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Sufficiency Conditions for Hypergeometric Functions to be in a Subclasses of Analytic Functions

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Abstract. The purpose of this paper is to introduce sufficient conditions for (Gaussian) hypergeometric functions to be in various subclasses of analytic functions. Also, we investigate several mapping properties involving these subclasses.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and let $\mathcal{S}$ be

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the subclass of all functions in $\mathcal{A}$, which are univalent. For $g(z) \in \mathcal{A}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{1.2}$$

the Hadamard product (or convolution) of the two power series $f(z)$ and $g(z)$ is given by (see [4]):

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n g_n z^n = (g * f)(z). \tag{1.3}$$

and the integral convolution is defined by (see [4]):

$$(f \circledast g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n g_n}{n} z^n = (g \circledast f)(z). \tag{1.4}$$

We recall some definitions which will be used in our paper.

Definition 1.1. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. A function $f(z) \in \mathcal{S}$ is called starlike of order α, denoted by $\mathcal{S}^*(\alpha)$, if $f(z)$ satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ \ (0 \leq \alpha < 1; \ z \in \mathbb{U}). \tag{1.5}$$

Also, a function $f(z) \in \mathcal{S}$ is called convex of order α, denoted by $\mathcal{K}(\alpha)$, if $f(z)$ satisfies the following condition:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ \ (0 \leq \alpha < 1; \ z \in \mathbb{U}). \tag{1.6}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were studied by MacGregor [12], Schild [17], Pinchuk [14] and others. From (1.5) and (1.6) we can see that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha). \tag{1.7}$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$, the classes of starlike and convex functions, respectively, (see Robertson [15]).

Definition 1.3.([7]) For $0 \le \alpha < 1$, $\beta \ge 0$, $-1 \le B < A \le 1$, $-1 \le B < 0$ and $g(z)$ is given by (1.2), we denote by $\mathcal{S}(f,g;A,B;\alpha,\beta)$ the subclass of $\mathcal{S}$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$\frac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right| \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha. \tag{1.8}$$

In other words, $f(z) \in \mathcal{S}(f,g;A,B;\alpha,\beta)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\left|\frac{\dfrac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\dfrac{z(f*g)'(z)}{(f*g)(z)} - 1\right| - 1}{B\left[\dfrac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\dfrac{z(f*g)'(z)}{(f*g)(z)} - 1\right|\right] - [B+(A-B)(1-\alpha)]}\right| < 1. \tag{1.9}$$

We note that:

(i) $\mathcal{S}(f,g;A,B;\alpha,0) = \mathcal{S}(f,g;A,B;\alpha)$

$$= \left\{ f(z) \in \mathcal{S} : \left|\frac{\dfrac{z(f*g)'(z)}{(f*g)(z)} - 1}{B\dfrac{z(f*g)'(z)}{(f*g)(z)} - [B+(A-B)(1-\alpha)]}\right| < 1 \ (z \in \mathbb{U}) \right\};$$

(ii) $\mathcal{S}(f,g;\gamma,-\gamma;\alpha,0) = \mathcal{S}(f,g;\gamma,\alpha)$

$$= \left\{ f(z) \in \mathcal{S} : \left|\frac{\dfrac{z(f*g)'(z)}{(f*g)(z)} - 1}{\dfrac{z(f*g)'(z)}{(f*g)(z)} + 1 - 2\alpha}\right| < \gamma \ (0 < \gamma \le 1;\ z \in \mathbb{U}) \right\}.$$

Definition 1.4.([9]) For $\delta < 1$ and $|\eta| \le \frac{\pi}{2}$, we define the class $R_\eta(\delta)$ which consists of functions $g(z)$ of the form (1.2) and satisfying the analytic criterion

$$\Re\left[e^{i\eta}\left(g'(z) - \delta\right)\right] > 0 \ (z \in \mathbb{U}). \tag{1.10}$$

Clearly, we have $R_\eta(\delta) \subset \mathcal{S}$ $(0 \le \delta < 1)$. Furthermore, if the function $g(z)$ of the form (1.2) belongs to the class $R_\eta(\delta)$, then

$$|g_n| \le \frac{2(1-\delta)\cos\eta}{n} \ (n \ge 2). \tag{1.11}$$

The class $R_\eta(\delta)$ was studied by Kanas and Srivastava [9].

Let ${}_2F_1(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \ (z \in \mathbb{U}), $$

where $c \neq 0, -1, -2, ...$ and

$$ (\lambda)_n = \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1) & \text{if } \ n \in \mathbb{N} = \{1, 2, ...\}. \end{cases} $$

We note that ${}_2F_1(a,b;c;1)$ converges for $\Re(c-a-b) > 0$ and is related to Gamma functions by

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.12} $$

We consider the functions

$$ e(a,b;c;z) = z\ {}_2F_1(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, \tag{1.13} $$

and

$$ \begin{aligned} h_\mu(a,b;c;z) &= (1-\mu)\left(e(a,b;c;z)\right) + \mu z \left(e(a,b;c;z)\right)' \\ &= z + \sum_{n=2}^{\infty} \left[1 + \mu(n-1)\right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n \ (\mu \geq 0). \end{aligned} \tag{1.14} $$

The mapping properties of a function $h_\mu(a,b;c;z)$ was studied by Shukla and Shukla [18].

Using the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ given by (1.13), Hohlov [8] introduced a convolution operator $I_{a,b,c}$ as

$$ \begin{aligned} \left[I_{a,b,c}f\right](z) &= \left[z {}_2F_1(a,b;c;z)\right] * f(z) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt * \frac{z}{(1-z)^a}. \end{aligned} \tag{1.15} $$

The operator $I_{a,b,c}$ contains as a special case most of the known linear integral and differential operators. For $b = 1$ in (1.15), the operator $I_{a,1,c}$ reduces to Carlson-Shaffer operator [3]. Also, it is a generalization of Bernardi operator [2] and Ruscheweyh operator [16].

Furthermore, Hohlov operator is a very specialized case of Dziok-Srivastava linear operator which introduced and studied by Dziok and Srivastava (see [5] and [6]) and consequently, Srivastava-Wright operator (see [11] and [19]).

On the other hand, Aouf et al. [1] introduced and studied the operator

$$[M_{a,b,c}f](z) = [z{}_2F_1(a,b;c;z)] \circledast f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{f(tz)}{t}dt \circledast \frac{z}{(1-z)^a}. \tag{1.16}$$

In this paper, we define the linear operator $I_{a,b,c}(f*g) : \mathcal{A} \to \mathcal{A}$ by the convolution as:
(1.17)

$$[I_{a,b,c}(f*g)](z) = [z{}_2F_1(a,b;c;z)] * [(f*g)(z)] = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n g_n z^n,$$

and the linear operator $M_{a,b,c}(f*g) : \mathcal{A} \to \mathcal{A}$ by the integral convolution as:

$$[M_{a,b,c}(f*g)](z) = [z{}_2F_1(a,b;c;z)] \circledast [(f*g)(z)] = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{a_n g_n}{n} z^n. \tag{1.18}$$

The operator $I_{a,b,c}$ was introduced by Hohlov [8] when $g(z) = \frac{z}{1-z}$ and the operator $M_{a,b,c}$ was introduced by Aouf et al. [1] when $g(z) = \frac{z}{1-z}$.

Also, we define the linear operator $L_\mu(f*g) : \mathcal{A} \to \mathcal{A}$ by the convolution as:

$$[L_\mu(f*g)](z) = h_\mu(a,b;c;z) * [(f*g)(z)] = z + \sum_{n=2}^{\infty} [1+\mu(n-1)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n g_n z^n, \tag{1.19}$$

and the linear operator $N_\mu(f*g) : \mathcal{A} \to \mathcal{A}$ by the integral convolution as:

$$[N_\mu(f*g)](z) = h_\mu(a,b;c;z) \circledast [(f*g)(z)] = z + \sum_{n=2}^{\infty} [1+\mu(n-1)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{a_n g_n}{n} z^n. \tag{1.20}$$

The operator L_μ was introduced by Kim and Shon [10] when $g(z) = \frac{z}{1-z}$ and the operator N_μ was introduced by Aouf et al. [1] when $g(z) = \frac{z}{1-z}$.

The main objective in this paper is to introduce sufficient conditions for (Gaussian) hypergeometric functions to be in various subclasses of analytic functions. Also, we investigate several mapping properties involving these subclasses.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 \le \alpha < 1$, $\beta \ge 0$, $\delta < 1$, $-1 \le B < A \le 1$, $-1 \le B < 0$, $|\eta| \le \frac{\pi}{2}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $e(a,b;c;z)$ is given by (1.13) and $h_\mu(a,b;c;z)$ is given by (1.14).

To establish our results, we need the following lemma.

Lemma 2.1.([7, Theorem 1]) *A sufficient condition for $f(z)$ defined by* (1.1) *to be in the class $\mathcal{S}(f,g;A,B;\alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,|a_n g_n| \le (A-B)(1-\alpha). \tag{2.1}$$

By using Lemma 2.1, we get the following results.

Theorem 2.1. *Let $a,b \in \mathbb{C}^*$ $(|a| \ne 1;\ |b| \ne 1)$ and c be a real number such that $c > \max\{0, |a|+|b|-1\}$. If $g(z) \in R_\eta(\delta)$ and the inequality*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(1-B)(1+\beta)+\{(A-B)(1-\alpha)-(1-B)(1+\beta)\}\,.\right.\\ &\left.\frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] \le (A-B)(1-\alpha)\left[1+\frac{1}{2(1-\delta)\cos\eta}\right]\\ &+\frac{[(A-B)(1-\alpha)-(1-B)(1+\beta)]\,(c-1)}{(|a|-1)(|b|-1)},\end{aligned} \tag{2.2}$$

satisfied, then $e(a,b;c;z)$ is in the class $\mathcal{S}(e,g;A,B;\alpha,\beta)$.

Proof. Let $g(z)$ of the form (1.2) belong to the class $R_\eta(\delta)$. According to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}g_n\right| \le (A-B)(1-\alpha). \tag{2.3}$$

Taking into account the sufficient condition (1.11) and

$$|(d)_n| \le (|d|)_n\,, \tag{2.4}$$

then, the left hand side of (2.3) is less than or equal to

$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\,\frac{2(1-\delta)\cos\eta}{n} = T_0.$$

Now

$$\begin{aligned} T_0 &= 2(1-\delta)(1-B)(1+\beta)\cos\eta\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right] \\ &\quad+\frac{2(1-\delta)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-1)\cos\eta}{(|a|-1)(|b|-1)} \\ &\quad\cdot\left[\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right] \\ &= 2(1-\delta)\cos\eta\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[[(A-B)(1-\alpha)-(1-B)(1+\beta)]\,.\right. \\ &\quad\left.\frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}+(1-B)(1+\beta)\right]-2(1-\delta)(A-B)(1-\alpha)\cos\eta \\ &\quad-\frac{2(1-\delta)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-1)\cos\eta}{(|a|-1)(|b|-1)}, \end{aligned}$$

and this last expression is bounded above by $(A-B)(1-\alpha)$ if (2.2) holds. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $a, b \in \mathbb{C}^*$ $(|a| \neq 1;\ |b| \neq 1)$ and c be a real number such that $c > |a|+|b|+1$. If $g(z) \in R_\eta(\delta)$ and the inequality*

$$\begin{aligned} &\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\{\mu(A-B)(1-\alpha)+(1-\mu)(1-B)(1+\beta)\}\right. \\ &\quad+\mu(1-B)(1+\beta)\frac{|ab|}{(c-|a|-|b|-1)} \\ &\quad\left.+\frac{(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] \\ \le\ &(A-B)(1-\alpha)\left[1+\frac{1}{2(1-\delta)\cos\eta}\right] \\ &\quad+\frac{(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-1)}{(|a|-1)(|b|-1)}, \end{aligned} \tag{2.5}$$

holds, then $h_\mu(a,b;c;z)$ is in the class $\mathcal{S}(h_\mu, g; A, B; \alpha, \beta)$.

Proof. Let $g(z)$ of the form (1.2) belong to the class $R_\eta(\delta)$. According to Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}\left[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)\right]\left[1+\mu(n-1)\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}g_n\right| \le (A-B)(1-\alpha). \tag{2.6}$$

Using (2.4) and the sufficient condition (1.11), the left hand side of (2.6) is less than or equal to

$$2(1-\delta)\cos\eta.$$

$$\sum_{n=2}^{\infty}\left[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)\right]\left[1+\mu(n-1)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}=T_1$$

and

$$\begin{aligned}T_1 &= 2(1-\delta)\cos\eta\left[\mu(1-B)(1+\beta)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}\right.\\ &\quad+\left[\mu(A-B)(1-\alpha)+(1-\mu)(1-B)(1+\beta)\right]\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\ &\quad\left.+(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right]\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}\right]\end{aligned}$$

$$\begin{aligned}&= 2(1-\delta)\cos\eta\left[\{\mu(A-B)(1-\alpha)+(1-\mu)(1-B)(1+\beta)\}\right.\\ &\quad\left(\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right)+\mu(1-B)(1+\beta)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\ &\quad+\frac{(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-1)}{(|a|-1)(|b|-1)}\\ &\quad\left.+\left(\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right)\right]\\ &= 2(1-\delta)\cos\eta\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\{\mu(A-B)(1-\alpha)+(1-\mu)(1-B)(1+\beta)\}\right.\\ &\quad+\frac{\mu(1-B)(1+\beta)(|ab|)}{(c-|a|-|b|-1)}\\ &\quad\left.+\frac{(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-|a|-|b|)}{(|a|-1)(|b|-1)}\right]\\ &\quad-\frac{(1-\mu)\left[(A-B)(1-\alpha)-(1-B)(1+\beta)\right](c-1)}{(|a|-1)(|b|-1)}-(A-B)(1-\alpha).\end{aligned}$$

But this last expression is bounded above by $(A-B)(1-\alpha)$ if (2.5) holds. Thus the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Let $a,b\in\mathbb{C}^*$ and c be a real number such that $c>|a|+|b|+2$. If the following inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}&\left[(A-B)(1-\alpha)+\frac{(1-B)(1+\beta)\,(|a|)_2\,(|b|)_2}{(c-|a|-|b|-2)_2}\right.\\ &\left.+\frac{\left[(A-B)(1-\alpha)+2(1-B)(1+\beta)\right](|ab|)}{(c-|a|-|b|-1)}\right]\le 2(A-B)(1-\alpha),\end{aligned}\tag{2.7}$$

is true, then $\left[I_{a,b,c}(f*g)\right](z)$ maps $(f*g)\in\mathcal{S}$ (or $\mathcal{S}^*$) to $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Proof. We need only to prove that

(2.8)
$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n g_n\right|\leq (A-B)(1-\alpha).$$

Using (2.4) and the fact that $|a_n g_n| \leq n$ for $(f*g)\in \mathcal{S}$ (or $\mathcal{S}^*$), the left hand side of (2.8) is less than or equal to

$$\sum_{n=2}^{\infty} n\,[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = T_2.$$

and

$$\begin{aligned} T_2 &= (1-B)(1+\beta)\frac{(|a|)_2\,(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ &\quad +[(A-B)(1-\alpha)+2(1-B)(1+\beta)]\,\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ &\quad +(A-B)(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right] \\ &= \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(A-B)(1-\alpha)+\frac{(1-B)(1+\beta)\,(|a|)_2\,(|b|)_2}{(c-|a|-|b|-2)_2}\right. \\ &\quad \left.+\frac{[(A-B)(1-\alpha)+2(1-B)(1+\beta)]\,(|ab|)}{(c-|a|-|b|-1)}\right]-(A-B)(1-\alpha). \end{aligned}$$

Thus, from (2.7), we obtain the required result. □

Theorem 2.4. *Let $a,b\in\mathbb{C}^*$ and c be a real number such that $c>|a|+|b|+1$. If the following inequality*

(2.9)
$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(A-B)(1-\alpha)+\frac{(1-B)(1+\beta)(|ab|)}{(c-|a|-|b|-1)}\right] \leq 2(A-B)(1-\alpha),$$

satisfied, then (i) $[I_{a,b,c}(f*g)](z)$ *maps* $(f*g)\in\mathcal{K}$ *to* $\mathcal{S}(f,g;A,B;\alpha,\beta)$,
(ii) $[M_{a,b,c}(f*g)](z)$ maps $(f*g)\in\mathcal{S}$ (or $\mathcal{S}^*$) to $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Proof. (i) It suffices to prove that

(2.10)
$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n g_n\right|\leq (A-B)(1-\alpha).$$

The left hand side of (2.10), by (2.4) and the fact that $|a_n g_n|\leq 1$ for $(f*g)\in\mathcal{K}$, is less than or equal to

$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = T_3.$$

Now

$$
\begin{aligned}
T_3 &= (1-B)(1+\beta)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + (A-B)(1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= (1-B)(1+\beta)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\quad +(A-B)(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right] \\
&= \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(A-B)(1-\alpha)+\frac{(1-B)(1+\beta)(|ab|)}{(c-|a|-|b|-1)}\right] \\
&\quad -(A-B)(1-\alpha).
\end{aligned}
$$

But this last expression is bounded above by $(A-B)(1-\alpha)$ if (2.9) holds. The rest of the proof of (ii) is the same as in the proof of (i), so, we omit it. This ends the proof of Theorem 2.4. □

Theorem 2.5. *Let* $a, b \in \mathbb{C}^*$ $(|a| \neq 1,\ |b| \neq 1)$ *and* c *be a real number such that* $c > \max\{0, |a|+|b|-1\}$. *If the following inequality*

$$
\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \qquad (2.11)\\
&\left[(1-B)(1+\beta)+\frac{[(A-B)(1-\alpha)-(1-B)(1+\beta)](c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] \\
&\leq 2(A-B)(1-\alpha)+\frac{[(A-B)(1-\alpha)-(1-B)(1+\beta)](c-1)}{(|a|-1)(|b|-1)},
\end{aligned}
$$

satisfied, then $[M_{a,b,c}(f*g)](z)$ *maps* $(f*g) \in \mathcal{K}$ *to* $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Proof. It is enough to show that

(2.12)
$$
\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{a_n g_n}{n}\right| \leq (A-B)(1-\alpha).
$$

The left hand side of (2.12) is less than or equal to

$$
\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}} = T_4.
$$

and

$$\begin{aligned}
T_4 &= (1-B)(1+\beta)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad + [(A-B)(1-\alpha)-(1-B)(1+\beta)]\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}} \\
&= (1-B)(1+\beta)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right] \\
&\quad + [(A-B)(1-\alpha)-(1-B)(1+\beta)] \\
&\quad \cdot\frac{(c-1)}{(|a|-1)(|b|-1)}\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\quad - [(A-B)(1-\alpha)-(1-B)(1+\beta)]\left(1+\frac{(c-1)}{(|a|-1)(|b|-1)}\right) \\
&= \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(1-B)(1+\beta)+\{(A-B)(1-\alpha)-(1-B)(1+\beta)\}\,.\right. \\
&\quad \left.\cdot\frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right]-\frac{[(A-B)(1-\alpha)-(1-B)(1+\beta)]\,(c-1)}{(|a|-1)(|b|-1)} \\
&\quad -(A-B)(1-\alpha).
\end{aligned}$$

The proof now follows by (2.11). □

Theorem 2.6. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a|+|b|+3$. If the following inequality*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{[(1+2\mu)(A-B)(1-\alpha)+2(1+\mu)(1-B)(1+\beta)](|ab|)}{(c-|a|-|b|-1)}\right. \\
&\quad + \frac{[\mu(A-B)(1-\alpha)+(1+4\mu)(1-B)(1+\beta)](|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} \\
&\quad \left.+\frac{\mu(1-B)(1+\beta)(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}+(A-B)(1-\alpha)\right] \le 2(A-B)(1-\alpha),
\end{aligned} \tag{2.13}$$

satisfied, then $[L_\mu(f*g)](z)$ maps $(f*g)\in \mathcal{S}$ (or $\mathcal{S}^*$) to $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Proof. It is enough to prove that

$$\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,[1+\mu(n-1)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n g_n\right| \le (A-B)(1-\alpha). \tag{2.14}$$

The left hand side of (2.14) is less than or equal to

$$\sum_{n=2}^{\infty} n\,[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,[1+\mu(n-1)]\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = T_5.$$

and

$$T_5 = [\mu(A-B)(1-\alpha) + (1+4\mu)(1-B)(1+\beta)] \sum_{n=3}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}}$$
$$+ [(1+2\mu)(A-B)(1-\alpha) + 2(1+\mu)(1-B)(1+\beta)] \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}$$
$$+\mu(1-B)(1+\beta) \sum_{n=4}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-4}} + (A-B)(1-\alpha) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$= [\mu(A-B)(1-\alpha) + (1+4\mu)(1-B)(1+\beta)]$$
$$\cdot \frac{(|a|)_2 (|b|)_2}{(c)_2} \frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$+ [(1+2\mu)(A-B)(1-\alpha) + 2(1+\mu)(1-B)(1+\beta)]$$
$$\cdot \frac{|ab|}{c} \frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$+\mu(1-B)(1+\beta) \frac{(|a|)_3 (|b|)_3}{(c)_3} \frac{\Gamma(c+3)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$+(A-B)(1-\alpha) \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right]$$
$$= \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[\frac{[(1+2\mu)(A-B)(1-\alpha) + 2(1+\mu)(1-B)(1+\beta)] (|ab|)}{(c-|a|-|b|-1)}\right.$$
$$+ \frac{[\mu(A-B)(1-\alpha) + (1+4\mu)(1-B)(1+\beta)] (|a|)_2 (|b|)_2}{(c-|a|-|b|-2)_2}$$
$$\left. + \frac{\mu(1-B)(1+\beta) (|a|)_3 (|b|)_3}{(c-|a|-|b|-3)_3} + (A-B)(1-\alpha)\right] - (A-B)(1-\alpha).$$

It is easy to see that this last expression is bounded above by $(A-B)(1-\alpha)$ if (2.13) holds. □

Theorem 2.7. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a| + |b| + 2$. If the following inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[\frac{[(1+\mu)(1-B)(1+\beta) + \mu(A-B)(1-\alpha)] (|ab|)}{(c-|a|-|b|-1)}\right.$$
$$\left. + \frac{\mu(1-B)(1+\beta) (|a|)_2 (|b|)_2}{(c-|a|-|b|-2)_2} + (A-B)(1-\alpha)\right]$$
$$\leq 2(A-B)(1-\alpha), \tag{2.15}$$

satisfied, then (i) $[L_\mu(f*g)](z)$ *maps* $(f*g) \in \mathcal{K}$ *to* $\mathcal{S}(f,g;A,B;\alpha,\beta)$, (ii) $[N_\mu(f*g)](z)$ maps $(f*g) \in \mathcal{S}$ (or $\mathcal{S}^*$) to $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Proof. It suffices for (i) and (ii) to show that

$$\sum_{n=2}^{\infty} [(1-B)(1+\beta)(n-1) + (A-B)(1-\alpha)] [1+\mu(n-1)] \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$\leq (A-B)(1-\alpha).$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(1-B)(1+\beta)(n-1)+(A-B)(1-\alpha)]\,[1+\mu(n-1)]\,\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
=\;& [(1+\mu)(1-B)(1+\beta)+\mu(A-B)(1-\alpha)]\,\frac{|ab|}{c}\,\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+\mu(1-B)(1+\beta)\frac{(|a|)_2\,(|b|)_2}{(c)_2}\,\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&+(A-B)(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\\
=\;& \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{[(1+\mu)(1-B)(1+\beta)+\mu(A-B)(1-\alpha)]\,(|ab|)}{(c-|a|-|b|-1)}\right.\\
&\left.+\frac{\mu(1-B)(1+\beta)\,(|a|)_2\,(|b|)_2}{(c-|a|-|b|-2)_2}+(A-B)(1-\alpha)\right]-(A-B)(1-\alpha).
\end{aligned}$$

Now, this last expression is bounded above by $(A-B)(1-\alpha)$ if (2.15) holds. □

Using similar arguments to those in the proof of the above theorems, we obtain the following theorem.

Theorem 2.8. *Let* $a, b \in \mathbb{C}^*$ $(|a| \neq 1;\ |b| \neq 1)$ *and* c *be a real number such that* $c > \max\{0,\ |a|+|b|-1\}$. *If the following inequality*

$$\begin{aligned}
(2.16)\qquad & \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\,[\{\mu(A-B)(1-\alpha)+(1-\mu)(1-B)(1+\beta)\}\\
&+\mu(1-B)(1+\beta)\frac{|ab|}{(c-|a|-|b|-1)}\\
&\left.+\frac{(1-\mu)\,[(A-B)(1-\alpha)-(1-B)(1+\beta)]\,(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right]\\
&\leq 2(A-B)(1-\alpha)+\frac{(1-\mu)\,[(A-B)(1-\alpha)-(1-B)(1+\beta)]\,(c-1)}{(|a|-1)(|b|-1)},
\end{aligned}$$

satisfied, then $[N_\mu(f*g)](z)$ *maps* $(f*g) \in \mathcal{K}$ *to* $\mathcal{S}(f,g;A,B;\alpha,\beta)$.

Remark. By specializing A, B and β in the above theorems, we will obtain new results for the classes $\mathcal{S}(f,g;A,B;\alpha)$ and $\mathcal{S}(f,g;\gamma,\alpha)$ mentioned in the introduction.

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