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Quantum Super Theta Vectors and Theta Functions

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Abstract. There are many concepts around classical theta functions, theta vectors and quantum theta functions. Manin clarified the relation among these concepts with the symmetry of functional equations. We extend his results to the super torus.

1. Introduction

Theta functions are the functions on complex spaces, more precisely the sections of line bundles on complex torus. Noncommutative versions of theta functions appeared recently[8]. However, the concept of noncommutative torus was already developed in terms of Heisenberg group and Schrödinger representation[6]. Noncommutative tori are used in physics in the toroidal compactification by Connes, Douglas and Schwarz[1]. Later the concept of theta vectors was introduced by Schwarz[9]. Lastly Manin studied operator version of theta functions, called quantum theta functions[5]. Classically theta functions play the role of observables and states. However, in quantum mechanics, the roles of observables and states are seperated. Theta vectors are the vacuum states in the Hilbert space and quantum theta functions are observables.

A classical theta function of $z \in \mathbb{C}^n$ is

$$\theta(z, T) = \sum_{l \in \mathbb{Z}^n} e^{\pi i l^t T l + 2\pi i l^t z},$$

where T is a symmetric complex matrix of size n with $\mathrm{Im}\, T > 0$. Then this function

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satisfies

$$\theta\,(z+k,\,T)=\theta\,(z,\,T)$$
$$\theta\,(z+Tk,\,T)=e^{-\pi ik^tTk-2\pi ik^tz}\,\theta\,(z,\,T)$$

for all $k\in\mathbb{Z}^n$, and

$$\theta(T^{-1}z,-T^{-1})=(\det(T/i))^{\frac{1}{2}}\,e^{\pi iz^tT^{-1}z}\theta(z,t).$$

Then the corresponding theta vectors are defined as $f_T(x)=e^{\pi ix^tTx}$ with the same T which is considered as a vacuum vector $L^2(\mathbb{R})$. A quantum theta function is a noncommutative algebra $C^\infty(D,\alpha)$ which consists of $\sum_{h\in D}a_h\,e_{D,\alpha}(h)$ over a lattice D in $\mathbb{R}^{2n}$, where $e_{D,\alpha}(h)$'s are generators satisfying

$$e_{D,\alpha}(h)e_{D,\alpha}(g)=\alpha(h,g)e_{D,\alpha}(h+g)$$

and $a_h\in\mathbb{C}$ satisfies the Schwarz condition.

Two questions were raised by Schwarz[9]. The first one is the connection between quantum theta functions and theta vectors, and the second one is the existence of a quantum analogue of the classical functional equation for thetas. Manin answered these two types of questions affirmatively[4]. We extend his result to the super torus.

2. Quantum Theta Vectors and Quantum Theta Functions

We define a Heisenberg group, $Heis(\mathbb{R}^{2n},\psi)$. As a set $Heis(\mathbb{R}^{2n},\psi)$ is $\mathbb{R}\times\mathbb{R}^{2n}$. For $t,t'\in\mathbb{R}$, and $(x,y),(x',y')\in\mathbb{R}^{2n}$, we define the multiplication of $(t,x,y),(t',x',y')\in Heis(\mathbb{R}^2,\psi)$ by

$$(t,x,y)\cdot(t',x',y')=(t+t'+\psi(x,y;\;x',y'),x+x',y+y'),$$

where $\psi:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$, satisfies the cocycle condition

$$\psi(x,y;\;x',y')+\psi(x+x',y+y';\;x'',y'')=\psi(x,y;\;x'+x'',y'+y'')+\psi(x',y';\;x'',y''),$$

which is a necessary and sufficient condition for the multiplication to be associative. Then there is an exact sequence

$$0\to\mathbb{R}\xrightarrow{i}Heis(\mathbb{R}^{2n},\psi)\xrightarrow{j}\mathbb{R}^{2n}\to 0,$$

called a central extension, with the inclusion $i(t)=(t,0,0)$ and the projection $j(t,x,y)=(x,y)$, where $i(\mathbb{R})$ is the center in $Heis(\mathbb{R}^{2n},\psi)$.

Let ψ be a skew symmetric form

$$\psi(x,y)=\frac{1}{2}(x_1^ty_2-y_1^tx_2).$$

Then it satisfies the cocycle condition. We define the unitary representation π of this Heisenberg group on $L^2(\mathbb{R}^n)$ as follows.

$$\pi(t,x,y)f(s) = e^{2\pi i(t+s^t\cdot y)+\pi i(x^t\cdot y)}f(s+x).$$

Then

$$\pi(t_1,x_1,y_1)\pi(t_2,x_2,y_2) = e^{\pi i(x_1^t y_2 - x_2^t y_1)}\pi(t_1+t_2, x_1+x_2, y_1+y_2),$$

so that

$$\pi(t_1,x_1,y_1)\pi(t_2,x_2,y_2) = e^{2\pi i(x_1^t y_2 - x_2^t y_1)}\pi(t_2,x_2,y_2)\pi(t_1,x_1,y_1).$$

Let D be a lattice in $\mathbb{R}^{2n}$ and we consider $C^\infty(D,\alpha)$ of smooth functions consisting of infinite series $\sum_{h\in D} a_h e_{D,\alpha}(h)$, where $e_{D,\alpha}(h)\cdot e_{D,\alpha}(g) = \alpha(h,g)e_{D,\alpha}(h+g)$ with $\alpha(h,g) = e^{2\pi i\psi(h,g)}$ and $a_h \in \mathbb{C}$ satisfies the condition of Schwarz space.

For $f,g \in L^2(\mathbb{R}^n)$, we define an inner product $\langle\langle f,g\rangle\rangle \in C^\infty(D,\alpha)$ as

$$\langle\langle f,g\rangle\rangle = \sum_{h\in D}\langle f,\pi_h g\rangle e_{D,\alpha}(h),$$

where $\langle f,\pi_h g\rangle$ is the inner product in $L^2(\mathbb{R}^n)$. For T, a complex matrix of size n with $T^t = T$ and $\operatorname{Im} T > 0$, we consider $f_T(s) = e^{\pi i s^t T s}$. Let

$$H(\underline{h},\underline{g}) = e^{\pi i \underline{h} T \underline{g}^*},$$

where $\underline{h} = Th_1 + h_2$ and $*$ is the complex conjugate. Then there is a theorem by Manin[4].

Theorem. *We have*

$$\langle\langle f_T,f_T\rangle\rangle = \frac{1}{\sqrt{2^n \det \operatorname{Im} T}}\sum_{h\in D} e^{-\frac{\pi}{2}H(\underline{h},\underline{h})}\, e_{D,\alpha}(h).$$

Moreover,

$$\Theta_D := \sum_{h\in D} e^{-\frac{\pi}{2}H(\underline{h},\underline{h})}\, e_{D,\alpha}(h)$$

is a quantum theta function in the ring $C^\infty(D,\alpha)$ satisfying the following functional equations:

$$\forall\, g\in D,\ \ c_g e_{D,\alpha}(g) s_g^*(\Theta_D) = \Theta_D,$$

where

$$c_g = e^{-\frac{\pi}{2}H(\underline{g},\underline{g})},\ \ s_g^*(e_{D,\alpha}(h)) = e^{-\pi H(\underline{g},\underline{h})}\, e_{D,\alpha}(h).$$

Also he showed that

$$\sum_{h\in D} e^{-\pi H(\underline{h},\underline{h})-\pi H(\underline{s},\underline{h})} = \sum_{g\in D^!} e^{-\pi H(\underline{g},\underline{g})-\pi H(\underline{s},\underline{g})}$$

as functions of variable s, where $D^! = \{x \in \mathbb{R}^{2n} \mid 2\psi(x,y) = x_1^t y_2 - y_1^t x_2 \in \mathbb{Z}\}$.

3. Quantum Super Thetas

Now, we consider the extension of the work in the previous section to the super case. As in the bosonic case, we define the super Heisenberg group, $sHeis(\mathbb{R}^{2n|2m},\psi)$, as follows. For $t,t' \in \mathbb{R}$, and $(x,\alpha),(y,\beta),(x',\alpha'),(y',\beta') \in \mathbb{R}^{n|m}$, we define the multiplication of $(t,x,y,\alpha,\beta),(t',x',y',\alpha',\beta') \in sHeis$ $(\mathbb{R}^{2n|2m},\psi)$ by

$$\begin{aligned}&(t,x,y,\alpha,\beta)\cdot(t',x',y',\alpha',\beta')\\&\quad=(t+t'+\psi(x,y,\alpha,\beta;\ x',y',\alpha',\beta'),x+x',y+y',\alpha+\alpha',\beta+\beta'),\end{aligned}$$

where $\psi : \mathbb{R}^{2n|2m}\times\mathbb{R}^{2n|2m}\to\mathbb{R}$, satisfies the cocycle condition

$$\begin{aligned}&\psi(x,y,\alpha,\beta;\ x',y',\alpha',\beta')+\psi(x+x',y+y',\alpha+\alpha',\beta+\beta';\ x'',y'',\alpha'',\beta'')\\&=\psi(x,y,\alpha,\beta;\ x'+x'',y'+y'',\alpha'+\alpha'',\beta'+\beta'')+\psi(x',y',\alpha',\beta';\ x'',y'',\alpha'',\beta''),\end{aligned}$$

a necessary and sufficient for the associative multiplication. Then there is an exact sequence

$$0\to\mathbb{R}\xrightarrow{i} sHeis(\mathbb{R}^{2n|2m},\psi)\xrightarrow{j}\mathbb{R}^{2n|2m}\to 0,$$

a central extension, with the inclusion $i(t)=(t,0)$, the projection $j(t,z)=z$, for $z\in\mathbb{R}^{2n|2m}$, where $i(\mathbb{R})$ is the center in $sHeis(\mathbb{R}^{2n|2m},\psi)$. As in the bosonic case, we can introduce the unitary representation of the super Heisenberg group.

Let $L^2(\mathbb{R}^{n|m}) = L^2(\mathbb{R}^n)\otimes\Lambda^\bullet(\mathbb{R}^m)$, which is the completion of the Schwarz space $S(\mathbb{R}^n)\otimes\Lambda^\bullet(\mathbb{R}^m)$. Here $\Lambda^\bullet(\mathbb{R}^m)$ is the Grassmann algebra spanned by $\{\eta_1\wedge\cdots\wedge\eta_l \mid \eta_i\in\mathbb{R}^m, l\le m\}$. Here we use a modified Berezin integral $\int\sqrt{2\pi}\eta_i\, d\eta_i = 1$ compatible to the bosonic case. Let

$$\psi(x,y,\alpha,\beta;\ x',y',\alpha',\beta')=\frac{1}{2}(x^ty'-x'^ty-\alpha^t\beta'+\alpha'^t\beta).$$

Then ψ satisfies the cocycle condition. Note that $\alpha'^t\beta=-\beta^t\alpha'$ since α' and β are odd elements. We define

$$(\pi_{(t,x,y,\alpha,\beta)}f)(s,\eta)=e^{2\pi i(t+s^ty-\eta^t\beta)+\pi i(x^ty-\alpha^t\beta)}\cdot f(s+x,\eta+\alpha).$$

Then

$$\begin{aligned}&\pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)}\pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)}\\&\quad=e^{\pi i(x_1^ty_2-x_2^ty_1-\alpha_1^t\beta_2+\alpha_2^t\beta_1)}\pi_{(t_1+t_2,x_1+x_2,y_1+y_2,\alpha_1+\alpha_2,\beta_1+\beta_2)},\end{aligned}$$

so that

$$\begin{aligned}&\pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)}\pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)}\\&\quad=e^{2\pi i(x_1^ty_2-x_2^ty_1-\alpha_1^t\beta_2+\alpha_2^t\beta_1)}\pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)}\pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)}.\end{aligned}$$

Let D be a lattice in $\mathbb{R}^{2n|2m}$. As in the bosonic case, we define $C^\infty(D,\alpha)$ of infinite series

$$\sum_{(h,\delta)\in D} a_{(h,\delta)}\, e_{D,\alpha}(h,\delta),$$

where

$$e_{D,\alpha}(h,\delta)e_{D,\alpha}(g,\mu) = \alpha((h,\delta),(g,\mu))e_{D,\alpha}(h+g,\delta+\mu),$$

where

$$\alpha((h,\delta),(g,\mu)) = e^{2\pi i\psi(h,\delta;\, g,\mu)}.$$

First we deal with only the odd case and later we combine both the even part and the odd part. Let D be a lattice in $\mathbb{R}^{2m}$, and $g_R(\eta) = e^{-\pi i\eta^t R\eta}$, where $R(=R_1+iR_2)$ is skew symmetric and R_2 is positive of full rank m. So, we assume that m is an even integer. For $\delta = (\delta_1,\delta_2)\in D\subset \mathbb{R}^{2m}$,

$$\begin{aligned}
&\langle g_R(\eta), \pi_\delta g_R(\eta)\rangle\\
&\quad = \int g_R(\eta)\overline{\pi_\delta g_R(\eta)}\, d\eta\\
&\quad = \int e^{-\pi i\eta^t R\eta} e^{\pi i(\eta+\delta_1)^t\overline{R}(\eta+\delta_1)} e^{2\pi i\eta^t\delta_2} e^{\pi i\delta_1^t\delta_2}\, d\eta\\
&\quad = \int e^{2\pi\left(\eta+R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)\right)^t R_2\left(\eta+R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)\right)} e^{-2\pi\left(R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)\right)^t R_2R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)} e^{\pi i\delta_1^t(\underline{\delta}^*)}\, d\eta\\
&\quad = \int e^{2\pi\left(\eta+R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)\right)^t R_2\left(\eta+R_2^{-1}\left(\frac{i}{2}\underline{\delta}^*\right)\right)} e^{-\frac{\pi}{2}(\underline{\delta})^t R_2^{-1}\underline{\delta}^*}\, d\eta,
\end{aligned}$$

where $\underline{\delta} = R\delta_1+\delta_2$, $\underline{\delta}^* = \overline{R}\delta_1+\delta_2$. Denote $\underline{\delta}^t R_2^{-1}\underline{\mu}^*$ by $K(\underline{\delta},\underline{\mu})$. Then

$$\begin{aligned}
K(\underline{\delta},\underline{\mu}) &= (R\delta_1+\delta_2)^t R_2^{-1}(\overline{R}\,\mu_1+\mu_2)\\
&= (R_1\delta_1+\delta_2+iR_2\delta_1)^t R_2^{-1}(R_1\,\mu_1+\mu_2-iR_2\,\mu_1)\\
&= K(\underline{\mu},\underline{\delta})^*.
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{Im}(K(\underline{\delta},\underline{\mu})) &= -(R_1\delta_1+\delta_2)^t R_2^{-1}R_2\,\mu_1 + (R_2\delta_1)^t R_2^{-1}(R_1\,\mu_1+\mu_2)\\
&= -(\delta_2^t+\delta_1^t R_1^t)\,\mu_1 - \delta_1^t(R_1\,\mu_1+\mu_2)\\
&= -\delta_2^t\,\mu_1 - \delta_1^t\,\mu_2\\
&= \begin{pmatrix}\delta_1^t & \delta_2^t\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\\
&= 2\psi(\delta,\mu),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}(K(\underline{\delta},\underline{\mu})) &= (R_1\delta_1+\delta_2)^t R_2^{-1}(R_1\,\mu_1+\mu_2) + (R_2\delta_1)^t R_2^{-1}(R_2\,\mu_1)\\
&= (\delta_2^t+\delta_1^t R_1^t)R_2^{-1}(R_1\,\mu_1+\mu_2) + \delta_1^t R_2^t\,\mu_1\\
&= (\delta_2^t+\delta_1^t R_1^t)R_2^{-1}(R_1\,\mu_1+\mu_2) - \delta_1^t R_2\,\mu_1\,.
\end{aligned}$$

If we use $\int \sqrt{2\pi}\eta_i \, d\eta_i = 1$,

$$\int e^{\pi\eta^t R_2 \eta} \, d\eta = \mathrm{Pf}(R_2),$$

where the Pfaffian of R_2, $\mathrm{Pf}(R_2)$ is defined by using only upper half part of skew symmetric R_2, such that $(\mathrm{Pf}(R_2))^2 = \det(R_2)$. Then for D, a lattice in $\mathbb{R}^{2m}$,

$$\begin{aligned}\langle\langle g_R(\eta), g_R(\eta)\rangle\rangle &= \sum_{\delta\in D} \langle g_R(\eta), \pi_\delta g_R(\eta)\rangle e_{D,\alpha}(\delta) \\ &= \sum_{\delta\in D} 2^{\frac{m}{2}} \,\mathrm{Pf}(R_2) e^{-\frac{\pi}{2}K(\underline{\delta},\underline{\delta})} e_{D,\alpha}(\delta).\end{aligned}$$

Let

$$\Theta_D := \sum_{\delta\in D} e^{-\frac{\pi}{2}K(\underline{\delta},\underline{\delta})} e_{D,\alpha}(\delta).$$

Then

$$\begin{aligned}& e^{-\frac{\pi}{2}K(\underline{\mu},\underline{\mu})} e_{D,\alpha}(\mu) \sum_{\delta\in D} e^{-\frac{\pi}{2}K(\underline{\delta},\underline{\delta})} e^{-\pi K(\underline{\mu},\underline{\delta})} e_{D,\alpha}(\delta) \\ &= \sum_{\delta\in D} e^{-\frac{\pi}{2}K(\underline{\mu},\underline{\mu}) - \frac{\pi}{2}K(\underline{\delta},\underline{\delta}) - \pi\,\mathrm{Re}(K(\underline{\mu},\underline{\delta}))} e^{\pi i\{\delta,\mu\}} e_{D,\alpha}(\mu) e_{D,\alpha}(\delta) \\ &= \sum_{\delta\in D} e^{-\frac{\pi}{2}K(\underline{\delta+\mu},\underline{\delta+\mu})} e_{D,\alpha}(\delta+\mu).\end{aligned}$$

That is

$$c_\mu e_{D,\alpha}(\mu) \eta_\mu^*(\Theta_D) = \Theta_D,$$

where

$$c_\mu = e^{-\frac{\pi}{2}K(\underline{\mu},\underline{\mu})}, \quad \eta_\mu^*(e_{D,\alpha}(\delta)) = e^{-\pi K(\underline{\mu},\underline{\delta})} e_{D,\alpha}(\delta).$$

Here we used

$$e_{D,\alpha}(\mu) e_{D,\alpha}(\delta) = e^{-\pi i\{\mu,\delta\}} e_{D,\alpha}(\delta+\mu),$$

where $-\{\mu,\delta\} = 2\psi(\mu,\delta)$.

Now we combine both the even and odd parts. Let $F_{T,R}(s,\eta) = f_T(s) g_R(\eta) = e^{\pi i(s^t T s - \eta^t R\eta)}$ and D be a lattice in $\mathbb{R}^{2n|2m}$. Then we have

Theorem.

$$\begin{aligned}&\langle\langle F_{T,R}(s,\eta), \pi_{h,\delta} F_{T,R}(s,\eta)\rangle\rangle \\ &= \frac{2^{\frac{m}{2}}\,\mathrm{Pf}(\mathrm{Im}\, R)}{\sqrt{2^n \det(\mathrm{Im}\, T)}} \sum_{(h,\delta)\in D} e^{-\frac{\pi}{2}(H(\underline{h},\underline{h}) + K(\underline{\delta},\underline{\delta}))} e_{D,\alpha}(h,\delta).\end{aligned}$$

Let

$$\Theta_D = \sum_{(h,\delta)\in D} e^{-\frac{\pi}{2}(H(\underline{h},\underline{h})+K(\underline{\delta},\underline{\delta}))} e_{D,\alpha}(h,\delta).$$

Then $\forall (g,\mu)\in D$,

$$c_{(g,\mu)} e_{D,\alpha}(g,\mu)(s,\eta)^*_{(g,\mu)}\Theta_D = \Theta_D,$$

where

$$c_{(g,\mu)} = e^{-\frac{\pi}{2}(H(\underline{g},\underline{g})+K(\underline{\mu},\underline{\mu}))}$$

$$(s,\eta)^*_{(g,\mu)} e_{D,\alpha}(h,\delta) = e^{-\pi(H(\underline{g},\underline{h})+K(\underline{\mu},\underline{\delta}))} e_{D,\alpha}(h,\delta).$$

Remark. Here $\dfrac{\mathrm{Pf}(\mathrm{Im}\,R)}{\sqrt{\det(\mathrm{Im}\,T)}}$ is just

$$\frac{1}{\sqrt{\mathrm{sdet}\begin{pmatrix}\mathrm{Im}\,T & 0\\ 0 & \mathrm{Im}\,R\end{pmatrix}}},$$

where $\mathrm{sdet}\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$ is the super determinant defined by $\det(A)(\det B)^{-1}$. Then

$$\frac{2^{\frac{m}{2}}\,\mathrm{Pf}(\mathrm{Im}\,R)}{\sqrt{2^n\det(\mathrm{Im}\,T)}} = \frac{1}{\sqrt{\mathrm{sdet}\,2\begin{pmatrix}\mathrm{Im}\,T & 0\\ 0 & \mathrm{Im}\,R\end{pmatrix}}}.$$

We can also show the extended version

$$\sum_{(h,\delta)\in D} e^{-\pi(H(\underline{h},\underline{h})+K(\underline{\delta},\underline{\delta}))-\pi(H(\underline{s},\underline{h})+K(\underline{\eta},\underline{\delta}))}$$
$$= \sum_{(g,\mu)\in D^!} e^{-\pi(H(\underline{g},\underline{g})+K(\underline{\mu},\underline{\mu}))-\pi((H(\underline{s},\underline{h})+K(\underline{\eta},\underline{\mu})))},$$

as functions of variables (s,η). The proof is just the extended version of Manin, by using Fourier transform of

$$F_{s,\eta}(h,\delta) = e^{-\pi(H(\underline{h},\underline{h})+K(\underline{\delta},\underline{\delta}))-\pi(H(\underline{s},\underline{h})+K(\underline{\eta},\underline{\delta}))}$$

and checking that

$$\widehat{F}_{s,\eta}(g,\mu) = e^{-\pi(H(\underline{g},\underline{g})+K(\underline{\mu},\underline{\mu}))-\pi(H(\underline{s},\underline{g})+K(\underline{\eta},\underline{\mu}))}$$

and using Poisson summation formular.

Remark. Here we assumed that R is complex anti-symmetric. We want to find what is the natural candidate for R and its interpretation.

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