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Approximating Coupled Solutions of Coupled PBVPs of Non-linear First Order Ordinary Differential Equations

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Abstract. The present paper proposes a new monotone iteration method for existence as well as approximation of the coupled solutions for a coupled periodic boundary value problem of first order ordinary nonlinear differential equations. A new hybrid coupled fixed point theorem involving the Dhage iteration principle is proved in a partially ordered normed linear space and applied to the coupled periodic boundary value problems for proving the main existence and approximation results of this paper. An algorithm for the coupled solutions is developed and it is shown that the sequences of successive approximations defined in a certain way converge monotonically to the coupled solutions of the related differential equations under some suitable mixed hybrid conditions. A numerical example is also indicated to illustrate the abstract theory developed in the paper.

1. Introduction

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$, consider the coupled periodic boundary value problems (in short CPBVPs) of nonlinear first order ordinary nonlinear differential equations (in short DEs) of the form

$$x'(t) + \lambda x(t) = f(t, x(t), y(t)), \;\; x(0) = x(T), \tag{1.1}$$

and

$$y'(t) + \lambda y(t) = f(t, y(t), x(t)), \;\; y(0) = y(T), \tag{1.2}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *coupled solution* of the CPBVPs (1.1) and (1.2) we mean an ordered pair

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of differentiable functions $(u, v) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ that satisfy the DEs (1.1) and (1.2), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The coupled PBVPs (1.1) and (1.2) are well-known and the existence of the coupled solutions for them have been proved using the coupled fixed point theorems based on the properties of cones in the solution space $C(J, \mathbb{R})$. See Guo and Lakshmikantham [12], Heikkilä and Lakshmikantham [13], Sun [14] and the references therein. Recnetly, Bhaskar and lakshmikantham [11] proved the existence and uniqueness results for the coupled solutions of the CBVPs (1.1) and (1.2) without using the properties of the cones, however in this case the nonlinearity f involved in (1.1) and (1.2) is required to satisfy a weak Lipschitz condition which is considered to be strong in the theory of nonlinear differential and integral equations. Very recently, Dhage and Dhage [7] proved the existence as well as approximation of the coupled solutions for the coupled initial value problems (in short CIVPs) of the nonlinear ordinary differential equations,

$$x'(t) = f(t, x(t), y(t)), \;\; x(0) = x_0, \tag{1.3}$$

and

$$y'(t) = f(t, y(t), x(t)), \;\; y(0) = y_0, \tag{1.4}$$

for all $t \in J$, using the Dhage iteration principle which does not require any type of Lipschitz condition as well as any property of the cones in a appropriate Banach space. The aim of the present paper is to extend the method involving the Dhage iteration principle to the CPBVPs (1.1) and (1.2). Therefore, our approach to the considered CPBVPs (1.1) and (1.2) is different from the earlier ones discussed in the literature. Moreover, when $x = y$ in (1.1) or (1.2) and $f(t, x, x) = f_1(t, x)$ for all $t \in J$ and $x \in \mathbb{R}$, the results of this paper include the existence and approximation results of Dhage and Dhage [6] as special cases.

2. Auxiliary Results

Let $(E, \preceq)$ be a partial ordered set and let d be a metric on E such that $(E, \preceq, d)$ becomes a partially ordered metric space. By $E \times E$ we denote a metric space with the metric d^* defined by

$$d^*\big((x, y), (w, z)\big) = d(x, w) + d(y, z) \tag{2.1}$$

for $(x, y), (w, z) \in E \times E$. We define a partial order $\preceq$ in $E \times E$ as follows. Let $(x_1, x_2), (y_1, y_2) \in E \times E$. Then,

$$(x_1, x_2) \preceq (y_1, y_2) \Longleftrightarrow x_1 \preceq y_1 \;\text{ and }\; x_2 \succeq y_2. \tag{2.2}$$

Then, the triplet $(E \times E, \preceq, d^*)$ again becomes a partially ordered metric space. Let $\mathcal{F} : E \times E \to E$ be a mapping and consider the coupled mapping equations,

$$\mathcal{F}(x, y) = x \;\text{ and }\; \mathcal{F}(y, x) = y. \tag{2.3}$$

A point $(x^*, y^*) \in E \times E$ is said to be a coupled solution or coupled fixed point for the coupled mapping equation (2.3) if

$$\mathcal{F}(x^*, y^*) = x^* \quad \text{and} \quad \mathcal{F}(y^*, x^*) = y^*. \tag{2.4}$$

We need the following definitions in what follows.

Definition 2.1. A partially ordered normed metric space $(E, \preceq, d)$ is called regular if every nondecreasing (resp. nonincreasing) sequence $\{x_n\}$ converges to x^*, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$.

The details of the regularity property of the ordered sets may be found in Heikkilä and Lakshmikantham [13] and the references therein.

Definition 2.2.(Dhage [2]) A mapping $\mathcal{F} : E \times E \to E$ is called partially continuous at a point $(a, b) \in E \times E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d^*\big(\mathcal{F}(x, y), \mathcal{F}(a, b)\big) < \epsilon$$

whenever (x, y) is comparable to (a, b) and

$$d^*\big((x, y), (a, b)\big) < \delta.$$

If $\mathcal{F}$ is partially continuous at every point of $E \times E$, we say that $\mathcal{F}$ is partially continuous on $E \times E$.

Remark 2.1. If $\mathcal{F}$ is partially continuous on $E \times E$, then it is continuous on every totally ordered set or chain in $E \times E$.

Definition 2.3. A mapping $\mathcal{F}$ is called *mixed monotone* if $\mathcal{F}(x, y)$ is nondecreasing in x for each $y \in E$ and nonincreasing in y for each $x \in E$ with respect to the order relation $\preceq$ in E.

Remark 2.2. If $\mathcal{F}$ is mixed monotone, then it is a nondecreasing mapping on $E \times E$ with respect to the order relation $\preceq$ defined in $E \times E$.

Definition 2.4. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is compact. A mixed monotone mapping $\mathcal{F} : E \times E \to E$ is called partially compact if $\mathcal{F}(C_1 \times C_2)$ is a relatively compact subset of E for all chains C_1 and C_2 in E.

The details of compact and continuous operators may be found in the monograph by Heikkilä and Lakshmikantham [13] and the references therein.

Definition 2.5.(Dhage [2, 3]) The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence of points in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the metric defined by the absolute value function has above compatibility property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

The Dhage iteration principle developed in Dhage [2, 3, 4] may be stated as "**the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution converges monotonically to its solution**." The above convergence principle forms a powerful tool in th existence theory of such equations. The details of Dhage iteration principle are given in Dhage [2, 3, 4, 5] and Dhage and Dhage [9, 10]. The following applicable coupled hybrid fixed point theorem is a slight improvement of the coupled hybrid fixed point theorem proved in Dhage and Dhage [8] containing the Dhage iteration principle.

Theorem 2.1. *Let $(E, \preceq, d)$ be a regular partially ordered complete metric space such that the metric d and the order relation $\preceq$ are compatible in every compact chain C of E. Let $\mathcal{F} : E \times E \to E$ is a mixed monotone, partially continuous and partially compact mapping. If there exist elements $x_0 \in E$ and $y_0 \in E$ such that $x_0 \preceq \mathcal{F}(x_0, y_0)$ and $y_0 \succeq \mathcal{F}(y_0, x_0)$, then $\mathcal{F}$ has a coupled fixed point (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = \mathcal{F}(x_{n-1}, y_{n-1}) = \mathcal{F}^n(x_0, y_0)$ and $y_n = \mathcal{F}(y_{n-1}, x_{n-1}) = \mathcal{F}^n(y_0, x_0)$ converge monotonically to x^* and y^*respectively.*

Proof. Define the sequences $\{x_n\}$ and $\{y_n\}$ of points in E as follows. Choose

$$x_1 = \mathcal{F}(x_0, y_0) \text{ and } y_1 = \mathcal{F}(y_0, x_0).$$

Then, $x_0 \preceq x_1$ and $y_1 \preceq y_0$. Again, choose

$$x_2 = \mathcal{F}^2(x_0, y_0) = \mathcal{F}(x_1, y_1) = \mathcal{F}(\mathcal{F}(x_0, y_0), \mathcal{F}(y_0, x_0)) \succeq \mathcal{F}(x_0, y_0) = x_1.$$

Similarly, choose $y_2 = \mathcal{F}^2(y_0, x_0) = \mathcal{F}(y_1, x_1)$ so that

$$y_2 \preceq \mathcal{F}(\mathcal{F}(y_0, x_0), \mathcal{F}(x_0, y_0)) \preceq \mathcal{F}(y_0, x_0) = y_1.$$

Proceeding in this way, by induction, define

$$x_{n+1} = \mathcal{F}(x_n, y_n) = \mathcal{F}^n(x_0, y_0) \text{ and } y_{n+1} = \mathcal{F}(y_n, x_n) = \mathcal{F}^n(y_0, x_0), \tag{2.5}$$

for $n = 0, 1, 2, \ldots$, so that

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots, \tag{2.6}$$

and

$$y_0 \succeq y_1 \succeq \cdots \succeq y_n \succeq \cdots. \tag{2.7}$$

Thus, $\{x_n\}$ and $\{y_n\}$ are respectively monotone nondecreasing and monotone nonincreasing sequences and so are chains in E. From the construction of $\{x_n\}$ and $\{y_n\}$, it follows that

$$\{x_n\} \subseteq \mathcal{F}(\{x_n\}, \{y_n\}) \subseteq \mathcal{F}(\{x_n\} \times \{y_n\}).$$

Since $\mathcal{F}$ is partially compact on $E \times E$, one has $\mathcal{F}(\{x_n\} \times \{y_n\})$ is a relatively compact subset of E. As a result, $\overline{\mathcal{F}(\{x_n\} \times \{y_n\})}$ is compact and that $\{x_n\}$ has a convergent subsequence converging to a point, say $x^* \in E$. Since d and $\preceq$ are compatible in every compact chain C of E, the original sequence $\{x_n\}$ converges to x^*. Similarly, the sequence $\{y_n\}$ converges to a point say $y^* \in E$. Equivalently, $(x_n, y_n) \to (x^*, y^*)$ in the topology of the norm in $E \times E$. As E is a regular, we have that $x_n \preceq x^*$ and $y_n \succeq y^*$ for all $n \in \mathbb{N}$. Therefore, we obtain $(x_n, y_n) \preceq (x^*, y^*)$ for all $n \in \mathbb{N}$. Finally, by the partial continuity of $\mathcal{F}$, we obtain

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{F}(x_n, y_n) = \mathcal{F}(x^*, y^*)$$

and

$$y^* = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \mathcal{F}(y_n, x_n) = \mathcal{F}(y^*, x^*).$$

Thus (x^*, y^*) is a coupled fixed point of the mapping $\mathcal{F}$ on $E \times E$ into itself. This completes the proof. □

Remark 2.3. The regularity of the partially ordered metric space E may be replaced with a stronger condition of continuity than the partial continuity of the mappings $\mathcal{F}$ on $E \times E$. Again, the condition of compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

The simple fact concerning the compactibility of the order relation and the norm mentioned in Remark 2.3 has been used in formulating the main results of this paper. In the following sectin we prove the main existence and approximation results for the CBVP (1.1) and (1.2) defined on J.

3. Existence and Approximations Results

We place our considerations of the CPBVPs (1.1) and (1.2) in the function space $C(J, \mathbb{R})$. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t \in J$. Clearly, $(C(J,\mathbb{R}), \|\cdot\|, \leq)$ is a partially ordered complete normed linear space and has compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in certain subsets of it. The following lemma in this connection is useful in what follows.

Lemma 3.1.(Dhage and Dhage [6]) *Let $\big(C(J,\mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by* (3.1) *and* (3.2)*. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset S of $C(J,\mathbb{R})$.*

We need the following definition in the sequel.

Definition 3.1. An ordered pair of differentiable functions $(u, v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a coupled lower solution of the CPBVPs of coupled differential equations (1.1) and (1.2) if

$$u'(t) + \lambda u(t) \leq f(t, u(t), v(t)), \;\; u(0) \leq u(T),$$

and

$$v'(t) + \lambda v(t) \geq f(t, v(t), u(t)), \;\; v(0) \geq v(T),$$

for all $t \in J$. Similarly, an ordered pair of differentiable functions $(p, q) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a coupled upper solution of the CPBVPs (1.1) and (1.2) if the above inequalities are satisfied with reverse sign.

We consider the following set of hypotheses in what follows.

(H_1) f is bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M.

(H_2) The function $f(t, x, y)$ is nondecreasing in x and nonincreasing in y for each $t \in J$.

(H_3) The CPBVPs (1.1) and (1.2) have a lower coupled solution $(u, v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

(H_4) The CPBVPs (1.1) and (1.2) have a upper coupled solution $(p, q) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The following useful lemma is obvious and may be found in Dhage [1] and the references therein.

Lemma 3.2. *For any $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= \sigma(t), \;\; t \in J,\\ x(0) &= x(T), \end{aligned}\right\} \tag{3.3}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G(t,s)\,\sigma(s)\,ds \tag{3.4}$$

where,

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le s \le t \le T, \\ \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{3.5}$$

Notice that the Green's function G is continuous and nonnegative on $J \times J$ and therefore, the number $K := \max\{\, |G(t,s)| \,:\, t, s \in [0,T] \,\}$ exists.

An application of above Lemma 3.2 we obtain the following useful result.

Lemma 3.3. *A pair of function $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is a coupled solution of the CPBVPs* (1.1) *and* (1.2) *if and only if u and v are the solutions of the nonlinear integral equations,*

$$x(t) = \int_0^T G(t,s) f(s, x(s), y(s))\, ds \tag{3.6}$$

and

$$y(t) = \int_0^T G(t,s) f(s, y(s), x(s))\, ds \tag{3.7}$$

for all $t \in J$, where the Green's function $G(t,s)$ is given by (3.5).

Lemma 3.4. *If there exists a function $u \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + \lambda u(t) &\le \sigma(t), \quad t \in J, \\ u(0) &\le u(T), \end{aligned}\right\} \tag{3.8}$$

then

$$u(t) \le \int_0^T G(t,s)\, \sigma(s)\, ds \tag{3.9}$$

where, $G(t,s)$ is a Green's function given by (3.5).

Proof. Suppose that the function $u \in C(J,\mathbb{R})$ satisfies the inequalities given in (3.8). Multiplying the first inequality in (3.8) by $e^{\lambda t}$,

$$\left(e^{\lambda t} u(t)\right)' \le e^{\lambda t} \sigma(t).$$

A direct integration of above inequality from 0 to t yields

$$e^{\lambda t} u(t) \le u(0) + \int_0^t e^{\lambda s} \sigma(s)\, ds, \tag{3.10}$$

for all $t \in J$. Therefore, in particular,

$$e^{\lambda T}u(T) \leq u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds. \tag{3.11}$$

Now $u(0) \leq u(T)$, so one has

$$u(0)e^{\lambda T} \leq u(T)e^{\lambda T}. \tag{3.12}$$

From (3.11) and (3.12) it follows that

$$e^{\lambda T}u(0) \leq u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds \tag{3.13}$$

which further yields

$$u(0) \leq \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)}\sigma(s)\,ds. \tag{3.14}$$

Substituting (3.14) in (3.10) we obtain

$$u(t) \leq \int_0^T G(t,s)\sigma(s)\,ds,$$

for all $t \in J$. This completes the proof. □

Now we are well equipped with all necessary details to state our main existence and approximation result of this paper.

Theorem 3.1. *Assume that the hypotheses* (H_1) *through* (H_3). *Then the CPBVPs* (1.1) *and* (1.2) *have a coupled solution* (x^*, y^*) *defined on* J *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)f(s,x_n(s),y_n(s))\,ds \tag{3.15}$$

and

$$y_{n+1}(t) = \int_0^T G(t,s)f(s,y_n(s),x_n(s))\,ds \tag{3.16}$$

for each $t \in J$ *converge monotonically to* x^* *and* y^* *respectively.*

Proof. Set $E = C(J,\mathbb{R})$. Then, by Lemma 3.1, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Consider the mapping $\mathcal{F}$ on $E \times E$ defined as

$$\mathcal{F}(x,y)(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds, \quad t \in J, \tag{3.17}$$

and

$$\mathcal{F}(y,x)(t) = \int_0^T G(t,s)f(s,y(s),x(s))\,ds, \quad t \in J. \tag{3.18}$$

Since the Green's function $G(t,s)$ is continuous on $J \times J$, we have that $\mathcal{F}(x,y), \mathcal{F}(y,x) \in E$. As a result, $\mathcal{F}$ defines a mapping $\mathcal{F} : E \times E \to E$. We shall show that $\mathcal{F}$ satisfies all the conditions of Theorem 2.1. This will be achieved in a series of following steps.

Step I : $\mathcal{F}$ *is a mixed monotone operator on* $E \times E$.

Let $x_1, x_2 \in S$ be such that $x_1 \leq x_2$. Then, by hypothesis (H_2),

$$\begin{aligned}\mathcal{F}(x_1,y)(t) &= \int_0^T G(t,s)f(s,x_1(s),y(s))\,ds \\ &\leq \int_0^T G(t,s)f(s,x_2(s),y(s))\,ds \\ &= \mathcal{F}(x_2,y)(t)\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{F}(x,y)$ is monotone nondecreasing in x for all $t \in J$ and $y \in S$. Next, let $y_1, y_2 \in E$ be such that $y_1 \leq y_2$. Then,

$$\begin{aligned}\mathcal{F}(x,y_1)(t) &= \int_0^T G(t,s)f(s,x(s),y_1(s))\,ds \\ &\geq \int_0^T G(t,s)f(s,x(s),y_2(s))\,ds \\ &= \mathcal{F}(x,y_2)(t)\end{aligned}$$

for all $t \in J$ and $x \in S$. Hence $\mathcal{F}(x,y)$ is monotone nonincreasing in y for all $x \in E$. Thus $\mathcal{F}$ is a mixed monotone mapping on $E \times E$.

Step II: $\mathcal{F}$ *is partially continuous mixed monotone operator on* $E \times E$ into E.

Let $\{X_n\}_{n\in\mathbb{N}} = \{(x_n, y_n)\}$ be a monotone nondecreasing sequence in a chain $C = C_1 \times C_2$ of $E \times E$ such that $X_n = (x_n, y_n) \to (x, y) = X$ and $X_n \leq X$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{F}(X_n)(t) &= \int_0^T G(t,s)\left[\lim_{n\to\infty} f(s,x_n(s),y_n(s))\right] ds \\ &= \int_0^T G(t,s)f(s,x(s),y(s))\,ds \\ &= \mathcal{F}(X)(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{F}(X_n)$ converges monotonically to $\mathcal{F}(X)$ pointwise on J.

Next, we will show that $\{\mathcal{F}(X_n)\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary. Then, by hypothesis (H_1),

$$\begin{aligned}
&|\mathcal{F}(X_n)(t_2) - \mathcal{F}(X_n)(t_1)| \\
&\qquad \le \Big| \int_0^T G(t_2,s) f(s,x_n(s),y_n(s))\,ds \\
&\qquad\qquad - \int_0^T G(t_1,s) g(s,x_n(s),y_n(s))\,ds \Big| \\
&\qquad \le \int_0^T \big|G(t_2,s) - G(t_1,s)\big|\,\big|f(s,x_n(s),y_n(s))\big|\,ds \\
&\qquad \le M_f \int_0^T \big|G(t_2,s) - G(t_1,s)\big|\,ds \\
&\qquad \to 0 \quad as \quad t_2 \to t_1,
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{F}(X_n) \to \mathcal{F}(X)$ is uniform and hence $\mathcal{F}$ is a partially continuous on $E \times E$.

Step III: *$\mathcal{F}$ is a partially compact mixed monotone operator on $E \times E$.*

Let C_1 and C_2 be two arbitrary chains in E. We show that $\mathcal{F}(C_1 \times C_2)$ is a relatively compact subset of E. To finish it is enough to prove that $\mathcal{F}(C_1 \times C_2)$ is uniformly bounded and equicontinuous set in E. Let $x \in C_1$ and $y \in C_2$ be arbitrary. Then, by (H_1),

$$|\mathcal{F}(x,y)(t)| \le \int_0^T G(t,s)|f(s,x(s),y(s))|\,ds \le M_f K\,T = r$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{F}(x,y)\| \le r$ for all $x \in C_1$ and $y \in C_2$. Hence, $\mathcal{F}(C_1 \times C_2)$ is a uniformly bounded subset of E. Next, we show that $\mathcal{F}(C_1 \times C_2)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary. Then, for any $z \in \mathcal{F}(C_1 \times C_2)$, there exist $x \in C_1$ and $y \in C_2$ such that $z = \mathcal{F}(x,y)$. Without loss of generality, we may assume that $x(t_1) \ge x(t_2)$ and $y(t_1) \le y(t_2)$. Therefore, by the definition of $\mathcal{F}$,

$$\begin{aligned}
|z(t_1) - z(t_2)| &= |\mathcal{F}(x,y)(t_1) - \mathcal{F}(x,y)(t_2)| \\
&= \left| \int_0^T G(_1, s) f(s, x(s), y(s))\, ds - \int_0^T G(t_2, s) f(s, x(s), y(s))\, ds \right| \\
&\leq \int_0^T \big|G(t_2, s) - G(t_1, s)\big| \, \big|f(s, x_n(s), y_n(s))\big|\, ds \\
&\leq M_f \int_0^T \big|G(t_2, s) - G(t_1, s)\big|\, ds \\
&\longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $x \in C_1$ and $y \in C_2$. As a result, we have

$$|\mathcal{F}(x,y)(t_1) - \mathcal{F}(x,y)(t_2)| \longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,$$

uniformly for all $(x,y) \in C_1 \times C_2$. Consequently $\mathcal{F}(C_1 \times C_2)$ is an equi-continuous set of E. We apply Arzeli-Ascoli theorem and deduce that $\mathcal{F}(C_1 \times C_2)$ is a relatively compact subset of E. Hence $\mathcal{F}$ is partially relatively compact on $E \times E$.

Now $\mathcal{F}$ is a partially continuous and partially compact mixed monotone operator on $E \times E$ into E. Again, by hypothesis (H$_3$), there exist elements u and v in E such that $u \leq \mathcal{F}(u,v)$ and $v \geq \mathcal{F}(v,u)$ in view of Lemma 3.4. Thus all the conditions of Theorem 2.1 are satisfied and hence the coupled equations $\mathcal{F}(x,y) = x$ and $y = \mathcal{F}(y,x)$ have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.15) and (3.16) converge monotonically to x^* and y^* respectively. This completes the proof. □

Remark 3.1. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H$_3$) with (H$_4$). The proof of Theorem 3.1 under this new hypothesis is obtained using the similar arguments with appropriate modifications.

Example 3.1. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the coupled PBVPs,

(3.19) $$x'(t) + x(t) = \tanh x(t) - \tanh y(t), \quad x(0) = x(1),$$

and

(3.20) $$y'(t) + y(t) = \tanh y(t) - \tanh x(t), \quad y(0) = x(1),$$

for all $t \in [0,1]$.

Here, the function f is given by

$$f(t,x,y) = \tanh x - \tanh y.$$

for all $t \in [0,1]$ and $x, y \in \mathbb{R}$. Clearly, f is uniformly continuous and bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$. Furthermore, $f(t,x,y)$ is nondecreasing in x for each $t \in J$ and $y \in \mathbb{R}$ and nonincreasing in y for each $t \in J$ and $x \in \mathbb{R}$. Finally, there exist functions $u(t) = -2e^{-t+1}(e-1)$ and $v(t) = 2e^{-t+1}(e-1)$ on $[0,1]$ such that

$$u'(t) \le \tanh u(t) - \tanh v(t) = f(t, u(t), v(t)), \quad u(0) \le u(1)$$

and

$$v'(t) \ge \tanh v(t) - \tanh u(t) = f(t, v(t), u(t)), \quad v(0) \ge v(1)$$

for all $t \in J$. Hence the hypothesis (H_3) is satisfied. Thus, the nonlinearity f satisfies all the hypotheses (H_1) through (H_3) of Theorem 3.1. Hence, the CPBVPs (3.19) and (3.20) have a coupled solution (x^*, y^*) defined on $[0,1]$ and the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0 = u, \quad x_{n+1}(t) = \int_0^1 G(t,s)\big[\tanh x_n(s) - \tanh y_n(s)\big]\,ds,$$

and

$$y_0 = v, \quad y_{n+1}(t) = \int_0^1 G(t,s)\big[\tanh y_n(s) - \tanh x_n(s)\big]\,ds,$$

for all $t \in [0,1]$, where the Green's function $G(t,s)$ is defined by

$$G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & \text{if} \quad 0 \le s \le t \le 1, \\ \dfrac{e^{s-t}}{e-1}, & \text{if} \quad 0 \le t < s \le 1, \end{cases}$$

converge monotonically to x^* and y^* respectively.

Remark 3.2. Finally, we mention that Theorem 2.1 may be applied to various nonlinear initial and boundary value problems of ordinary coupled differential equations for proving the existence as well as algorithms for the coupled solutions under suitable mixed monotonic and partial compactness type conditions.

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