+\tfrac{3\lambda_2\lambda_3\lambda_4}{4}\}\{(6-54\varepsilon)H^2 \\ -\tfrac{(5+8\varepsilon)}{4}\rho\}+2e_1^2(H)(6-54\varepsilon)H=\epsilon_1(\rho-24H^2)\{H^3(12-20\varepsilon) \\ -\tfrac{\rho H(2+20\varepsilon)}{4}\}+\epsilon_1\{3H^2(12-20\varepsilon)-\tfrac{\rho(2+20\varepsilon)}{4}\} \\ \{H^3(6-18\varepsilon)-\tfrac{\rho H(1+8\varepsilon)}{4}+\tfrac{3\lambda_2\lambda_3\lambda_4}{4}\}.\end{aligned} \tag{3.93}$$

From (3.86) and (3.87), we have

$$\begin{aligned}&\epsilon_1 e_1^2(H)\{(6-54\varepsilon)H^2-\tfrac{(5+8\varepsilon)}{4}\rho\} \\ &=\{H^3(12-20\varepsilon)-\tfrac{\rho H(2+20\varepsilon)}{4}\}\{H^3(6-18\varepsilon)-\tfrac{\rho H(1+8\varepsilon)}{4}+\tfrac{3\lambda_2\lambda_3\lambda_4}{4}\}.\end{aligned} \tag{3.94}$$

Then we have following cases:

(i) For spacelike normal vector ξ: In this case $\varepsilon=1$. Then eliminating $e_1^2(H)$ using (3.93) and (3.94), we get

$$\begin{aligned}\{8(\rho-24H^2)(16H^3+11\rho H)+2(48H^2+11\rho)(48H^3 \\ -9\rho H+3\lambda_2\lambda_3\lambda_4)\}(192H^2+13\rho)=(16H^3-13\rho H \\ +3\lambda_2\lambda_3\lambda_4)(192H^2+13\rho)^2+788H(48H^3+9\rho H \\ -3\lambda_2\lambda_3\lambda_4)(16H^3+11\rho H),\end{aligned} \tag{3.95}$$

or

$$\begin{aligned}\lambda_2\lambda_3\lambda_4(-18432H^4+26784\rho H^2+351\rho^2)=884736H^7 \\ +37136\rho^2H^3+1521\rho^3H-370176\rho H^5.\end{aligned} \tag{3.96}$$

On the other hand, differentiating (3.94) along e_1 and using (3.91) and (3.87), we obtain

$$\begin{aligned}\{8(\rho+12H^2)(16H^3+11\rho H)+(192H^2+13\rho)(-32H^3-22\rho H \\ +6\lambda_2\lambda_3\lambda_4)+2(48H^2+11\rho)(-48H^3-9\rho H+3\lambda_2\lambda_3\lambda_4)\} \\ (192H^2+13\rho)=2(16H^3-13\rho H+3\lambda_2\lambda_3\lambda_4)(192H^2+13\rho)^2 \\ -768H(48H^3+9\rho H-3\lambda_2\lambda_3\lambda_4)(16H^3+11\rho H),\end{aligned} \tag{3.97}$$

which on solving, gives

$$\begin{aligned}\lambda_2\lambda_3\lambda_4(9216H^4-4464\rho H^2+429\rho^2)=1179648H^7 \\ -81408\rho H^5-27248\rho^2H^3+377\rho^3H.\end{aligned} \tag{3.98}$$

Now, eliminating $\lambda_2\lambda_3\lambda_4$ from equations (3.96) and (3.98), we get

$$f(H,\rho)=2H(-260091\rho^5-4304040H^2\rho^4-227385216H^4\rho^3-1819201536H^6\rho^2+20228603900H^8\rho-14948499460H^{10})=0.$$

Now, $f(H,\rho)$ is a polynomial in H with constant coefficients and as a real function satisfying a polynomial equation with constant coefficients must be constant, we get that H is constant.
medskip

(ii) As above, for timelike normal vector ξ, we get the same result that H must be constant.

Case B: Let (3.77) hold. Then, from (3.5) and (3.7), we find

$$(\lambda_2-\lambda_3)\omega_{42}^3=(\lambda_4-\lambda_3)\omega_{24}^3=(\lambda_2-\lambda_4)\omega_{32}^4. \tag{3.99}$$

Using (3.99) and (3.5), we get

$$\omega_{42}^3\omega_{24}^3+\omega_{32}^4\omega_{23}^4+\omega_{43}^2\omega_{34}^2=0. \tag{3.100}$$

Adding (3.27), (3.30), (3.33) and using (3.11), (3.42), (3.100) and Lemma 3.2, we get

$$-\epsilon_2\epsilon_3\omega_{22}^1\omega_{33}^1-\epsilon_2\epsilon_4\omega_{22}^1\omega_{44}^1-\epsilon_4\epsilon_3\omega_{44}^1\omega_{33}^1=\epsilon_1(12H^2+\frac{\rho}{2}). \tag{3.101}$$

Using (3.5), (3.7) and (3.100), we obtain

$$\lambda_2\omega_{43}^2\omega_{34}^2+\lambda_3\omega_{42}^3\omega_{24}^3+\lambda_4\omega_{32}^4\omega_{23}^4=0. \tag{3.102}$$

Multiplying (3.27), (3.30), (3.33) by $\lambda_4\epsilon_2\epsilon_3$, $\lambda_3\epsilon_2\epsilon_4$, and $\lambda_2\epsilon_4\epsilon_3$ respectively, and adding these equations and using (3.102), we find

$$\epsilon_2\epsilon_3\lambda_4\omega_{22}^1\omega_{33}^1+\epsilon_2\epsilon_4\lambda_3\omega_{22}^1\omega_{44}^1+\epsilon_4\epsilon_3\lambda_2\omega_{44}^1\omega_{33}^1=-3\epsilon_1\lambda_2\lambda_3\lambda_4. \tag{3.103}$$

Using (3.100), (3.77), (3.7) and (3.5), we get

$$\epsilon_2\omega_{22}^1\omega_{43}^2\omega_{34}^2+\epsilon_3\omega_{33}^1\omega_{42}^3\omega_{24}^3+\epsilon_4\omega_{44}^1\omega_{32}^4\omega_{23}^4=0. \tag{3.104}$$

Multiplying (3.27), (3.30), (3.33) by $\epsilon_2\epsilon_3\epsilon_4\omega_{44}^1$, $\epsilon_2\epsilon_3\epsilon_4\omega_{33}^1$, and $\epsilon_2\epsilon_3\epsilon_4\omega_{22}^1$ respectively, and adding these equations and using (3.104), we get

$$\epsilon_4\lambda_2\lambda_3\omega_{44}^1+\epsilon_3\lambda_2\lambda_4\omega_{33}^1+\epsilon_2\lambda_4\lambda_3\omega_{22}^1=-3\epsilon_1\omega_{22}^1\omega_{33}^1\omega_{44}^1. \tag{3.105}$$

Differentiating (3.11) along e_1 and using (3.6), we have

$$\epsilon_2\lambda_2\omega_{22}^1+\epsilon_3\lambda_3\omega_{33}^1+\epsilon_4\lambda_4\omega_{44}^1=6\epsilon e_1(H)-2\epsilon H(\epsilon_2\omega_{22}^1+\epsilon_3\omega_{33}^1+\epsilon_4\omega_{44}^1). \tag{3.106}$$

Again, differentiating (3.42) along e_1 and eliminating $e_1(H)$ using (3.106) and (3.6), we obtain

$$\lambda_2^2\omega_{22}^1\epsilon_2+\lambda_3^2\omega_{33}^1\epsilon_3+\lambda_4^2\omega_{44}^1\epsilon_4=4H^2(\epsilon_2\omega_{22}^1+\epsilon_3\omega_{33}^1+\epsilon_4\omega_{44}^1). \tag{3.107}$$

Also, from (3.11) and (3.42), we find

$$\lambda_2\lambda_3 = 12H^2 + \frac{\rho}{2} - \lambda_4(6\epsilon H - \lambda_4), \tag{3.108}$$

$$\lambda_2\lambda_4 = 12H^2 + \frac{\rho}{2} - \lambda_3(6\epsilon H - \lambda_3), \tag{3.109}$$

$$\lambda_3\lambda_4 = 12H^2 + \frac{\rho}{2} - \lambda_2(6\epsilon H - \lambda_2). \tag{3.110}$$

Now, multiplying (3.108), (3.109) and (3.110) by $\epsilon_4\omega_{44}^1$, $\epsilon_3\omega_{33}^1$ and $\epsilon_2\omega_{22}^1$, respectively and adding these equations and using (3.106) and (3.107), we get

$$\begin{aligned}\epsilon_4\lambda_2\lambda_3\omega_{44}^1 + \epsilon_3\lambda_2\lambda_4\omega_{33}^1 + \epsilon_2\lambda_4\lambda_3\omega_{22}^1 = (\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 \\ +\epsilon_4\omega_{44}^1)(28H^2 + \tfrac{\rho}{2}) - 36He_1(H).\end{aligned} \tag{3.111}$$

Equating (3.105) and (3.111), we get

$$-3\epsilon_1\omega_{22}^1\omega_{33}^1\omega_{44}^1 = (\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1)(28H^2 + \frac{\rho}{2}) - 36He_1(H). \tag{3.112}$$

Using (3.6), (3.106) and (3.111), we find

$$e_1(\lambda_2\lambda_3\lambda_4) = (\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1)(\lambda_2\lambda_3\lambda_4 + 56H^3\epsilon + \rho H) - 36He_1(H). \tag{3.113}$$

Eliminating $e_1e_1(H)$ form (3.43) and (3.84), and using (3.6), (3.101) and (3.103), we get

$$-4\epsilon_1e_1(H)(\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1) = 3\lambda_2\lambda_3\lambda_4 - 24H^3(1+7\epsilon) + \rho H(1-2\epsilon). \tag{3.114}$$

Using (3.43) and (3.114), we obtain

$$-4\epsilon_1e_1e_1(H) = 3\lambda_2\lambda_3\lambda_4 - 8H^3(1+9\epsilon) - \rho H(1-\epsilon). \tag{3.115}$$

Now, using (3.11), (3.18)∼(3.20) and (3.101), we find

$$\begin{aligned}e_1(\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1) = -12H^2\epsilon_1 \\ +(\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1)^2 + 2\epsilon_1(12H^2 + \tfrac{\rho}{2}).\end{aligned} \tag{3.116}$$

Differentiating (3.114) and using (3.113), (3.115) and (3.116), we obtain

$$\begin{aligned}(\epsilon_2\omega_{22}^1 + \epsilon_3\omega_{33}^1 + \epsilon_4\omega_{44}^1)(-8H^3(14+63\epsilon) + \rho H(2-3\epsilon) + 3\lambda_2\lambda_3\lambda_4) \\ = e_1(H)(-12H^2(2+60\epsilon) + \rho(5-2\epsilon)).\end{aligned} \tag{3.117}$$

Using (3.114) and (3.117), we get

$$\begin{aligned}-9(\lambda_2\lambda_3\lambda_4)^2 + \lambda_2\lambda_3\lambda_4\{(408+2016\epsilon)H^3 + (-9+15\epsilon)\rho H\} \\ +(544\epsilon - 1352)\rho H^4 + (7\epsilon - 8)\rho^2H^2 - (30912\epsilon + 87360)H^6 \\ = 4\epsilon_1e_1^2(H)(\rho(5-2\epsilon) - 12H^2(60\epsilon + 2)).\end{aligned} \tag{3.118}$$

Differentiating (3.118) and using (3.113), (3.115) and (3.117), we obtain

$$\begin{aligned}(\lambda_2\lambda_3\lambda_4)^2\{(45\epsilon-27)\rho+(3672+22032\epsilon)H^2\}+\lambda_2\lambda_3\lambda_4\{(-1800576\\ -4076352\epsilon)H^5+(-23040+4896\epsilon)\rho H^3+(-12-54\epsilon)\rho^2H\}\\ +(225437184\epsilon+144883200)H^8+(1510656\epsilon+160960)\rho H^6+(384\epsilon\\ +8808)\rho^2H^4+(-69\epsilon+89)\rho^3H^2=-96\epsilon_1e_1^2(H)H(60\epsilon+2)\\ (-8H^3(14+63)+\rho H(2-3\epsilon)+3\lambda_2\lambda_3\lambda_4).\end{aligned} \tag{3.119}$$

Differentiating (3.112) and using (3.11), (3.114) (3.18)∼(3.20), we get

$$\begin{aligned}36\epsilon_1e_1^2(H)=-6H\lambda_2\lambda_3\lambda_4(7-3\epsilon\epsilon_2\epsilon_3\epsilon_4)+24H^4(34+134\epsilon)\\ +\rho H^2(20+37\epsilon)+\tfrac{\rho^2}{2}.\end{aligned} \tag{3.120}$$

Using (3.120) to eliminate $e_1^2(H)$ from (3.118) and (3.119), we obtain

$$(\lambda_2\lambda_3\lambda_4)^2a_1+\lambda_2\lambda_3\lambda_4b_1=c_1, \tag{3.121}$$

and

$$(\lambda_2\lambda_3\lambda_4)^2a_2+\lambda_2\lambda_3\lambda_4b_2=c_2, \tag{3.122}$$

respectively, where

$$\begin{aligned}
a_1&=-162,\\
b_1&=6H[(888-4032\epsilon+\epsilon_2\epsilon_3\epsilon_4(4320+144\epsilon))H^2+(43+17\epsilon+\epsilon_2\epsilon_3\epsilon_4(12-30\epsilon))\rho],\\
c_1&=-(772992\epsilon+3097728)H^6-(11472\epsilon+34608)\rho H^4\\
&\quad-(556\epsilon-172)\rho^2H^2-\rho^3(2\epsilon-5),\\
a_2&=9[(15\epsilon-9)\rho+(1000+624\epsilon+\epsilon_2\epsilon_3\epsilon_4(2880+96\epsilon))H^2],\\
b_2&=6H[(158406-1383840\epsilon+\epsilon_2\epsilon_3\epsilon_4(-185472-731136\epsilon))H^4+(7376\\
&\quad+1160\epsilon+\epsilon_2\epsilon_3\epsilon_4(2736-4224\epsilon))\rho H^2+(-2+93\epsilon)\rho^2],\\
c_2&=-H^2[(-157914624\epsilon+38270672)H^6+(-9505792\epsilon-4894784)\rho H^4\\
&\quad+(-63616\epsilon-89848)\rho^2H^2+(249\epsilon-437)\rho^3].
\end{aligned}$$

Now, eliminating $(\lambda_2\lambda_3\lambda_4)^2$ and $\lambda_2\lambda_3\lambda_4$ from (3.121) and (3.122), we obtain

$$(a_1c_2-a_2c_1)^2-(c_1b_2-c_2b_1)(a_1b_2-a_2b_1)=0,$$

which is a polynomial equation in H of degree 16 with constant coefficients. Now a real function satisfying a polynomial equation with constant coefficients must be constant and therefore H is constant, which is a contradiction.

(b) *The case of three distinct principal curvatures*

Suppose that M is a biharmonic hypersurfaces with three distinct principal curvatures and constant scalar curvature with diagonal shape operator. We also assume that mean curvature is not constant and grad$H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset U of M_r^4, with grad$_pH \neq 0$ for all $p \in U$. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2\varepsilon H$. Without losing generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of the hypersurface will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2\varepsilon H & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.123}$$

From (3.11), (3.42) and (3.123), we get

$$2\lambda + \lambda_4 = 6\varepsilon H. \tag{3.124}$$

$$2\lambda^2 + \lambda_4^2 = 12H^2 - \rho. \tag{3.125}$$

We have the following cases:

(i) For spacelike normal vector ξ: In this case $\varepsilon = 1$. From (3.124) and (3.125), we find

$$e_1(\lambda) = e_1(\lambda_4) = 2e_1(H). \tag{3.126}$$

and

$$e_4(\lambda) = e_4(\lambda_4) = 0. \tag{3.127}$$

Now, equations (3.18), (3.20) and (3.33) reduce to

$$e_1(\frac{e_1(\lambda)}{\lambda + 2H}) - (\frac{e_1(\lambda)}{\lambda + 2H})^2 = -2\epsilon_1 H\lambda, \tag{3.128}$$

$$e_1(\frac{e_1(6H - 2\lambda)}{8H - 2\lambda}) - (\frac{e_1(6H - 2\lambda)}{8H - 2\lambda})^2 = -2\epsilon_1 H(6H - 2\lambda), \tag{3.129}$$

and,

$$(\frac{e_1(\lambda)}{\lambda + 2H})(\frac{e_1(6H - 2\lambda)}{8H - 2\lambda}) = -\epsilon_1\lambda(6H - 2\lambda). \tag{3.130}$$

Using (3.126) in (3.128), (3.129) and (3.130), we find

$$e_1e_1(H) - (\frac{6e_1^2(H)}{\lambda + 2H}) = -\epsilon_1 H\lambda(\lambda + 2H), \tag{3.131}$$

$$e_1e_1(H) - (\frac{6e_1^2(H)}{8H - 2\lambda}) = -\epsilon_1 H(6H - 2\lambda)(8H - 2\lambda), \tag{3.132}$$

and,

$$4e_1^2(H) = -\epsilon_1\lambda(\lambda + 2H)(6H - 2\lambda)(8H - 2\lambda). \tag{3.133}$$

From (3.131) and (3.132), we get

$$3e_1^2(H) = -\epsilon_1 H(\lambda + 2H)(\lambda - 8H)(\lambda - 4H). \tag{3.134}$$

Eliminating $e_1^2(H)$ from (3.133) and (3.134), we obtain

$$3\lambda^2 - 8\lambda H - 8H^2 = 0. \tag{3.135}$$

On solving (3.135), we get $\lambda = (\frac{4\pm 2\sqrt{10}}{3})H$, which gives $e_1(\lambda) = (\frac{4\pm 2\sqrt{10}}{3})e_1(H)$, thus contradicting (3.126).

(ii) Proceeding as above, for timelike normal vector ξ, we get a contradiction.

(c) *The case of two distinct principal curvatures*

Suppose that M is a nonminimal biharmonic hypersurface with two distinct principal curvatures and constant scalar curvature with shape operator diagonal. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2\varepsilon H$. Without losing generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2\varepsilon H & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}. \tag{3.136}$$

From (3.11) and (3.136), we get

$$\lambda = 2\varepsilon H. \tag{3.137}$$

Also, from (3.6) and (3.137), we obtain

$$\epsilon_2\omega_{22}^1 = \epsilon_3\omega_{33}^1 = \epsilon_4\omega_{44}^1 = \frac{e_1(H)}{2H}. \tag{3.138}$$

Also, from (2.5), $R(e_1, e_2, e_1, e_2)$ shows that

$$e_1(\omega_{22}^1)\epsilon_2 = (\omega_{22}^1)^2 - \epsilon_1 4H^2. \tag{3.139}$$

Using (3.138) and (3.139), we find

$$\epsilon_1 e_1 e_1(H) = \frac{3e_1^2(H)}{2H} - 8H^3. \tag{3.140}$$

On the other hand, from (2.8), (2.10), (3.136), and (3.138), we have

$$\epsilon_1 e_1 e_1(H) = \frac{3e_1^2(H)}{2H} + 16H^3. \tag{3.141}$$

From (3.140) and (3.141), we get that H must be zero, which is a contradiction.

Combining (a), (b) and (c), we have:

Theorem 3.4. *Every biharmonic non-degenerate hypersurfaces M_r^4, $r = 0, 1, 2, 3, 4$, of constant scalar curvature with diagonal shape operator in semi-Euclidean space E_s^5, $s = 0, 1, 2, 3, 4, 5$ has zero mean curvature.*

Acknowledgement. The authors are thankful to the referee for helpful suggestions to improve the original version of the article.

References

[1] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis, V. Papantoniou, *Biharmonic Lorentzian hypersurfaces in E_1^4*, Pac. J. Math., **229(2)**(2007), 293–305.

[2] A. Balmus, *Biharmonic Maps and Submanifolds Ph. D. thesis*, Universita degli Studi di Cagliari, Italy, (2007).

[3] A. Balmus, S. Montaldo, C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel. J. Math., **168**(2008), 201–220.

[4] A. Balmus, S. Montaldo, C. Oniciuc, *Classification results and new examples of proper biharmonic submanifolds in spheres*, Note Mat., **1(1)**(2008), 49–61.

[5] A. Balmus, S. Montaldo, C. Oniciuc, *Properties of biharmonic submanifolds in spheres*, J. Geom. Symmetry Phys., **17**(2010), 87–102.

[6] A. Balmus, S. Montaldo, C. Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr., **283(12)**(2010), 1696–1705.

[7] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., **17(2)**(1991), 169–188.

[8] B. Y. Chen, *Classification of marginally trapped Lorentzian flat surfaces in E_1^4 and its application to biharmonic surfaces*, J. Math. Anal. Appl., **340**(2008), 861–875.

[9] B. Y. Chen, S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ., **A 45**(1991), 323–347.

[10] B. Y. Chen, S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math., **52**(1998), 1–18.

[11] B. Y. Chen, M. I. Munteanu, *Biharmonic ideal hypersurfaces in Euclidean spaces*, Differ. Geom. Appl., **31**(2013), 1–16.

[12] Deepika, Ram Shankar Gupta, *Biharmonic hypersurfaces in E^5 with zero scalar curvature*, Afr. Diaspora J. Math., **18(1)**(2015), 12–26.

[13] F. Defever, G. Kaimakamis, V. Papantoniou, *Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space E_s^4*, J. Math. Anal. Appl., **315**(2006), 276–286.

[14] I. Dimitric, *Quadric representation and submanifolds of finite type, Doctoral thesis*, Michigan State University, (1989).

[15] I. Dimitric, *Submanifolds of E^n with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sin, **20**(1992), 53–65.

[16] J. Eells, J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86**(1964), 109–160.

[17] K. Akutagawa, S. Maeta, *Biharmonic properly immersed submanifolds in Euclidean spaces*, Geom. Dedicate, **164**(2013), 351–355.

[18] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds of S^3*, Int. J. Math., **12(8)**(2001), 867–876.

[19] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math., **130**(2002), 109–123.

[20] Ram Shankar Gupta, *On biharmonic hypersurfaces in Euclidean space of arbitrary dimension*, Glasgow Math. J., **57** (2015), 633–642.

[21] Ram Shankar Gupta, *Biharmonic hypersurfaces in E_s^5*, An. St. Univ. Al. I. Cuza, (accepted).

[22] T. Hasanis, T. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr., **172**(1995), 145–169.

[23] Yu Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in the Euclidean 5-space*, Journal of Geometry and Physics, **75**(2014), 113–119.