

Invariant Trace Fields of Chain Links

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ABSTRACT. In this paper, we compute the trace field of $C(2, s)$, the complement of two component chain link with s left half twists in \mathbb{S}^3 , for every s . As a result, for every $n \in \mathbb{N} \setminus \{1\}$, we can find $s \in \mathbb{Z}$ such that the degree of the trace field of $C(2, s)$ is n . We also prove that if for fixed p , the degree of the trace field of $C(p, s)$ runs over $\mathbb{N} \setminus \{1\}$, then p is contained in $\{1, 2, 4, 8\}$.

1. Introduction

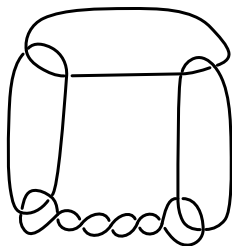
Let L be a link. If $\mathbb{S}^3 \setminus L$ admits a complete hyperbolic structure of finite volume, then there is a torsion-free Kleinian group Γ such that $\mathbb{S}^3 \setminus L$ is homeomorphic to \mathbb{H}^3/Γ , where \mathbb{H}^3 is the hyperbolic 3-space. By the Mostow-Prasad Rigidity Theorem, if $\mathbb{S}^3 \setminus L$ is homeomorphic to \mathbb{H}^3/Γ_1 and \mathbb{H}^3/Γ_2 , then Γ_1 is conjugate to Γ_2 . Hence the extension field $\mathbb{Q}(\{\text{tr}\gamma : \gamma \in \Gamma\})$, called the trace field of Γ , is a topological invariant, since trace is invariant under conjugation. While we know many features of trace fields, there are a few infinite families of links such that we can compute the trace fields.

Let $C(p, s)$ be the complement of a p chain link with s left half twists in \mathbb{S}^3 pictured in Figure 1. In [4], Neumann and Reid proved that $C(p, s)$ has a complete hyperbolic structure of finite volume if and only if $\{|p+s|, |s|\} \not\subset \{0, 1, 2\}$. Moreover, they computed the trace fields of $C(p, s)$ for $|p+s|, |s| \leq 13$. Based on their work, Hoste and Shanahan computed the trace fields of $C(1, s)$ in [1]. One of the purpose of this paper is to compute the trace fields of $C(2, s)$.

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Figure 1: $C(p, s)$ for $p = 4, s = 5$

To state the main theorem, define $\psi_s(x)$ to be

$$\begin{cases} (x^{\frac{s}{2}+1} - x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} + x^{-\frac{s}{2}}) & (\frac{s}{2}: \text{even}) \\ (x^{\frac{s}{2}+1} + x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} - x^{-\frac{s}{2}}) & (\frac{s}{2}: \text{odd}) \\ (x^{s+1} + x^{-(s+1)}) + 2 \sum_{j=0}^{\frac{s-1}{2}} (-1)^j (x^{s-2j} - x^{-(s-2j)}) & (s: \text{odd}). \end{cases}$$

Since $x^{2j-1} - x^{-(2j-1)}$ and $x^{2j} + x^{-2j}$ can be written in $z = x - x^{-1}$, $\psi_s(x)$ can be written as a polynomial in $z = x - x^{-1}$, which we denote by $\Psi_s(z)$. The following theorem is the first main result in this paper.

Theorem 1.1.

1. $\Psi_s(z)$ is irreducible.
2. For $s \in \mathbb{Z} \setminus \{-2, -1, 0\}$, the trace field of $C(2, s)$ is $\mathbb{Q}(w)$, where w is a root of $\Psi_s(z)$ for $s > 0$, or $\Psi_{-(2+s)}(z)$ for $s < -2$.

Remark 1.2. The explicit form of $\Psi_s(z)$ is in the Appendix.

According to the computation done by Hoste and Shanahan, the degree of the trace field of $C(1, s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of \mathbb{Z} . By Theorem , we have:

Corollary 1.3. The degree of the trace field of $C(2, s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of \mathbb{Z} .

T. Chinburg proved that the trace fields of $C(p, s)$ have degree 2 if and only if $(|p+s|, |s|)$ or $(|s|, |p+s|)$ is in $\{(3, 0), (3, 1), (3, 2), (3, 3), (4, 0), (4, 2), (4, 4), (6, 0), (6, 6)\}$ (the proof can be found in [4]). Hence if for fixed $p \in \mathbb{N}$ the degree of the trace field of $C(p, s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of \mathbb{Z} , then p is contained in the set $\{1, 2, 3, 4, 5, 6, 8, 12\}$. The following theorem is the second main result.

Theorem 1.4. If for fixed $p \in \mathbb{Z}$ the degree of the trace field of $C(p, s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of \mathbb{Z} , then p is contained in the set $\{1, 2, 4, 8\}$.

This paper proceeds as follows. In section 2, we briefly recall some basic results on Kleinian groups and trace fields. Moreover, we recall the work done by Neumann and Reid. In section 3, we extend the theorem proved by Hoste and Shanahan, and prove the main results.

2. Preliminaries

In this section, we recall the results obtained by Neumann and Reid, which enable us to compute the invariant trace field of $C(p, s)$ for fixed p, s . Expositions mainly follow that of [4]. Texts for basic results on Kleinian groups and invariant trace fields are [3] and [2].

2.1 Kleinian groups

The group $\mathrm{PSL}(2, \mathbb{C})$ is the quotient of the group $\mathrm{SL}(2, \mathbb{C})$ of all 2×2 matrices with complex entries and determinant 1 by its center $\{\pm I\}$. $\mathrm{PSL}(2, \mathbb{C})$ acts on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by Möbius transformations. By the Poincaré extension, every Möbius transformation extends to an orientation-preserving isometry on the upper half 3-space \mathbb{H}^3 equipped with the hyperbolic metric (cf.[3]). Moreover, every orientation-preserving isometry of \mathbb{H}^3 is obtained by the Poincaré extension of some Möbius transformation. Hence $\mathrm{PSL}(2, \mathbb{C})$ can be identified with the group of orientation-preserving isometries of \mathbb{H}^3 .

Remark 2.1. $\mathrm{PSL}(2, \mathbb{C})$ can be identified with $\mathrm{PGL}(2, \mathbb{C})$.

Definition 2.2. A discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is called a Kleinian group.

Remark 2.3. This condition is equivalent to requiring that Γ acts properly discontinuously on \mathbb{H}^3 .

If Γ is torsion-free, then \mathbb{H}^3/Γ is an orientable hyperbolic 3-manifold. If Γ is a conjugate of Γ' in $\mathrm{PSL}(2, \mathbb{C})$, then \mathbb{H}^3/Γ and \mathbb{H}^3/Γ' are isometric. Conversely every orientable hyperbolic 3-manifold M has the form \mathbb{H}^3/Γ , where Γ is a torsion-free Kleinian group, uniquely determined by the orientation-preserving isometry class of M up to conjugacy.

2.2 Invariant trace fields

Let Γ be a Kleinian group of finite covolume and P be the projection of $\mathrm{SL}(2, \mathbb{C})$ onto $\mathrm{PSL}(2, \mathbb{C})$.

Definition 2.4. The smallest field containing \mathbb{Q} and $\{\mathrm{tr}\gamma \mid \gamma \in P^{-1}(\Gamma)\}$ is called the trace field of Γ and denoted by $\mathbb{Q}(\mathrm{tr}\Gamma)$.

Lemma 2.5.([2], Theorem 3.1.2) $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a finite extension of \mathbb{Q} .

Theorem 2.6.([2], Theorem 3.3.4) Let $\Gamma^{(2)}$ be the subgroup of Γ generated by the set $\{\gamma^2 \mid \gamma \in \Gamma\}$. Then the trace field $\mathbb{Q}(\mathrm{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of Γ .

Definition 2.7. $\mathbb{Q}(\mathrm{tr}\Gamma^{(2)})$ is called the invariant trace field of Γ .

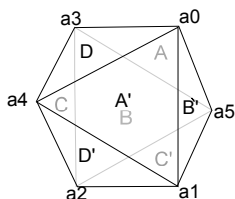


Figure 2: octahedron

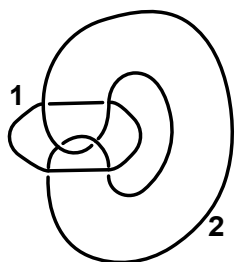


Figure 3: Whitehead link

Definition 2.8. For a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, $\mathbb{Q}(\text{tr}\Gamma)$ and $\mathbb{Q}(\text{tr}\Gamma^{(2)})$ are called the trace field of M and the invariant trace field of M , respectively. We denote the invariant trace field of M by kM .

Remark 2.9. The trace field and the invariant trace field of M is well-defined since trace is invariant under conjugation.

For the complement of a link in \mathbb{S}^3 , the trace field coincides with the invariant trace field by the following theorem.

Theorem 2.10. ([2], Corollary 4.2.2) *If $M = \mathbb{H}^3/\Gamma$ is the complement of a link in a \mathbb{Z}_2 -homology sphere, then the trace field coincides with the invariant trace field.*

2.3 Whitehead link

We denote the complement of the Whitehead link in \mathbb{S}^3 by W . W can be obtained by identifying the faces of an ideal octahedron, gluing A to A' , B to B' and so on, as shown in Figure 2. Now we consider the orbifold $W(p, q)$ obtained by performing (p, q) Dehn surgery of W at the toral end 1 in Figure 3 for $p, q \in \mathbb{Z}$. In this paper, we do not assume that p and q are coprime, so that $W(p, q)$ can be an orbifold. If the octahedron in Figure 2 is taken to be a regular ideal octahedron in \mathbb{H}^3 , then we obtain a complete hyperbolic structure on W . Hence for almost all $p, q \in \mathbb{Z}$, $W(p, q)$ admits a complete hyperbolic structure.

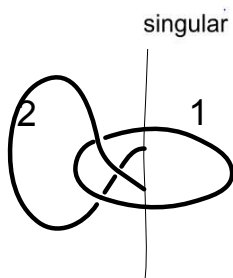


Figure 4: Orbifold W'

Let (u_1, v_1) and (u_2, v_2) be analytic Dehn surgery parameters for the toral ends 1 and 2. Since we only perform Dehn surgery on the toral end 1, we have $u_2 = 0$ and $v_2 = 0$, by the requirement that the hyperbolic structure on the toral end 2 is complete.

To represent (u_1, v_1) in terms of shape of octahedron, we consider a special ideal octahedron in \mathbb{H}^3 with vertices at $0, 1, \infty, -1, x,$ and x^{-1} on $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$. Denote these vertices by a_0, \dots, a_5 , respectively. Then

$$\begin{pmatrix} 1 & 1 \\ 1-z & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1+z \end{pmatrix}, \begin{pmatrix} 1 & 1-z \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1+z & 1 \\ 1 & 1 \end{pmatrix} \in \text{PGL}(2, \mathbb{C})$$

translate A to A', B to B', C to $C',$ and D to $D',$ respectively, where $z = x - x^{-1}$. We denote these matrices by $a, b, c,$ and $d,$ respectively. W inherits a hyperbolic structure for each $x \in \mathbb{H}$, where \mathbb{H} denotes the upper-half plane. A hyperbolic structure on W depends on the choice of $x \in \mathbb{H}$. However, it is known that the hyperbolic structure on the toral end 2 is complete for each x .

In [4], Neumann and Reid obtained the following equations.

$$(2.1) \quad u_1 = \log x + \log(x + 1) - \log(x - 1)$$

$$(2.2) \quad v_1 = 4 \log x - 2\pi i.$$

Here \log denotes the standard branch of natural log on the complex plane splitting along $(-\infty, 0]$. Let α be a solution of the equation $pu_1 + qv_1 = 2\pi i$ of x . For α , there exists a metric completion \overline{W} of W such that \overline{W} is orbifold homeomorphic to $W(p, q)$. By (2.2), we can determine the value of α .

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{PGL}(2, \mathbb{C})$$

rotates W about the axis through a_1 and a_3 . We denote the quotient space $W/\{e\}$ by W' . Then W' inherits a structure as a hyperbolic orbifold for each $x \in \mathbb{H}$.

For the orbifold W' , we can obtain

$$(2.3) \quad u'_1 = \log x + \log(x + 1) - \log(x - 1)$$

$$(2.4) \quad v'_1 = 2 \log x - \pi i,$$

where (u'_1, v'_1) is an analytic Dehn surgery parameter for the toral end 1 of W' in the case that the hyperbolic structure on the toral end 2 is complete.

For each $x \in \mathbb{H}$, the real Dehn surgery parameter $(p_1(x), q_1(x))$ takes their value in $\mathbb{R}^2 - \mathcal{N}$ where \mathcal{N} denotes the closed parallelogram in \mathbb{R}^2 with vertices $\pm(-4, 1)$, $\pm(0, 1)$. Hence $W(p, q)$ has a hyperbolic structure for integer pair $(p, q) \in \mathbb{R}^2 - \mathcal{N}$. In this paper, we only consider the case $W(p, q)$ for $(p, q) \in \mathbb{Z}^2$. The following theorem allows us to compute the invariant trace field of $W(p, q)$ and $W'(p, 2q)$.

Theorem 2.11.([Neumann-Reid[4], Theorem 6.2])

$$kW(p, q) = \mathbb{Q}(\alpha - \alpha^{-1}) \quad (q \in \mathbb{Z})$$

$$kW'(p, 2q) = \mathbb{Q}(\alpha - \alpha^{-1}) \quad (q \in \frac{1}{2}\mathbb{Z}).$$

2.4 Chain links

Let $C(p, s)$ denote the complement of a p chain link in \mathbb{S}^3 with s left half twists. $C(p, 2q)$ is homeomorphic to the manifold obtained by performing (p, q) Dehn filling at the toral end 1 of W , and then taking the p -fold cover of the resulting manifold or orbifold. Hence, if $W(p, q)$ admits a complete hyperbolic structure, then $C(p, 2q)$ admits a complete hyperbolic structure. Therefore, there exist Kleinian groups Γ and Γ' such that \mathbb{H}^3/Γ and \mathbb{H}^3/Γ' are isometric to $C(p, 2q)$ and $W(p, q)$ respectively. Since $C(p, 2q)$ is a cover of $W(p, q)$, Γ' is conjugate to a finite index subgroup of Γ . Hence we have $kW(p, q) = kC(p, 2q)$.

To obtain $C(p, s)$ for s odd, we use W' . We obtain the orbifold $W'(p, s)$ by performing (p, s) Dehn filling of the toral end of W' . This orbifold is a quotient of $C(p, s)$. Neumann and Reid proved the following theorem in [4].

Theorem 2.12.([Neumann-Reid[2], Theorem 5.1])

1. $C(p, s)$ has a hyperbolic structure if and only if $\{|p + s|, |s|\} \not\subset \{0, 1, 2\}$.
2. $C(p, s)$ and $C(p', s')$ have same invariant trace field if $(p' + s', s') = \pm(p + s, s)$ or $(p' + s', s') = \pm(-s, p + s)$.

By this theorem, if we obtain the invariant trace fields of $C(2, s)$ in the case $s > 0$, then we can obtain the invariant trace fields of $C(2, s)$ in the case $s < -2$. Hence we compute the invariant trace field (which coincides with the trace field by Theorem) of $C(2, s)$ only in the case $s > 0$ in the next section.

3. Main Theorem

In this section, we compute the trace fields of $C(2, s)$. To prove $\Psi_s(z)$ is indeed

irreducible, we need the lemma proved by Hoste and Shanahan. Hence we first recall the lemma with the proof.

Definition 3.1. Let $p(x) \in \mathbb{Z}[x]$ be a polynomial with no repeated roots. $p(x)$ is called complete if $p(\alpha) = 0$ implies $p(-\alpha^{-1}) = 0$.

The following lemma is a part of the lemma 3 in [1]. Since we need the proof of the lemma to obtain Theorem , we recall it with the proof.

Lemma 3.2.([Hoste-Shanahan]) *Let $p(x)$ be a complete polynomial which has no roots on the imaginary axis. If the leading and the next coefficients of $p(x)$ are 1 and the norm of every root in the first quadrant is < 1 , then $p(x)$ does not factor into two complete factors in $\mathbb{Z}[x]$.*

Proof. Assume that there is a complete factor $g(x)$ of $p(x)$. If α is a root of $g(x)$, then $-\alpha^{-1}$, $\bar{\alpha}$, and $-\bar{\alpha}^{-1}$ are also roots of $g(x)$. Since the norm of every non-real root α in the first quadrant is < 1 , $\alpha + \bar{\alpha} - \alpha^{-1} - \bar{\alpha}^{-1}$ is contained in \mathbb{R} and < 0 . If a root α' is real, then $\alpha' - \alpha'^{-1} < 0$. Hence if $g(x)$ has a form

$$g(x) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_0,$$

then $b_{n-1} = -\sum(\alpha_i + \bar{\alpha}_i - \alpha_i^{-1} - \bar{\alpha}_i^{-1}) - \sum(\alpha'_i - \alpha_i'^{-1}) > 0$, where α_i are non-real roots and α'_i are real. Moreover, $b_{n-1} < 1$ since the sum of the all roots of $p(x)$ is -1 . However, this is a contradiction to the fact that b_{n-1} is an integer. \square Before

we prove the main theorems in this paper, we prove the following lemma.

Lemma 3.3. *The polynomial $f(x) = (x+1)^p x^{p+4q} - (x-1)^p \in \mathbb{Z}[x]$ has no repeated roots*

Proof. If $f(x)$ has repeated roots, then $f(x)$ and $f'(x)$ have a common root. From

$$f'(x) = p(x+1)^{p-1}x^{p+4q} + (p+4q)(x+1)^p x^{p+4q-1} - p(x-1)^{p-1},$$

we have

$$(x-1)f'(x) - pf(x) = (x+1)^{p-1}x^{p+4q-1}\{(p+4q)x^2 - 2px - (p+4q)\}.$$

Neither 0 nor -1 is a root of $f(x)$. Thus, if $f(x)$ has a repeated root, then the repeated root is a root of the polynomial $g(x) = (p+4q)x^2 - 2px - (p+4q)$.

$\beta \neq 1$ is a root of $f(x)$ if and only if β is a solution of

$$(3.1) \quad \left(\frac{x(x+1)}{(x-1)}\right)^p x^{4q} = 1.$$

If β is a solution of (3.1) in the first quadrant, then $\left|\frac{\beta+1}{\beta-1}\right| \geq 1$. Thus if $|\beta| > 1$, then $\left|\frac{\beta(\beta+1)}{\beta-1}\beta^s\right| > 1$. Hence $|\beta| \leq 1$. However, the roots of $g(x)$ are real and the positive root of $g(x)$ is > 1 . Therefore $f(x)$ has no repeated roots. \square

First, we prove Theorem in the case s is even. In this case, we can apply Lemma directly.

Theorem 3.4. *If $s \in \mathbb{Z}$ is even, then the trace field of $C(2, s)$ is $\mathbb{Q}(w)$, where w is a root of the irreducible polynomial $\Psi_s(z)$ in Theorem .*

Proof. $kC(2, s)$ is equal to $kW(2, \frac{s}{2})$, since $C(2, s)$ is a cover of $W(2, \frac{s}{2})$. Hence we compute $kW(2, \frac{s}{2})$. By Theorem , we have $kW(2, \frac{s}{2}) = \mathbb{Q}(\alpha - \alpha^{-1})$, where α is a solution of the following equation

$$(3.2) \quad 2 \log \frac{x(x+1)}{(x-1)} + \frac{s}{2}(4 \log x - 2\pi i) = 2\pi i.$$

Since s is even, α is a solution of

$$\left(\frac{x(x+1)}{(x-1)} \right)^2 x^{2s} = 1.$$

This equation can be factored as

$$\left(\frac{x(x+1)}{(x-1)} x^s + 1 \right) \left(\frac{x(x+1)}{(x-1)} x^s - 1 \right) = 0.$$

If $\frac{s}{2}$ is even, then β satisfying

$$(3.3) \quad \frac{\beta(\beta+1)}{(\beta-1)} \beta^s - 1 = 0$$

is not a solution of the equation (3.2) since β is a solution of

$$\log \frac{x(x+1)}{(x-1)} + \frac{s}{4}(4 \log x - 2\pi i) = 2n\pi i$$

for some $n \in \mathbb{Z}$. Hence α is a solution of

$$\frac{x(x+1)}{(x-1)} x^s + 1 = 0.$$

Therefore, α is a root of

$$(3.4) \quad p(x) = x^{s+2} + x^{s+1} + x - 1 = 0.$$

If $\frac{s}{2}$ is odd, then α is a solution of

$$\frac{x(x+1)}{(x-1)} x^s - 1 = 0.$$

Therefore, α is a root of

$$(3.5) \quad p(x) = x^{s+2} + x^{s+1} - x + 1 = 0.$$

Every root of $p(x)$ is a solution of

$$\left(\frac{x(x+1)}{x-1}\right)^2 x^{2s} = 1.$$

By Lemma , $f(x) = (x+1)^2 x^{2s+2} - (x-1)^2$ has no repeated roots. Thus $p(x)$ has no repeated roots. Hence it is clear that $p(x)$ is complete.

Now we prove the polynomials (3.4) and (3.5) satisfy the conditions in Lemma . Let β be a root of the polynomial (3.4) or (3.5) in the first quadrant not on the imaginary axis. Then, $\left|\frac{\beta+1}{\beta-1}\right| > 1$. Hence if $|\beta| \geq 1$, then $\left|\left(\frac{\beta(\beta+1)}{\beta-1}\right)^2 \beta^{2s}\right| > 1$. This is a contradiction to the fact that β is a solution of

$$\left(\frac{x(x+1)}{x-1}\right)^2 x^{2s} = 1.$$

Hence $|\beta| < 1$ or β is on the imaginary axis. On the other hand, assume that $p(x)$ has a root β on the imaginary axis. Then β is a solution of

$$\left(\frac{x(x+1)}{x-1}\right)^2 x^{2s} = 1.$$

Since β is on the imaginary axis, $\left|\frac{\beta+1}{\beta-1}\right|$ is 1. Thus $|\beta|$ is 1. Furthermore, we have $\beta = \pm i$. However, $p(x)$ does not have $\pm i$ as solutions. Hence $p(x)$ has no roots on the imaginary axis. Therefore, the polynomials (3.4) and (3.5) satisfy the conditions in Lemma .

By the polynomials (3.4) and (3.5), we obtain the equations

$$\begin{cases} (x^{\frac{s}{2}+1} - x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} + x^{-\frac{s}{2}}) = 0 & (\frac{s}{2}: \text{even}) \\ (x^{\frac{s}{2}+1} + x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} - x^{-\frac{s}{2}}) = 0 & (\frac{s}{2}: \text{odd}). \end{cases}$$

Denote $z = x - x^{-1}$. Then $x^j - x^{-j}$ can be written as a polynomial in z in the case when j is odd, and $x^j + x^{-j}$ can be written as a polynomial in z in the case when j is even. Hence by these equations, we obtain polynomials in z . We denote these polynomials in z by $\Psi_s(z)$. Any factoring of $\Psi_s(z)$ will induce a factoring of the polynomial (3.4) or (3.5). Hence $\Psi_s(z)$ is irreducible by Lemma . Since $\alpha - \alpha^{-1}$ is a solution of $\Psi_s(z)$, the result follows. \square

Next we prove Theorem in the case that s is odd. In this case, we cannot apply Lemma directly, so we extend Lemma in the proof of Theorem .

Theorem 3.5. *If $s \in \mathbb{Z}$ is odd, then the trace field of $C(2, s)$ is $\mathbb{Q}(w)$, where w is a root of the irreducible polynomial $\Psi_s(z)$.*

Proof. Since s is odd, by the equation (3.2) we obtain

$$\left(\frac{x(x+1)}{x-1}\right)^2 x^{2s} = -1.$$

Hence α is a root of

$$x^{2s+4} + 2x^{2s+3} + x^{2s+2} + x^2 - 2x + 1 = 0.$$

This polynomial has $x^2 + 1$ as a factor. We set

$$(3.6) \quad p(x) = \frac{x^{2s+4} + 2x^{2s+3} + x^{2s+2} + x^2 - 2x + 1}{x^2 + 1}$$

$$(3.7) \quad = x^{2s+2} + 2x^{2s+1} + \cdots - 2x + 1.$$

Every root of $p(x)$ is a solution of

$$\left(\frac{x(x+1)}{x-1} \right)^4 x^{4s} = 1.$$

By Lemma , $f(x) = (x+1)^4 x^{4s+4} - (x-1)^4$ has no repeated roots. Thus, $p(x)$ has no repeated roots. Hence $p(x)$ is complete. Thus, if β is a non-real root of $p(x)$, then $-\beta^{-1}, \bar{\beta}$ and $-\bar{\beta}^{-1}$ are also roots of $p(x)$. If β' is a real root of $p(x)$, then β'^{-1} is also a root of $p(x)$.

If β is in the first quadrant on the imaginary axis, then $\left| \frac{\beta+1}{\beta-1} \right| > 1$. Hence if $|\beta| \geq 1$, then $\left| \left(\frac{\beta(\beta+1)}{\beta-1} \right)^2 \beta^{2s} \right| > 1$. This is a contradiction to the fact that β is a solution of $\left(\frac{x(x+1)}{x-1} \right)^2 x^{2s} = -1$. Hence if β is in the first quadrant, then $|\beta| < 1$ or β is on the imaginary part. On the other hand, assume that β is on the imaginary axis. Then β is a solution of

$$\left(\frac{x(x+1)}{x-1} \right)^2 x^{2s} = -1.$$

Since β is on the imaginary axis, $\left| \frac{\beta+1}{\beta-1} \right|$ is 1. Thus $|\beta|$ is 1. Furthermore, we have $\beta = \pm i$. However, $p(x)$ does not have $\pm i$ as solutions. Hence β is not on the imaginary axis. Therefore, if β is a non-real root of $p(x)$, then $\beta + \bar{\beta} - \beta^{-1} - \bar{\beta}^{-1}$ is contained in \mathbb{R} and < 0 . If β' is a real root of $p(x)$, then $\beta - \beta^{-1} < 0$.

We now prove that $p(x)$ does not factor into two complete factors. Assume that there is a complete factor $g_1(x)$ of $p(x)$. If $g_1(x)$ has a form

$$g_1(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0,$$

then $b_{m-1} = -\sum(\beta_i + \bar{\beta}_i - \beta_i^{-1} - \bar{\beta}_i^{-1}) - \sum(\beta'_i - \beta'^{-1}_i) > 0$ and < 2 , since the sum of all roots of $p(x)$ is -2 . Hence we obtain $b_{m-1} = 1$. Therefore, if there are complete factors $g_1(x)$ and $g_2(x)$ of $p(x)$, then $g_1(x)$ and $g_2(x)$ have the forms

$$g_1(x) = x^m + x^{m-1} + c_{m-2}x^{m-2} \cdots + c_0$$

$$g_2(x) = x^n + x^{n-1} + d_{n-2}x^{n-2} \cdots + d_0.$$

Since $p(x)$ is factorized into

$$\left\{ \begin{array}{l} \frac{x^{s+1}(x+1)+i(x-1)}{x-i} \frac{x^{s+1}(x+1)-i(x-1)}{x+i} \quad \left(\frac{s+1}{4} \in \mathbb{Z}\right) \\ \frac{x^{s+1}(x+1)+i(x-1)}{x+i} \frac{x^{s+1}(x+1)-i(x-1)}{x-i} \quad \left(\frac{s+1}{4} \notin \mathbb{Z}\right) \end{array} \right\},$$

there is $h(x) \in \mathbb{Z}[i][x]$ such that $p(x)$ is equal to $h(x)\overline{h}(x)$. Since $\mathbb{Z}[i]$ is an unique factorization domain, $\mathbb{Z}[i][x]$ is also an unique factorization domain. Thus g_1 is factored as $q_1(x)q_2(x) \cdots q_n(x) \in \mathbb{Z}[i][x]$, where $q_k(x)$ is prime in $\mathbb{Z}[i][x]$ for each k . Since $g_1(x)$ is a factor of $p(x)$, $q_k(x)$ is a factor of $h(x)$ or $\overline{h}(x)$ for each k . However, $p(x)$ has no repeated roots. Hence there exists

$$p_1(x) = x^l + a_{l-1}x^{l-1} + \cdots + a_0 \in \mathbb{Z}[i][x]$$

such that $g_1(x)$ is equal to $p_1(x)\overline{p_1(x)}$. Since $g_1(x)$ is complete, $p_1(\beta) = 0$ implies that $p_1(-\beta^{-1}) = 0$ or $p_1(-\overline{\beta}^{-1}) = 0$. However, both $\text{Re}(\beta - \beta^{-1})$ and $\text{Re}(\beta - \overline{\beta}^{-1})$ are < 0 . Hence the sum of the real parts of the roots of g_1 is < 0 and > -1 . Therefore, we have $0 < \text{Re}(a_{l-1}) < 1$. This is a contradiction to the fact that a_{l-1} is in $\mathbb{Z}[i]$. Hence $p(x)$ does not factor into two complete factors.

Denote $z = x - x^{-1}$. $p(x)$ has a form

$$x^{2s+2} + 2x^{2s+1} - 2x^{2s-1} + 2x^{2s-3} - \cdots + 2x^3 - 2x + 1.$$

By dividing it by x^{s+1} , we have

$$(x^{s+1} + x^{-(s+1)}) + 2 \sum_{j=0}^{\frac{s-1}{2}} (-1)^j (x^{s-2j} - x^{-(s-2j)}).$$

Hence we obtain the polynomial $\Psi_s(z)$ in z . Any factoring of $\Psi_s(z)$ will induce a factoring of $p(x)$. Hence $\Psi_s(z)$ is irreducible. Since $\alpha - \alpha^{-1}$ is a solution of $\Psi_s(z)$, the result follows. □

By Theorem and , we obtain Theorem .

To prove Theorem , we first prove the following theorem.

Theorem 3.6. *Fix $p \in \mathbb{N}$. For any $n \in \mathbb{N}$ there exists s_0 such that if $s > s_0$, then the degree of $kC(p, s)$ is $> n$.*

Proof. Let $\alpha \in \mathbb{C}$ be a solution of the equation $\left(\frac{x(x+1)}{(x-1)}\right)^p x^{2s} = \pm 1$ of x . If α is in the first quadrant not on the imaginary axis, then $\left|\frac{\alpha+1}{\alpha-1}\right| > 1$. Hence if $|\alpha| \geq 1$, then $\left|\left(\frac{\alpha(\alpha+1)}{(\alpha-1)}\right)^p \alpha^{2s}\right| > 1$. This is a contradiction to the fact that α satisfies $\left(\frac{\alpha(\alpha+1)}{(\alpha-1)}\right)^p \alpha^{2s} = \pm 1$. Hence if α is in the first quadrant not on the imaginary axis,

then $|\alpha| < 1$. Hence we have

$$\begin{aligned} 1 &= \left| \frac{\alpha(\alpha + 1)}{(\alpha - 1)} \right|^p |\alpha|^{2s} \\ &\leq \left(\frac{1 + |\alpha|}{1 - |\alpha|} \right)^p |\alpha|^{p+2s}. \end{aligned}$$

Since the function

$$f_s(x) = \left(\frac{1 + x}{1 - x} \right)^p x^{p+2s}$$

on the interval $(0, 1)$ is an increasing continuous function, there exists $\beta_s \in (0, 1)$ satisfying $f_s(\beta_s) = 1$ and $\beta_s < |\alpha|$. Since $\beta_s < 1$ and satisfies $f_s(\beta_s) = 1$, β_s converges to 1, when s goes to ∞ . Hence there exists $s_0 \in \mathbb{N}$ such that if $s > s_0$, then $\beta_s^{-1} - \beta_s < \frac{1}{n}$. Since $\beta_s < |\alpha|$, $|\alpha^{-1}| - |\alpha| < \frac{1}{n}$. Note that β_s is independent of the choice of α satisfying $\left(\frac{\alpha(\alpha+1)}{\alpha-1} \right)^p \alpha^{2s} = \pm 1$. Therefore, if α is a solution of $p(x) = x^p(x + 1)^p x^{2s} \pm (x - 1)^p$ in the first quadrant, then $|\alpha^{-1}| - |\alpha| < \frac{1}{n}$ for $s > s_0$.

Every root of $p(x)$ is a solution of

$$\left(\frac{x(x + 1)}{x - 1} \right)^{2p} x^{4s} = 1.$$

By Lemma , $f(x) = (x + 1)^{2p} x^{2p+4s} - (x - 1)^{2p}$ has no repeated roots. Thus $p(x)$ has no repeated roots. Hence $p(x)$ is complete.

Every root of $p(x)$ is a solution of the equation $\left(\frac{x(x+1)}{x-1} \right)^p x^{2s} = \pm 1$ of x . Hence the norm of every root of $p(x)$ in the first quadrant is < 1 or the root is on the imaginary axis. On the other hand, assume that a root α of $p(x)$ is on the imaginary axis. Then α is a solution of

$$\left(\frac{x(x + 1)}{x - 1} \right)^p x^{2s} = \pm 1.$$

Since α is on the imaginary axis, $\left| \frac{\alpha+1}{\alpha-1} \right|$ is 1. Thus $|\alpha|$ is 1. Furthermore, we have $\alpha = \pm i$.

For $m \leq n$, if $p(x)$ has a complete factor

$$p_1(x) = x^{2m} + a_{2m-1}x^{2m-1} + \dots + a_0,$$

then we have

$$0 \leq a_{2m-1} < - \sum (\alpha_i + \bar{\alpha}_i - \alpha_i^{-1} - \bar{\alpha}_i^{-1}) < m(\beta_s^{-1} - \beta_s) < 1.$$

Thus we obtain $a_{2m-1} = 0$. Hence every root of $p_1(x)$ is on the imaginary axis. Since $p(x)$ has no repeated roots, $p_1(x)$ must be $x^2 + 1$. However the trace fields of

$C(p, s)$ have degree 2 if and only if $(|p + s|, |s|)$ or $(|s|, |p + s|)$ is in $\{(3, 0), (3, 1), (3, 2), (3, 3), (4, 0), (4, 2), (4, 4), (6, 0), (6, 6)\}$ by Proposition 7.1 in [4]. Since we can assume that $s_0 > 12$, the degree of $kC(p, s)$ is $> n$. \square

We now prove Theorem 1.4.

Proof. First, we prove that the trace field of $C(3, s)$ does not have degree 5 for any $s \in \mathbb{N}$. To do it, we will find s_0 such that if $s > s_0$, $\beta_s^{-1} - \beta_s < \frac{1}{5}$, where β_s is as in the proof of Theorem .

We choose a number less than $\frac{1}{5}$, for example we choose $\frac{20}{19} - \frac{19}{20}$. If

$$(3.8) \quad f_s\left(\frac{19}{20}\right) = \left(\frac{1 + \frac{19}{20}}{1 - \frac{19}{20}}\right)^5 \left(\frac{19}{20}\right)^{5+2s}$$

is less than $1 = f_s(\beta_s)$, then $\frac{19}{20} < \beta_s$ since $f_s(x)$ is an increasing function. Hence we obtain

$$\beta_s^{-1} - \beta_s < \frac{20}{19} - \frac{19}{20} < \frac{1}{5}.$$

Therefore, if we choose s_0 such that $f_{s_0}(\frac{19}{20}) < 1$, then for $s > s_0$ the trace field of $C(3, s)$ does not have degree 5, by the proof of Theorem . If $s > 200$, then $f_s(\frac{19}{20}) < 1$. Hence if $s > 200$, then the trace field of $C(3, s)$ does not have degree 5. Using *Mathematica*, we can check that the trace field of $C(3, s)$ does not have degree 5 for $s < 200$. Therefore, the trace field of $C(3, s)$ does not have degree 5 for any $s \in \mathbb{N}$.

In a similar way, we can prove that the trace field of $C(5, s)$ does not have degree 9 for any $s \in \mathbb{N}$, and that of $C(6, s)$ does not have degree 3, and that of $C(12, s)$ does not have degree 3. Hence the result follows. \square

Remark 3.7. If p is 1 or 2, then the degree of the trace field of $C(p, s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of \mathbb{Z} . However, in the case p is 4 or 8, whether this property holds or not is still unknown.

4. Appendix

The explicit form of $\Psi_s(z)$ in Theorem is as follows.

$$z^{\frac{s}{2}+1} + z^{\frac{s}{2}} + \sum_{i=1}^{\frac{s}{4}} \left\{ \sum_{n=0}^i \sum_{i_1+\dots+i_n=i} C_{i_1\dots i_n} \right\} z^{\frac{s}{2}+1-2i} + \sum_{i=1}^{\frac{s}{4}} \left\{ \sum_{n=0}^i (-1)^n \sum_{i_1+\dots+i_n=i} D_{i_1\dots i_n} \right\} z^{\frac{s}{2}-2i} \quad \left(\frac{s}{2}: \text{even}\right)$$

$$\begin{aligned}
& z^{\frac{s}{2}+1} + z^{\frac{s}{2}} + \sum_{i=1}^{\frac{s+2}{4}} \left\{ \sum_{n=0}^i (-1)^n \sum_{i_1+\dots+i_n=i} C_{i_1\dots i_n} \right\} z^{\frac{s}{2}+1-2i} \\
& + \sum_{i=1}^{\frac{s-2}{4}} \left\{ \sum_{n=0}^i \sum_{i_1+\dots+i_n=i} D_{i_1\dots i_n} \right\} z^{\frac{s}{2}-2i} \quad \left(\frac{s}{2}: \text{ odd}\right) \\
& z^{s+1} + \sum_{i=1}^{\frac{s+1}{2}} \left\{ \sum_{n=0}^i \sum_{i_1+\dots+i_n=i} E_{i_1\dots i_n} \right\} z^{s+1-2i} \\
& + \sum_{i=1}^{\frac{s-1}{2}} \left\{ 2 \sum_j^{\frac{s-1}{2}} \sum_{n=0}^i \sum_{i_1+\dots+i_n=i} F_{i_1\dots i_n} \right\} z^{s-2i} \quad (s: \text{ odd}).
\end{aligned}$$

where

$$C_{i_1\dots i_n} = (-1)^{1+i_1} \dots (-1)^{1+i_n} \binom{\frac{s}{2}+1}{i_1} \dots \binom{\frac{s}{2}+1-2i_1-\dots-2i_{n-1}}{i_n},$$

$$D_{i_1\dots i_n} = \binom{\frac{s}{2}}{i_1} \dots \binom{\frac{s}{2}-2i_1-\dots-2i_{n-1}}{i_n},$$

$$E_{i_1\dots i_n} = (-1)^n \binom{s+1}{i_1} \dots \binom{s+1-2i_1-\dots-2i_{n-1}}{i_n},$$

and

$$F_{i_1\dots i_n} = (-1)^j (-1)^{1+i_1} \dots (-1)^{1+i_n} \binom{s-2j}{i_1} \dots \binom{s-2j-2i_1-\dots-2i_{n-1}}{i_n}.$$

These polynomials can be simplified by letting $z = x - x^{-1}$ as follows:

$$\begin{cases}
(x^{\frac{s}{2}+1} - x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} + x^{-\frac{s}{2}}) & (\frac{s}{2}: \text{ even}) \\
(x^{\frac{s}{2}+1} + x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} - x^{-\frac{s}{2}}) & (\frac{s}{2}: \text{ odd}) \\
(x^{s+1} + x^{-(s+1)}) + 2 \sum_{j=0}^{\frac{s-1}{2}} (-1)^j (x^{s-2j} - x^{-(s-2j)}) & (s: \text{ odd})
\end{cases}$$

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