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L^p-Boundedness for the Littlewood-Paley g-Function Connected with the Riemann-Liouville Operator

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Abstract. We study the Gauss and Poisson semigroups connected with the Riemann-Liouville operator defined on the half plane. Next, we establish a principle of maximum for the singular partial differential operator

$$\Delta_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2};\ (r,x,t) \in]0,+\infty[\times\mathbb{R}\times]0,+\infty[.$$

Later, we define the Littlewood-Paley g-function and using the principle of maximum, we prove that for every $p \in]1,+\infty[$, there exists a positive constant C_p such that for every $f \in L^p(d\nu_\alpha)$,

$$\frac{1}{C_p}\, ||f||_{p,\nu_\alpha} \leqslant ||g(f)||_{p,\nu_\alpha} \leqslant C_p\, ||f||_{p,\nu_\alpha}.$$

1. Introduction

The usual Littlewood-Paley g-function is defined in the Euclidean space [27] by

$$\forall x \in \mathbb{R}^n;\ g(f)(x) = \Big(\int_0^{+\infty} \big|\nabla P^t f(x)\big|^2 t dt\Big)^{\frac{1}{2}},$$

where $(P^t)_{t>0}$ is the usual Poisson semigroup defined by

$$P^t f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n} \frac{tf(y)}{\big(t^2+|x-y|^2\big)^{\frac{n+1}{2}}} dy,$$

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and ∇ is the gradient given by

$$\nabla = \Big(\frac{\partial}{\partial x_1},\ ...,\ \frac{\partial}{\partial x_n},\ \frac{\partial}{\partial t}\Big).$$

It is well known (see for example [27]) that the mapping

$$f \longmapsto g(f)$$

is bounded from the Lebesgue space $L^p(\mathbb{R}^n, dx),\ p \in\ ]1, +\infty[$ into itself. Moreover, the Littlewood-Paley theory plays an important role in the study of many function spaces as the Hardy space H^p. Many aspects of the Littlewood-Paley g-function connected with several hypergroups are studied [1, 2, 6, 25, 29]. The authors have been especially interested by the boundedness of such operator when acting on the Lebesgue space $L^p;\ p \in\ ]1, +\infty[$.
In [7], the authors have defined the Riemann-Liouville operator $\mathscr{R}_\alpha;\ \alpha \geqslant 0$, by

$$(1.1)\quad \mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \qquad\qquad\qquad \times(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0; \end{cases}$$

where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.
The dual ${}^t\mathscr{R}_\alpha$ is defined by

$$(1.2)\ {}^t\mathscr{R}_\alpha(g)(r,x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\ \dfrac{1}{2^\alpha\ \Gamma(\alpha+1)}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u, x+v) \\ \qquad\qquad\qquad \times(u^2-v^2-r^2)^{\alpha-1}u\ du\ dv, & \text{if } \alpha > 0, \\ \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} g\big(\sqrt{r^2+(x-y)^2}, y\big)dy, & \text{if } \alpha = 0; \end{cases}$$

where g is any continuous function on $\mathbb{R}^2$, even with respect to the first variable and with compact support.
In particular, for $\alpha = 0$ and by a change of variables, we get

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, x + r\sin\theta)d\theta.$$

This means that $\mathscr{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and with radius r.
The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [17, 18] or in the linearized inverse scattering problem in acoustics [15].

The operators $\mathscr{R}_\alpha$ and its dual ${}^t\mathscr{R}_\alpha$ have the same properties as the Radon transform [16], for this reason, $\mathscr{R}_\alpha$ is called sometimes the generalized Radon transform.
The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined by

$$(1.3)\ \forall(\mu,\lambda)\in\Upsilon,\ \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^\infty\int_{\mathbb{R}} f(r,x)\mathscr{R}_\alpha\big(\cos(\mu\cdot)e^{-i\lambda\cdot}\big)(r,x)d\nu_\alpha(r,x) = \int_0^\infty\int_{\mathbb{R}} f(r,x)\ j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x}d\nu_\alpha(r,x),$$

where

- Υ is the set given by

$$(1.4)\qquad \Upsilon = \mathbb{R}^2\cup\big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2;\ |\mu|\leqslant|\lambda|\big\}.$$

- $d\nu_\alpha(r,x)$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$(1.5)\qquad d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}dr}{2^\alpha\Gamma(\alpha+1)}\otimes\frac{dx}{\sqrt{2\pi}}.$$

- j_α is the modified Bessel function that will be defined in the second section.

Many harmonic analysis results have been established for the Fourier transform $\mathscr{F}_\alpha$ [5, 7, 9, 10, 11, 24]. Also, many uncertainty principles related to the Fourier transform $\mathscr{F}_\alpha$ have been proved [3, 4, 8, 20, 22, 23].

In [2], we have defined the Gauss and Poisson semigroups associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$. These semigroups are denoted by $(\mathscr{G}^t)_{t>0}$ and $(\mathscr{P}^t)_{t>0}$. The Poisson semigroup $(\mathscr{P}^t)_{t>0}$ allows us to define the Littlewood-Paley g-function connected with $\mathscr{R}_\alpha$ by

$$g(f)(r,x)=\Big(\int_0^{+\infty}\Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2 tdt\Big)^{\frac12},$$

where

$$(1.6)\qquad \mathscr{U}(f)(r,x,t) = \mathscr{P}^t f(r,x),$$

and

$$\nabla=\Big(\frac{\partial}{\partial r},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial t}\Big).$$

Then, using the maximal functions associated with these semigroups and their mutual connection, we have established in [2] the following main result

For every $p\in\,]1,2]$; the mapping $f\longmapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f\in L^p(d\nu_\alpha)$, we have

$$(1.7)\qquad \big|\big|g(f)\big|\big|_{p,\nu_\alpha}\leqslant 2\,\frac{2^{\frac{2-p}{2}}}{p}\,\big(\frac{p}{p-1}\big)^{\frac1p}||f||_{p,\nu_\alpha}.$$

Where
. $L^p(d\nu_\alpha)$; $p\in [1,+\infty]$, is the Lebesgue space formed by the measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $||f||_{p,\nu_\alpha}<+\infty$, with

$$(1.8)\qquad ||f||_{p,\nu_\alpha} = \begin{cases} \Big(\displaystyle\int_0^{+\infty}\int_{\mathbb{R}}\big|f(r,x)\big|^p d\nu_\alpha(r,x)\Big)^{\frac{1}{p}}, & \text{if } p\in [1,+\infty[,\\ \displaystyle\operatorname*{ess\ sup}_{(r,x)\in [0,+\infty[\times\mathbb{R}}\big|f(r,x)\big|, & \text{if } p=+\infty,\end{cases}$$

and $d\nu_\alpha$ is given by the relation (1.5).
Our purpose in this work consists to extend the inequality (1.7) for every $p\in]1,+\infty[$.
In this context, we consider the singular partial differential operator

$$\Delta_\alpha=\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\,\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial t^2}.$$

Building on the Hopf principle of maximum, we have established the following principle of maximum for the operator Δ_α:
Let a_0,a_1,T be positive real numbers and $\Omega=]-a_0,a_0[\times]-a_1,a_1[\times]0,T[$. Let $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ such that

i. $\forall (r,x,t)\in\Omega$; $u(r,x,t)=u(-r,x,t)$.

ii. $\forall (r,x,t)\in\Omega$, $\Delta_\alpha u(r,x,t)\geqslant 0$.

Then, if u attains its maximum in Ω, u is constant on Ω.
Using the precedent principle of maximum, we have proved the following interesting result
Let $u\in C^2(\mathbb{R}^2\times]0,+\infty[)\cap C^0(\mathbb{R}^2\times[0,+\infty[)$ such that

i. $\forall (r,x,t)\in\mathbb{R}^2\times[0,+\infty[$; $u(r,x,t)=u(-r,x,t)$.

ii. $\forall (r,x)\in\mathbb{R}^2$, $u(r,x,0)\geqslant 0$.

iii. $\displaystyle\lim_{r^2+x^2+t^2\longmapsto+\infty}u(r,x,t)=0$.

vi. $\forall (r,x,t)\in\mathbb{R}^2\times[0,+\infty[$; $\Delta_\alpha u(r,x,t)\leqslant 0$.

Then u is nonnegative.
This theorem allows us to establish the well known inequality satisfied by the Poisson semigroup, that is
For every positive real number t and for every $f\in\mathscr{D}_e(\mathbb{R}^2)$, we have

$$\big|\nabla\big(\mathscr{U}(f)\big)(r,x,2t)\big|^2\leqslant\mathscr{P}^t\Big(\big|\nabla\big(\mathscr{U}(f)\big)(.,.,t)\big|^2\Big)(r,x),$$

where $\mathscr{U}(f)$ is given by the relation (1.6) and $\mathscr{D}_e(\mathbb{R}^2)$ is the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.
Combining the previous results together with the Riesz-Thorin theorem and our paper [2], we have establish the main result of this paper
For every $p \in \,]1, +\infty[$, the mapping: $f \longmapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f \in L^p(d\nu_\alpha)$, we have

$$||g(f)||_{p,\nu_\alpha} \quad \leqslant \quad B_p \; ||f||_{p,\nu_\alpha} \tag{1.9}$$

Finally, using the Plancherel theorem for the Fourier transform associated with the Riemann-Liouville operator, we have proved the "converse" inequality of (1.9), that is
For every $p \in \,]1, +\infty[$ and every $f \in L^p(d\nu_\alpha)$, we have

$$||f||_{p,\nu_\alpha} \quad \leqslant \quad 4 \; B_{\frac{p}{p-1}} \; ||g(f)||_{p,\nu_\alpha}. \tag{1.10}$$

The inequalities (1.9) and (1.10) show that
For every $p \in \,]1, +\infty[$, there exists a positive constant C_p such that for every $f \in L^p(d\nu_\alpha)$,

$$\frac{1}{C_p} \; ||f||_{p,\nu_\alpha} \leqslant ||g(f)||_{p,\nu_\alpha} \leqslant C_p \; ||f||_{p,\nu_\alpha}.$$

2. The Riemann-Liouville Transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with the Riemann-Liouville operator. For more details see [5, 7, 9, 10, 11, 24].
Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \dfrac{\partial}{\partial x}; \\ \Xi = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \quad (r,x) \in \,]0,+\infty[\times\mathbb{R}, \;\; \alpha \geqslant 0. \end{cases}$$

For all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} Du(r,x) = -i\lambda u(r,x); \\ \Xi u(r,x) = -\mu^2 u(r,x); \\ u(0,0) = 1, \;\; \dfrac{\partial u}{\partial r}(0,x) = 0; \; \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}, \qquad \varphi_{\mu,\lambda}(r,x) \quad = \quad j_\alpha\big(r\sqrt{\mu^2+\lambda^2}\big)\; e^{-i\lambda x}, \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\big(\frac{z}{2}\big)^{2k},$$

and J_α is the Bessel function of first kind and index α [13, 14, 21, 30] . The modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(-izt)dt. \tag{2.2}$$

Consequently, for every $k\in\ \mathbb{N}$ and $z\in\ \mathbb{C}$, we have

$$\big|j_\alpha^{(k)}(z)\big| \leqslant e^{|Im(z)|}. \tag{2.3}$$

Proposition 2.1. *The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties*
i. The function $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R}^2$ if, and only if $(\mu,\lambda)\in\ \Upsilon$, where Υ is the set given by the relation (1.4) *and in this case,*

$$\sup_{(r,x)\in\ \mathbb{R}^2}\big|\varphi_{\mu,\lambda}(r,x)\big| = 1. \tag{2.4}$$

ii. The function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1}\cos\big(\mu rs\sqrt{1-t^2}\big)\exp\big(-i\lambda(x+rt)\big) \\ \qquad\qquad \times(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; \quad \text{if } \alpha>0, \\ \frac{1}{\pi}\int_{-1}^{1}\cos\big(r\mu\sqrt{1-t^2}\big)\exp\big(-i\lambda(x+rt)\big) \\ \qquad\qquad \times\frac{dt}{\sqrt{1-t^2}}, \quad \text{if } \alpha=0. \end{cases} \tag{2.5}$$

Remark 2.2. The Mehler integral representation (2.5) of the eigenfunction $\varphi_{\mu,\lambda}$ allows us to define the integral transform $\mathscr{R}_\alpha$ by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1}f(rs\sqrt{1-t^2},x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \qquad\qquad \times(1-s^2)^{\alpha-1}\,dt\,ds, \text{ if } \alpha>0, \\ \frac{1}{\pi}\int_{-1}^{1}f(r\sqrt{1-t^2},x+rt)\frac{dt}{\sqrt{1-t^2}}, \quad \text{if } \alpha=0; \end{cases} \tag{2.6}$$

where f is any continuous function on $\mathbb{R}^2$; even with respect to the first variable. Then, the relations (2.5) and (2.6) show that

$$\varphi_{\mu,\ \lambda}(r,x) = \mathscr{R}_\alpha\big(\cos(\mu\cdot)e^{-i\lambda\cdot}\big)(r,x), \tag{2.7}$$

which gives the mutual connection between the functions $\varphi_{\mu,\ \lambda}$ and $\cos(\mu\cdot)e^{-i\lambda\cdot}$. For this reason, the operator $\mathscr{R}_\alpha$ is called the Riemann-Liouville transform associated with the operators D and Ξ.

The partial differential operators D and Ξ satisfy the intertwining properties with the Riemann-Liouville operator and its dual

$$ {}^t\mathscr{R}_\alpha\Xi(f) = \frac{\partial^2}{\partial r^2}\ {}^t\mathscr{R}_\alpha(f), \qquad {}^t\mathscr{R}_\alpha D(f) = D\ {}^t\mathscr{R}_\alpha(f), $$

$$ \Xi\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha\frac{\partial^2}{\partial r^2}(f), \qquad D\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha D(f), $$

where f is a sufficiently smooth function.
To define the translation operator associated with the Riemann-Liouville transform, we use the product formula for the eigenfunction $\varphi_{\mu,\lambda}$, that is for (r,x) and $(s,y)\in\ [0,+\infty[\times\mathbb{R}$,

$$ \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_{\mu,\lambda}\big(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\big)\sin^{2\alpha}\theta d\theta. $$

Definition 2.3. i) For every $(r,x)\in\ [0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville transform is defined on $L^p(d\nu_\alpha)$; $p\in\ [1,+\infty]$, by

$$ \tau_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\big(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\big)\sin^{2\alpha}(\theta)d\theta. \tag{2.8} $$

ii) The convolution product of $f,g\in\ L^1(d\nu_\alpha)$ is defined for every $(r,x)\in [0,+\infty[\times\mathbb{R}$, by

$$ f*g(r,x) = \int_o^{+\infty}\int_{\mathbb{R}}\tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \tag{2.9} $$

where $\check{f}(s,y)=f(s,-y)$.

The set $[0,+\infty[\times\mathbb{R}$ equipped with the convolution product $*$ is an hypergroup in the sense of [12].
We denote by $C_{0,e}(\mathbb{R}^2)$ the space of continuous function on $\mathbb{R}^2$, even with respect to the first variable such that

$$ \lim_{r^2+x^2\to+\infty} f(r,x)=0. $$

The space $C_{0,e}(\mathbb{R}^2)$ is equipped with the norm

$$ ||f||_{\infty,\nu_\alpha} = \sup_{(r,x)\in\ [0,+\infty[\times\mathbb{R}} |f(r,x)|. $$

Proposition 2.4. *i. For every* $f \in L^p(d\nu_\alpha)$*;* $1 \leqslant p \leqslant +\infty$*, and for every* $(r,x) \in [0,+\infty[\times\mathbb{R}$*, the function* $\tau_{(r,x)}(f)$ *belongs to* $L^p(d\nu_\alpha)$ *and we have*

$$||\tau_{(r,x)}(f)||_{p,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}. \tag{2.10}$$

ii. For every $f \in L^1(d\nu_\alpha)$ *and* $(r,x) \in [0,+\infty[\times\mathbb{R}$*,*

$$\int_0^\infty \int_{\mathbb{R}} \tau_{(r,x)}(f)(s,y)d\nu_\alpha(s,y) = \int_0^\infty \int_{\mathbb{R}} f(s,y)d\nu_\alpha(s,y). \tag{2.11}$$

iii. For every $f \in L^p(d\nu_\alpha)$*;* $p \in [1,+\infty[$*, we have*

$$\lim_{(r,x)\to(0,0)} ||\tau_{(r,x)}(f) - f||_{p,\nu_\alpha} = 0\ . \tag{2.12}$$

iv. For every $f \in C_{0,e}(\mathbb{R}^2)$ *and every* $(r,x) \in \mathbb{R}^2$*, the function* $\tau_{(r,x)}(f)$ *belongs to* $C_{0,e}(\mathbb{R}^2)$ *and*

$$\lim_{(r,x)\to(0,0)} ||\tau_{(r,x)}(f) - f||_{\infty,\nu_\alpha} = 0. \tag{2.13}$$

v. Let φ *be a nonnegative measurable function on* $\mathbb{R}\times\mathbb{R}$*, even with respect to the first variable, such that*

$$\int_0^{+\infty} \int_{\mathbb{R}} \varphi(r,x)d\nu_\alpha(r,x) = 1.$$

Then the family $(\varphi_{(a,b)})_{(a,b)\in(\mathbb{R}_+^*)^2}$ *defined by*

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R},\ \varphi_{(a,b)}(r,x) = \frac{1}{a^{2\alpha+2}b}\varphi\big(\frac{r}{a},\frac{x}{b}\big)$$

is an approximation of the identity in $L^p(d\nu_\alpha)$*;* $p \in [1,+\infty[$*, that is for every* $f \in L^p(d\nu_\alpha)$*, we have*

$$\lim_{(a,b)\to(0^+,0^+)} ||f*\varphi_{(a,b)} - f||_{p,\nu_\alpha} = 0. \tag{2.14}$$

vi. For every $f \in C_{0,e}(\mathbb{R}^2)$*,*

$$\lim_{(a,b)\to(0^+,0^+)} ||f*\varphi_{(a,b)} - f||_{\infty,\nu_\alpha} = 0. \tag{2.15}$$

vii. If $1 \leqslant p,q,r \leqslant +\infty$ *are such that* $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1$ *and if* $f \in L^p(d\nu_\alpha)$*,* $g \in L^q(d\nu_\alpha)$*, then the function* $f*g$ *belongs to* $L^r(d\nu_\alpha)$*, and we have the Young's inequality*

$$||f*g||_{r,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}||g||_{q,\nu_\alpha}. \tag{2.16}$$

In the following, we need the notations
. Υ_+ is the subset of Υ given by

$$\Upsilon_+ = \mathbb{R}_+ \times \mathbb{R} \cup \big\{(it,x);\ (t,x) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant |x|\big\}.$$

. $\mathscr{B}_{\Upsilon_+}$ is the σ-algebra defined on Υ_+ by

$$\mathscr{B}_{\Upsilon_+} = \big\{\theta^{-1}(B),\ B \in \mathscr{B}_{\text{Or}}\big([0,+\infty[\times\mathbb{R}\big)\big\},$$

where θ is the bijective function defined on the set Υ_+ by

$$\theta(\mu,\lambda) = \big(\sqrt{\mu^2+\lambda^2},\lambda\big), \tag{2.17}$$

and $\mathscr{B}_{\text{Or}}\big([0,+\infty[\times\mathbb{R}\big)$ is the usual Borel σ-algebra on $[0,+\infty[\times\mathbb{R}$.
. $d\gamma_\alpha$ is the measure defined on $\mathscr{B}_{\Upsilon_+}$ by

$$\forall A \in \mathscr{B}_{\Upsilon_+},\ \gamma_\alpha(A) = \nu_\alpha\big(\theta(A)\big).$$

Proposition 2.5. *i. For all nonnegative measurable function g on Υ_+, we have*

$$\begin{aligned}\int\int_{\Upsilon_+} g(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) &= \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu d\lambda \\ &+ \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu d\lambda\Big).\end{aligned}$$

ii. For all nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$), $f\circ\theta$ is a nonnegative measurable function on Υ_+ (respectively integrable on Υ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\int\int_{\Upsilon_+} (f\circ\theta)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)d\nu_\alpha(r,x). \tag{2.18}$$

Now, using the eigenfunction $\varphi_{\mu,\lambda}$ given by the relation (2.1), we can define the Fourier transform.

Definition 2.6. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda) \in \Upsilon, \qquad \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x).$$

Proposition 2.7. *i. For every $f \in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have*

$$\big|\big|\mathscr{F}_\alpha(f)\big|\big|_{\infty,\gamma_\alpha} \leqslant ||f||_{1,\nu_\alpha}. \tag{2.19}$$

ii. Let $f \in L^1(d\nu_\alpha)$. For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$\forall(\mu,\lambda) \in \Upsilon, \qquad \mathscr{F}_\alpha\big(\tau_{(r,x)}(f)\big)(\mu,\lambda) = \overline{\varphi_{\mu,\lambda}(r,x)}\mathscr{F}_\alpha(f)(\mu,\lambda).$$

iii. For $f,g \in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda) \in \Upsilon, \qquad \mathscr{F}_\alpha(f*g)(\mu,\lambda) = \mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda). \tag{2.20}$$

vi. For $f \in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda) \in \Upsilon, \qquad \mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda), \tag{2.21}$$

where for every $(\mu,\lambda) \in \mathbb{R}^2$,

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)d\nu_\alpha(r,x), \tag{2.22}$$

and θ is the function defined by the relation (2.17)*.*

We denote by $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable. The space $\mathscr{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) = \sup_{\substack{(r,x)\in [0,+\infty[\times\mathbb{R}\\ k+|\beta|\leqslant m}} \big(1+r^2+x^2\big)^k\big|D^\beta(\varphi)(r,x)\big|. \tag{2.23}$$

Theorem 2.8. *i. Let $f \in L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,*

$$\begin{aligned} f(r,x) &= \int\int_{\Upsilon_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda) \\ &= \int_0^\infty\int_{\mathbb{R}} \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda). \end{aligned} \tag{2.24}$$

ii. ([19]) *The transform $\widetilde{\mathscr{F}_\alpha}$ is a topological isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself and we have*

$$\widetilde{\mathscr{F}_\alpha}^{-1}(f)(r,x) = \int_0^\infty\int_{\mathbb{R}} f(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}d\nu_\alpha(\mu,\lambda).$$

iii. (Plancherel theorem) The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$ and for every $f \in L^2(d\nu_\alpha)$,

$$||\mathscr{F}_\alpha(f)||_{2,\gamma_\alpha} \quad = \quad ||f||_{2,\nu_\alpha}. \tag{2.25}$$

Using the relations (2.19), (2.25) and the Riesz-Thorin theorem's [26, 28], we deduce that for every $f \in L^p(d\nu_\alpha)$; $p \in [1,2]$, the function $\mathscr{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$; $p' = \dfrac{p}{p-1}$, and we have

$$\big|\big|\mathscr{F}_\alpha(f)\big|\big|_{p',\gamma_\alpha} \quad \leqslant \quad ||f||_{p,\nu_\alpha}. \tag{2.26}$$

3. Gauss and Poisson Semigroups associated with the Riemann-Liouville Operator

In our paper [2], we have defined and studied the Gauss and Poisson semigroups connected with the operator $\mathscr{R}_\alpha$. In this section, we recall some properties of these operators to simplify the reading of this paper. Also, we establish some new results that we use in the next section.

Definition 3.1. i. The Gauss kernel g_t, $t > 0$, associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} g_t(r,x) &= \frac{e^{-\frac{(r^2+x^2)}{4t}}}{(2t)^{\alpha+\frac{3}{2}}} = \int\int_{\Upsilon_+} e^{-t(\mu^2+2\lambda^2)}\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda) \\ &= \widetilde{\mathscr{F}}_\alpha^{-1}\big(e^{-t(s^2+y^2)}\big)(r,x). \end{aligned} \tag{3.1}$$

ii. For every $t > 0$, the Poisson kernel p_t associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} p_t(r,x) &= \int\int_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}}\ \overline{\varphi_{s,y}(r,x)}d\gamma_\alpha(s,y) \\ &= \widetilde{\mathscr{F}_\alpha}^{-1}\Big(e^{-t\sqrt{s^2+y^2}}\Big)(r,x) = \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\ \frac{t}{(r^2+x^2+t^2)^{\alpha+2}}. \end{aligned} \tag{3.2}$$

Definition 3.2. The Gauss (respectively the Poisson) semigroup $(\mathscr{G}^t)_{t>0}$ (respectively $(\mathscr{P}^t)_{t>0}$) is defined by

$$\mathscr{G}^t(f)(r,x) \quad = \quad g_t * f(r,x)\ \big(\text{respectively } \mathscr{P}^t(f)(r,x) = p_t * f(r,x)\big). \tag{3.3}$$

Proposition 3.3. *i. For every $p \in [1,+\infty]$; the operator $\mathscr{G}^t$ (respectively $\mathscr{P}^t$) ; $t > 0$, is a bounded positive operator from $L^p(d\nu_\alpha)$ into itself and for every $f \in L^p(d\nu_\alpha)$, we have*

$$\big|\big|\mathscr{G}^t(f)\big|\big|_{p,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}\ \Big(\textit{respectively } \big|\big|\mathscr{P}^t(f)\big|\big|_{p,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}\Big).$$

ii. For every $p \in [1,+\infty[$, the family $(\mathscr{G}^t)_{t>0}$ (respectively $(\mathscr{P}^t)_{t>0}$) is a strongly continuous semigroup of operators on $L^p(d\nu_\alpha)$, that is

. *For $s,t>0$; $\mathscr{G}^s \circ \mathscr{G}^t = \mathscr{G}^{s+t}$, (respectively $\mathscr{P}^s \circ \mathscr{P}^t = \mathscr{P}^{s+t}$).*

. *For every $f \in L^p(d\nu_\alpha)$, $\lim_{t\to 0^+} ||\mathscr{G}^t(f) - f||_{p,\nu_\alpha} = 0$, (respectively $\lim_{t\to 0^+} ||\mathscr{P}^t(f) - f||_{p,\nu_\alpha} = 0$).*

Lemma 3.4. *i. We have the following connection between the Gauss and Poisson semigroups, that is*

$$\mathscr{P}^t(f)(r,x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)du.$$

ii. For every $p \in]1,+\infty[$ and every $f \in \mathscr{D}_e(\mathbb{R}^2)$, the maximal function f^* defined by

$$f^*(r,x) \quad = \quad \sup_{t>0} \left|\mathscr{P}^t(f)(r,x)\right| \tag{3.4}$$

belongs to the space $L^p(d\nu_\alpha)$ and we have

$$||f^*||_{p,\nu_\alpha} \quad \leqslant \quad 2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} ||f||_{p,\nu_\alpha}. \tag{3.5}$$

Lemma 3.5. *Let $f \in \mathscr{D}_e(\mathbb{R}^2)$; $supp(f) \subset B_a = \big\{(r,x) \in \mathbb{R}^2;\ r^2+x^2 \leqslant a^2\big\}, a>0$ and let*

$$\begin{aligned} \mathscr{U}(f)(r,x,t) &= p_t * f(r,x) \\ &= \mathscr{P}^t(f)(r,x). \end{aligned} \tag{3.6}$$

i. For every $(r,x) \in \mathbb{R}^2$; $r^2+x^2 \geqslant 4a^2$,

$$\Big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big| \leqslant \frac{2^{2\alpha+5}\ \Gamma(\alpha+2)(2\alpha+5)a^{2\alpha+3}}{\sqrt{\pi}\ \Gamma(\alpha+\frac{3}{2})(2\alpha+3)}\ \frac{||f||_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+2}}.$$

ii. For every $(r,x,t) \in \mathbb{R}^2 \times]0,+\infty[$,

$$\Big|\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,t)\Big| \leqslant \frac{\Gamma(\alpha+\frac{3}{2})\ \Gamma(\alpha+\frac{5}{2})\ 2^{\alpha+\frac{5}{2}}}{\sqrt{\pi}\ \Gamma(\alpha+1)}\ \frac{||f||_{1,\nu_\alpha}}{t^{2\alpha+4}},$$

and

$$\Big|\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,t)\Big| \leqslant \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+\frac{5}{2})}{\pi}\ \frac{||f||_{1,\nu_\alpha}}{t^{2\alpha+4}}.$$

Proof. i) From the relations (2.8) and (3.2), we have

$$\begin{aligned} \tau_{(r,-x)}(p_t)(s,y) \quad &= \quad \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\pi}\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \\ &\times \int_o^\pi \frac{t\sin^{2\alpha}(\theta)d\theta}{\big(t^2+(r^2+s^2+2rs\cos(\theta))+(x-y)^2\big)^{\alpha+2}}. \end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial}{\partial t}\big(\tau_{(r,-x)}(p_t)\big)(s,y) &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\pi}\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\\ &\times \int_o^\pi \frac{\partial}{\partial t}\Big(\frac{t\sin^{2\alpha}(\theta)}{\big(t^2+(r^2+s^2+2rs\cos(\theta))+(x-y)^2\big)^{\alpha+2}}\Big)d\theta.\end{aligned}$$

By a standard computation, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial t}\big(\tau_{(r,-x)}(p_t)\big)(s,y)\Big| &\leqslant \frac{2^{\alpha+\frac{3}{2}}\ (2\alpha+5)\Gamma(\alpha+2)}{\pi}\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\\ &\times \int_o^\pi \frac{\sin^{2\alpha}(\theta)}{\big(t^2+(r^2+s^2+2rs\cos(\theta))+(x-y)^2\big)^{\alpha+2}}d\theta\\ &\leqslant \frac{2^{\alpha+\frac{3}{2}}\ (2\alpha+5)\Gamma(\alpha+2)}{\pi}\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\\ &\times \frac{1}{\big(t^2+(r-s)^2+(x-y)^2\big)^{\alpha+2}}\int_o^\pi \sin^{2\alpha}(\theta)d\theta\\ &= \frac{2^{\alpha+\frac{3}{2}}(2\alpha+5)\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{1}{\big(t^2+(r-s)^2+(x-y)^2\big)^{\alpha+2}}. \qquad (3.7)\end{aligned}$$

Let $f\in\ \mathscr{D}_e(\mathbb{R}^2)$; $\text{supp}(f)\subset B_a$, let $B_a^+=\big\{(r,x)\in\ B_a;\ r\geqslant 0\big\}$. From the relation (2.9),

$$\mathscr{U}(f)(r,x,t)=\int\int_{B_a^+}\tau_{(r,-x)}(p_t)(s,y)\ f(s,y)d\nu_\alpha(s,y),$$

consequently,

$$\Big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big|\leqslant\int\int_{B_a^+}\Big|\frac{\partial}{\partial t}\tau_{(r,-x)}(p_t)(s,y)\Big|\ \big|f(s,y)\big|d\nu_\alpha(s,y),$$

and from the relation (3.7), it follows that

$$\begin{aligned}\Big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big| &\leqslant \frac{2^{\alpha+\frac{3}{2}}\ (2\alpha+5)\Gamma(\alpha+2)}{\sqrt{\pi}}\ ||f||_{\infty,\nu_\alpha}\\ &\times \int\int_{B_a^+}\frac{d\nu_\alpha(s,y)}{\big(t^2+(r-s)^2+(x-y)^2\big)^{\alpha+2}}.\end{aligned}$$

However, for $r^2+x^2\geqslant 4a^2$ and $(s,y)\in\ B_a^+$, we have

$$||(r,x)-(s,y)||\geqslant\frac{1}{2}\ ||(r,x)||,$$

thus, for every $(r,x)\in\mathbb{R}^2$; $r^2+x^2\geqslant 4a^2$ and $t>0$,

$$\begin{aligned}\Big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big| &\leqslant \frac{2^{\alpha+\frac{3}{2}}\ (2\alpha+5)\Gamma(\alpha+2)}{\sqrt{\pi}}\ ||f||_{\infty,\nu_\alpha}\frac{\nu_\alpha(B_a^+)}{\big(t^2+\frac{1}{4}(r^2+x^2)\big)^{\alpha+2}}\\ &\leqslant \frac{2^{\alpha+\frac{3}{2}}\ 4^{\alpha+2}\ (2\alpha+5)\Gamma(\alpha+2)}{\sqrt{\pi}}\ \frac{\nu_\alpha(B_a^+)}{(r^2+x^2+t^2)^{\alpha+2}},\end{aligned}$$

and using the fact that

$$\nu_\alpha(B_a^+)=\frac{a^{2\alpha+3}}{(2\alpha+3)\ 2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{3}{2})},$$

we obtain

$$\Big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big|\leqslant\frac{2^{2\alpha+5}\Gamma(\alpha+2)(2\alpha+5)a^{2\alpha+3}}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})(2\alpha+3)}\ \frac{||f||_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+2}}.$$

ii) From the relations (2.20), (2.21), (2.24), (3.2), (3.6), we deduce that for every $f\in L^1(d\nu_\alpha)$ and every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$(3.8)\quad \mathscr{U}(f)(r,x,t)=\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda).$$

So,

$$\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,t)=\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)\mu\frac{\partial}{\partial r}\big(j_\alpha(r\mu)\big)e^{i\lambda x}d\nu_\alpha(\mu,\lambda).$$

Consequently, for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$;

$$\Big|\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,t)\Big|\leqslant\frac{||f||_{1,\nu_\alpha}}{2^{\alpha+\frac{1}{2}}\sqrt{\pi}\ \Gamma(\alpha+1)}\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\ \mu^{2\alpha+2}d\mu d\lambda.$$

By the change of variables $\mu=\dfrac{\rho}{t}\ \cos(\theta)$, $\lambda=\dfrac{\rho}{t}\ \sin(\theta)$, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,t)\Big| &\leqslant\frac{||f||_{1,\nu_\alpha}}{2^{\alpha+\frac{1}{2}}\sqrt{\pi}\ \Gamma(\alpha+1)}\frac{1}{t^{2\alpha+4}}\ 2\int_0^{\frac{\pi}{2}}\cos^{2\alpha+2}(\theta)d\theta\int_0^\infty e^{-\rho}\ \rho^{2\alpha+3}\ d\rho\\ &=\frac{\Gamma(\alpha+\frac{3}{2})\ \Gamma(\alpha+\frac{5}{2})\ 2^{\alpha+\frac{5}{2}}}{\sqrt{\pi}\ \Gamma(\alpha+1)}\ \frac{||f||_{1,\nu_\alpha}}{t^{2\alpha+4}}.\end{aligned}$$

iii) For every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,t)=\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)(i\lambda)e^{i\lambda x}d\nu_\alpha(\mu,\lambda).$$

Consequently, for every $(r,x,t)\in\ \mathbb{R}^2\times]0,+\infty[$;

$$\Big|\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,t)\Big| \leqslant \frac{||f||_{1,\nu_\alpha}}{2^{\alpha+\frac{1}{2}}\sqrt{\pi}\ \Gamma(\alpha+1)}\int_0^\infty\int_{\mathbb{R}} e^{-t\sqrt{\mu^2+\lambda^2}}\ |\lambda|\ \mu^{2\alpha+1}d\mu d\lambda.$$

Again by the change of variables $\mu=\dfrac{\rho}{t}\ \cos(\theta),\ \lambda=\dfrac{\rho}{t}\ \sin(\theta)$, we have

$$\begin{aligned}\Big|\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,t)\Big| &\leqslant \frac{||f||_{1,\nu_\alpha}}{2^{\alpha+\frac{1}{2}}\sqrt{\pi}\ \Gamma(\alpha+1)}\frac{1}{t^{2\alpha+4}}\\ &\times\ 2\int_0^{\frac{\pi}{2}}\cos^{2\alpha+1}(\theta)\sin(\theta)d\theta\int_0^\infty e^{-\rho}\ \rho^{2\alpha+3}\ d\rho\\ &= \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+\frac{5}{2})}{\pi}\ \frac{||f||_{1,\nu_\alpha}}{t^{2\alpha+4}}.\end{aligned}$$

Proposition 3.6. *Let $f\in\ \mathscr{D}_e(\mathbb{R}^2)$. The function*

$$\begin{aligned}v(f)(r,x,t) &= \Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2\\ &= \Big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,t)\Big)^2+\Big(\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,t)\Big)^2\\ &+ \Big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\Big)^2\end{aligned}$$

satisfies the following properties
i. For every $t>0$, the function $v(f)(.,.,t)$ belongs to the space $C_{0,e}(\mathbb{R}^2)$.
ii. For every $t>0$, the function $(r,x)\longmapsto(1+r^2+x^2)^2\ v(f)(r,x,t)$ belongs to the space $L^1(d\nu_\alpha)$.
iii. For every $t>0$, the function $\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,t)\big)$ belongs to $C^2(\mathbb{R}^2)$. Moreover, the functions $\ell_\alpha\Big(\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,t)\big)\Big)$ and $\dfrac{\partial}{\partial\mu}\Big(\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,t)\big)\Big)$ are bounded on $\mathbb{R}^2$.
vi. $\displaystyle\lim_{r^2+x^2+t^2\to+\infty} v(f)(r,x,t)=0.$

Where ℓ_α is the Hankel operator defined by

$$\ell_\alpha = \frac{\partial^2}{\partial\mu^2}+\frac{2\alpha+1}{\mu}\ \frac{\partial}{\partial\mu}=\frac{1}{\mu^{2\alpha+1}}\ \frac{\partial}{\partial\mu}\big(\mu^{2\alpha+1}\ \frac{\partial}{\partial\mu}\big). \tag{3.9}$$

Proof. Let $f\in\ \mathscr{D}_e(\mathbb{R}^2)$; $\operatorname{supp}(f)\subset B_a$; $a>0$.
i) The assertion follows from [2, Lemma 4.2] and Lemma 3.5 i).
ii) From the relation (3.8), we have

$$\mathscr{U}(f)(r,x,t)=\int_0^\infty\int_{\mathbb{R}} e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda),$$

wish implies that for every $(r,x,t)\in\ [0,+\infty[\times\mathbb{R}\times[0,+\infty[$,

$$|v(f)(r,x,t)|\leqslant||\mu\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}+||\lambda\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}+||\sqrt{\mu^2+\lambda^2}\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}. \tag{3.10}$$

Again, from [2, Lemma 4.2] and Lemma 3.5 i), for every $(r,x)\in\mathbb{R}^2;\ r^2+x^2\geqslant 4a^2$, we have

$$\Big|(1+r^2+x^2)^2\ v(f)(r,x,t)\Big|\leqslant\ C_{1,\alpha}\ \frac{(1+r^2+x^2)^2}{(r^2+x^2+t^2)^{2\alpha+4}}.$$

Thus, from the relation (3.10), we get

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}(1+r^2+x^2)^2\ \big|v(f)(r,x,t)\big|d\nu_\alpha(r,x)\\ &=\int\int_{B^+_{2a}}(1+r^2+x^2)^2\ \big|v(f)(r,x,t)\big|d\nu_\alpha(r,x)\\ &+\int\int_{(B^+_{2a})^c}(1+r^2+x^2)^2\ \big|v(f)(r,x,t)\big|d\nu_\alpha(r,x)\\ &\leqslant(1+4a^2)^2\ \Big[||\mu\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}+||\lambda\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}+||\sqrt{\mu^2+\lambda^2}\ \widetilde{\mathscr{F}}_\alpha(f)||^2_{1,\nu_\alpha}\Big]\ \nu_\alpha(B^+_{2a})\\ &+C_{1,\alpha}\int_0^\infty\int_{\mathbb{R}}\frac{(1+r^2+x^2)^2}{(r^2+x^2+t^2)^{2\alpha+4}}d\nu_\alpha(r,x)<+\infty.\end{aligned}$$

iii) The result follows from ii).
vi) For every $(r,x,t)\in\ \mathbb{R}^2\times]0,+\infty[;\ r^2+x^2\geqslant 4a^2$, we have

$$\big|v(f)(r,x,t)\big|\ \leqslant\ \frac{C_{1,\alpha}}{(r^2+x^2+t^2)^{2\alpha+4}},\tag{3.11}$$

and for every $(r,x,t)\in\ \mathbb{R}^2\times]0,+\infty[$,

$$\big|v(f)(r,x,t)\big|\ \leqslant\ \frac{C_{3,\alpha}}{t^{4\alpha+8}}.\tag{3.12}$$

The relations (3.11) and (3.12) involve that

$$\lim_{r^2+x^2+t^2\to+\infty}v(f)(r,x,t)=0.$$

Lemme 3.7. *Let* $f\in\ \mathscr{D}_e(\mathbb{R}^2)$ *and* $v(f)(r,x,t)=\Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2$, *then, for every* $s>0$,

$$\lim_{r^2+x^2+t^2\to+\infty}\mathscr{U}\Big(v(f)(.,.,s)\Big)(r,x,t)=0.$$

Proof. Let $f\in\ \mathscr{D}_e(\mathbb{R}^2)$. From the relation (3.8) and Proposition 3.6, it follows that for every $s>0$, and $(r,x,t)\in\ \mathbb{R}^2\times]0,+\infty[$,

$$\begin{aligned}&\mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\\ &=\textstyle\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda).\end{aligned}\tag{3.13}$$

Thus, by Fubini's theorem, we get

$$\begin{aligned}
&r^2\ \mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\\
&=\int_{\mathbb{R}} e^{i\lambda x}\Big(\int_0^\infty e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\ r^2\ j_\alpha(r\mu)d\tau_\alpha(\mu)\Big)\ \frac{d\lambda}{\sqrt{2\pi}},
\end{aligned}$$

where

$$d\tau_\alpha(\mu)=\frac{\mu^{2\alpha+1}}{2^\alpha\ \Gamma(\alpha+1)}\ d\mu.$$

Using the fact that

$$(-\ell_\alpha)\big(j_\alpha(r\cdot)\big)(\mu)\quad=\quad r^2\ j_\alpha(r\mu) \tag{3.14}$$

where ℓ_α is given by the relation (3.9), we obtain

$$\begin{aligned}
&r^2\ \mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\\
&=\int_{\mathbb{R}} e^{i\lambda x}\int_0^\infty e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)(-\ell_\alpha)\big(j_\alpha(r\cdot)\big)(\mu)d\tau_\alpha(\mu)\ \frac{d\lambda}{\sqrt{2\pi}}\\
&=\int_{\mathbb{R}} e^{i\lambda x}\Big(\int_0^\infty (-\ell_\alpha)\Big[e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\Big]\ j_\alpha(r\mu)d\tau_\alpha(\mu)\Big)\ \frac{d\lambda}{\sqrt{2\pi}}.
\end{aligned}$$

. By computation,

$$\begin{aligned}
&-\ell_\alpha\Big(e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\Big)\\
&\qquad=\Big\{\Big[\frac{t\lambda^2}{(\mu^2+\lambda^2)^{\frac{3}{2}}}-\frac{t^2\mu^2}{\mu^2+\lambda^2}+\frac{(2\alpha+1)t}{\sqrt{\mu^2+\lambda^2}}\Big]\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\\
&\qquad-\ell_\alpha\Big(\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\Big)\\
&\qquad+\frac{2t\mu}{\sqrt{\mu^2+\lambda^2}}\ \frac{\partial}{\partial\mu}\Big(\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)(\mu,\lambda)\Big)\Big\}e^{-t\sqrt{\mu^2+\lambda^2}}.
\end{aligned}$$

Let

$$\begin{aligned}
M_1(\alpha,s)=\max\Big\{&||\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)||_{\infty,\nu_\alpha},\\
&||\ell_\alpha\Big(\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)\Big)||_{\infty,\nu_\alpha},\ ||\frac{\partial}{\partial\mu}\widetilde{\mathscr{F}}_\alpha\big(v(f)(.,.,s)\big)||_{\infty,\nu_\alpha}\Big\},
\end{aligned}$$

then,

$$\begin{aligned}
\Big|r^2\ \mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\Big|\leqslant M_1(\alpha,s)\int_0^\infty\int_{\mathbb{R}}\Big[&\frac{t\lambda^2}{(\mu^2+\lambda^2)^{\frac{3}{2}}}+\frac{t^2\mu^2}{\mu^2+\lambda^2}+\frac{(2\alpha+1)t}{\sqrt{\mu^2+\lambda^2}}\\
&+\quad 1+\frac{2t\mu}{\sqrt{\mu^2+\lambda^2}}\Big]e^{-t\sqrt{\mu^2+\lambda^2}}d\nu_\alpha(\mu,\lambda),
\end{aligned}$$

and by the change of variables $\mu = \dfrac{\rho}{t}\cos(\theta)$, $\lambda = \dfrac{\rho}{t}\sin(\theta)$, we deduce that there exists $C_1(\alpha, s)$ such that for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$,

$$\left| r^2 \,\mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\right| \leqslant C_1(\alpha,s)\,\frac{1+t+t^2}{t^{2\alpha+3}}. \tag{3.15}$$

As the same way, there exist $c_2(\alpha, s)$ and $c_3(\alpha, s)$ such that for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$,

$$\left| x^2 \,\mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\right| \leqslant C_2(\alpha,s)\,\frac{1+t+t^2}{t^{2\alpha+3}}, \tag{3.16}$$

$$\left| t^2 \,\mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\right| \leqslant C_3(\alpha,s)\,\frac{1+t+t^2}{t^{2\alpha+3}}. \tag{3.17}$$

Combining the relations (3.15), (3.16) and (3.17), we deduce that there exits a positive constant $C(\alpha, s)$ such that for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$,

$$\left| \mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)\right| \leqslant C(\alpha,s)\,\frac{1+t+t^2}{t^{2\alpha+3}}\,\frac{1}{r^2+x^2+t^2}. \tag{3.18}$$

From Proposition 3.6, for every $s > 0$, the function $v(f)(.,.,s)$ belongs to $C_{0,e}(\mathbb{R}^2)$. Since the family $(p_t)_{t>0}$ is an approximation of the identity in $C_{0,e}(\mathbb{R}^2)$ (2.15), we deduce that

$$\lim_{t\to 0^+} \mathscr{U}\Big(v(f)(.,.,s)\Big)(.,.,t) = v(f)(.,.,s) \text{ in } C_{0,e}(\mathbb{R}^2).$$

Consequently,

$$\lim_{\substack{r^2+x^2\to+\infty \\ t\to 0^+}} \mathscr{U}\Big(v(f)(.,.,s)\Big)(r,x,t) = 0. \tag{3.19}$$

On the other hand, from the relation (3.18), we deduce that for every $a > 0$,

$$\lim_{\substack{r^2+x^2+t^2\to+\infty \\ t\geqslant a}} \mathscr{U}\Big(v(f)(.,.,s)\Big)(r,x,t) = 0. \tag{3.20}$$

The relations (3.19) and (3.20) show that for every $s > 0$,

$$\lim_{r^2+x^2+t^2\to+\infty} \mathscr{U}\Big(v(f)(.,.,s)\Big)(r,x,t) = 0.$$

Lemma 3.8. *Let f be a bounded continuous function on $\mathbb{R}^2$, even with respect to the first variable. Then, the function*

$$\mathscr{U}(f)(r,x,t) = p_t * f(r,x)$$

is continuous on $\mathbb{R}^2 \times [0,+\infty[$, *even with respect to the first variable and we have*

$$\forall (r,x) \in \mathbb{R}^2, \ \mathscr{U}(f)(r,x,0) = f(r,x).$$

Proof. Let f be a bounded continuous function on $\mathbb{R}^2$, even with respect to the first variable. From the relations (2.9) and (3.2), we get

$$\begin{aligned}\mathscr{U}(f)(r,x,t) &= \frac{2(\alpha+1)}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{t}{(t^2+s^2+y^2)^{\alpha+2}}\ \tau_{(r,-x)}(\check{f})(s,y)\ s^{2\alpha+1}\ dsdy \\ &= \frac{2(\alpha+1)}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{\tau_{(r,-x)}(\check{f})(tu,tv)}{(1+u^2+v^2)^{\alpha+2}}\ u^{2\alpha+1}\ dudv.\end{aligned}$$

Since, for all $(r,x,t) \in \mathbb{R}^2 \times [0,+\infty[$, $(u,v) \in \mathbb{R}^2$, we have

$$\begin{aligned}\left|\frac{\tau_{(r,-x)}(\check{f})(tu,tv)}{(1+u^2+v^2)^{\alpha+2}}\ u^{2\alpha+1}\right| &\leqslant ||\tau_{(r,-x)}(\check{f})||_{\infty,\nu_\alpha}\ \frac{u^{2\alpha+1}}{(1+u^2+v^2)^{\alpha+2}} \\ &\leqslant ||f||_{\infty,\nu_\alpha}\ \frac{u^{2\alpha+1}}{(1+u^2+v^2)^{\alpha+2}},\end{aligned}$$

and since the function $(u,v) \longmapsto \dfrac{u^{2\alpha+1}}{(1+u^2+v^2)^{\alpha+2}}$ is integrable on $[0,+\infty[\times\mathbb{R}$, we deduce that the function $\mathscr{U}(f)$ is continuous on $\mathbb{R}^2 \times [0,+\infty[$.
Moreover, for every $(r,x) \in \mathbb{R}^2$,

$$\begin{aligned}\mathscr{U}(f)(r,x,0) &= \frac{2(\alpha+1)}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{\tau_{(r,-x)}(\check{f})(0,0)}{(1+u^2+v^2)^{\alpha+2}}\ u^{2\alpha+1}\ dudv \\ &= f(r,x)\ \frac{2(\alpha+1)}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{u^{2\alpha+1}}{(1+u^2+v^2)^{\alpha+2}}\ dudv \\ &= f(r,x).\end{aligned}$$

4. Principle of the Maximum

In this section, we will establish a principle of the maximum for the singular partial differential operator

$$\Delta_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}. \tag{4.1}$$

We use this principle to prove that the Poisson semigroup satisfies the inequality

$$\left|\nabla\big(\mathscr{U}(f)\big)(r,x,2t)\right|^2 \leqslant \mathscr{P}^t\Big(\left|\nabla\big(\mathscr{U}(f)\big)(.,.,t)\right|^2\Big)(r,x).$$

This inequality plays an important role in the next section.

Theorem 4.1.(Hopf) *Let*

$$L = \sum_{i,j=1}^{n} a_{i,j}(x) \ \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \ \frac{\partial}{\partial x_j}$$

be an uniformly elliptic operator on a bounded connected set $\Omega \subset \mathbb{R}^n$ *such that the functions* $a_{i,j}, b_{i,j}$ *are continuous on* $\overline{\Omega}$.
Let u *be a function in* $C^2(\Omega) \cap C^0(\overline{\Omega})$ *such that for every* $x \in \ \Omega,\ Lu(x) \geqslant 0$. *If there exists* $x_0 \in \ \Omega$ *such that* $\sup_{x \in \ \overline{\Omega}} u(x) = u(x_0)$. *Then,* u *is constant.*

Proposition 4.2. *Let* a_0, a_1, T *be positive real numbers and* $\Omega =]-a_0, a_0[\times]-a_1, a_1[\times]0, T[$. *Let* $u \in \ C^2(\Omega) \cap C^0(\overline{\Omega})$ *such that*
i. $\forall (r,x,t) \in \ \Omega;\ u(r,x,t) = u(-r,x,t)$.
ii. $\forall (r,x,t) \in \ \Omega;\ \Delta_\alpha u(r,x,t) \geqslant 0$.
If there exists $(r_0, x_0, t_0) \in \ \Omega;\ r_0 \neq 0$ *such that* $\sup_{(r,x,t) \in \ \overline{\Omega}} u(r,x,t) = u(r_0, x_0, t_0)$.
Then, u *is constant.*

Proof. Let u be a function satisfying the hypothesis. From i) we can assume that $r_0 > 0$.
Let $0 < \varepsilon < r_0$ and $\Omega_\varepsilon =]\varepsilon, a_0[\times]-a_1, a_1[\times]0, T[$. Then, it is clear that the operator Δ_α is uniformly elliptic on Ω_ε and we have

$$\sup_{\overline{\Omega_\varepsilon}} u(r,x,t) = \sup_{\overline{\Omega}} u(r,x,t) = u(r_0, x_0, t_0).$$

Since $(r_0, x_0, t_0) \in \ \Omega_\varepsilon$, then, from Theorem 4.1, we deduce that

$$\forall (r,x,t) \in \ \overline{\Omega_\varepsilon},\ u(r,x,t) = u(r_0, x_0, t_0).$$

This means that for every $\varepsilon > 0$ and $(r,x,t) \in]\varepsilon, a_0[\times]-a_1, a_1[\times]0, T[$,

$$u(r,x,t) = u(r_0, x_0, t_0).$$

On the other hand, the function u is continuous on $\overline{\Omega}$. Then,

$$\forall (r,x) \in \ ]-a_1, a_1[\times]0, T[, \quad u(0,x,t) = \lim_{r \longrightarrow 0^+} u(r,x,t) = u(r_0, x_0, t_0).$$

Hence, $\forall (r,x,t) \in \ [0, a_0[\times]-a_1, a_1[\times]0, T[,\ u(r,x,t) = u(r_0, x_0, t_0)$.
From the hypothesis i), we conclude that

$$\forall (r,x,t) \in \ \Omega,\ u(r,x,t) = u(|r|,x,t) = u(r_0, x_0, t_0).$$

Proposition 4.3. *Let* u *be a function satisfying the hypothesis of Proposition 4.2. If there exits* $(x_0, t_0) \in \ ]-a_1, a_1[\times]0, T[$ *such that*

$$\sup_{\overline{\Omega}} u(r,x,t) = u(0, x_0, t_0),$$

then u is constant on Ω.

Proof. Let $M_1 = \sup_{\overline{\Omega}} u(r,x,t) = u(0,x_0,t_0)$. We shall prove that there exists $(r_1,x_1,t_1) \in \Omega$; $r_1 \neq 0$, such that

$$u(r_1,x_1,t_1) = u(0,x_0,t_0) = M_1.$$

In fact, suppose that we have

$$\forall (r,x,t) \in \Omega;\ r \neq 0,\ u(r,x,t) \quad < \quad M_1. \tag{4.2}$$

Let us define the function ψ and the set K by

$$\psi(r,x,t) = e^{2r^2-(x-x_0)^2-(t-t_0)^2} - 1, \qquad K = \{(r,x,t) \in \Omega;\ \psi(r,x,t) \geqslant 0\}.$$

Since Ω is an open set, there exists $\varepsilon > 0$ such that

$$B'(\varepsilon) = \{(r,x,t) \in \mathbb{R}^3;\ r^2 + (x-x_0)^2 + (t-t_0)^2 \leqslant \varepsilon^2\} \subset \Omega.$$

The set $K \cap \partial B'(\varepsilon)$ is a compact one. Then there exists $(r_2,x_2,t_2) \in K \cap \partial B'(\varepsilon)$ such that

$$M_2 = \sup_{K\cap\partial B'(\varepsilon)} u(r,x,t) = u(r_2,x_2,t_2).$$

Since

$$r_2^2 + (x_2-x_0)^2 + (t_2-t_0)^2 = \varepsilon^2 \text{ and } \psi(r_2,x_2,t_2) = e^{2r_2^2-(x_2-x_0)^2-(t_2-t_0)^2} - 1 \geqslant 0,$$

then, $r_2 \neq 0$. Thus, by the assertion (4.2), $M_2 < M_1$.
On the other hand, let $M_3 = \sup_{(r,x,t)\in\ \partial B'\cap K} \psi(r,x,t)$, we have $M_3 \geqslant \psi(\varepsilon,x_0,t_0) = e^{2\varepsilon^2} - 1 > 0$.
Let

$$\delta \in \left]0, \frac{M_1 - M_2}{M_3}\right[\text{ and } \phi(r,x,t) = u(r,x,t) + \delta\psi(r,x,t).$$

By computation, for every $(r,x,t) \in \Omega$,

$$\Delta_\alpha \psi(r,x,t) = 4\Big[(2\alpha+1) + 4r^2 + (x-x_0)^2 + (t-t_0)^2\Big] e^{2r^2-(x-x_0)^2-(t-t_0)^2}.$$

Since $\Delta_\alpha u(r,x,t) \geqslant 0$ on Ω, we deduce that

$$\forall (r,x,t) \in \Omega,\ \Delta_\alpha \phi(r,x,t) \quad \geqslant \quad 4\delta(2\alpha+1)e^{2r^2-(x-x_0)^2-(t-t_0)^2} > 0. \tag{4.3}$$

Now,
.$\forall (r,x,t) \in \partial B'(\varepsilon) \cap K^c$; $\psi(r,x,t) < 0$; and then, $\phi(r,x,t) < M_1$.
.$\forall (r,x,t) \in \partial B'(\varepsilon) \cap K$, $\phi(r,x,t) \leqslant M_2 + \delta M_3 < M_1$, which shows that

$$\forall (r,x,t) \in \partial B'(\varepsilon);\ \phi(r,x,t) \quad < \quad M_1. \tag{4.4}$$

Let $(r_3, x_3, t_3) \in B'(\varepsilon)$ such that

$$\sup_{(r,x,t)\in B'(\varepsilon)} \phi(r, x, t) = \phi(r_3, x_3, t_3).$$

We have

$$\phi(r_3, x_3, t_3) \geqslant \phi(0, x_0, t_0) = M_1,$$

and from the relation (4.4), we deduce that the function φ attains its maximum in $(r_3, x_3, t_3) \in B(\varepsilon) = \{(r, x, t) \in \mathbb{R}^3;\ r^2 + (x - x_0)^2 + (t - t_0)^2 < \varepsilon^2\}$, but

- For $r_3 \neq 0$,

$$\Delta_\alpha \phi(r_3, x_3, t_3) = \frac{\partial^2 \phi}{\partial r^2}(r_3, x_3, t_3) + \frac{\partial^2 \phi}{\partial x^2}(r_3, x_3, t_3) + \frac{\partial^2 \phi}{\partial t^2}(r_3, x_3, t_3) \leqslant 0. \tag{4.5}$$

-

$$\begin{aligned} \Delta_\alpha \phi(0, x_3, t_3) &= (2\alpha + 2)\frac{\partial^2 \phi}{\partial r^2}(0, x_3, t_3) + \frac{\partial^2 \phi}{\partial x^2}(0, x_3, t_3) \\ &+ \frac{\partial^2 \phi}{\partial t^2}(0, x_3, t_3) \leqslant 0. \end{aligned} \tag{4.6}$$

The relations (4.5) and (4.6) contradict the relation (4.3) and show that the assertion (4.2) can not be true, that is there exists $(r_1, x_1, t_1) \in \Omega;\ r_1 \neq 0$ such that

$$\sup_{\overline{\Omega}} u(r, x, t) = u(r_1, x_1, t_1) = M_1,$$

and the proof is complete by applying Proposition 4.2.

Theorem 4.4. *Let a_0, a_1, T be positive real numbers and $\Omega =]-a_0, a_0[\times]-a_1, a_1[\times]0, T[$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that*
i. $\forall (r, x, t) \in \Omega;\ u(r, x, t) = u(-r, x, t)$.
ii. $\forall (r, x, t) \in \Omega;\ \Delta_\alpha u(r, x, t) \geqslant 0$.
Then, if u attains its maximum in Ω, u is constant.

Proof. The proof follows immediately from Proposition 4.2 and Proposition 4.3.
The Theorem 4.4 implies the following interesting result.

Theorem 4.5. *Let $u \in C^2(\mathbb{R}^2\times]0, +\infty[) \cap C^0(\mathbb{R}^2 \times [0, +\infty[)$ such that*
i. $\forall (r, x, t) \in \mathbb{R}^2 \times [0, +\infty[;\ u(r, x, t) = u(-r, x, t)$.
ii. $\forall (r, x) \in \mathbb{R}^2,\ u(r, x, 0) \geqslant 0$.
iii. $\lim\limits_{r^2+x^2+t^2\longmapsto+\infty} u(r, x, t) = 0$.
vi. $\forall (r, x, t) \in \mathbb{R}^2 \times [0, +\infty[;\ \Delta_\alpha u(r, x, t) \leqslant 0$. Then u is non negative.

Proof. Suppose that there exists $(r_0, x_0, t_0) \in \mathbb{R}^2 \times [0, +\infty[;\ u(r_0, x_0, t_0) < 0$. Then, u attains its minimum in $(r_1, x_1, t_1) \in \mathbb{R}^2 \times [0, +\infty[$.
Since $u(r_1, x_1, t_1) \leqslant u(r_0, x_0, t_0) < 0$, and using the hypothesis ii), we deduce that $t_1 > 0$.

Let a_0, a_1, T be positive real numbers such that $(r_1, x_1, t_1) \in \Omega =]-a_0, a_0[\times]-a_1, a_1[\times]0, T[$, let $v = -u$. Then, v satisfies the hypothesis of Theorem 4.4 on $\Omega =]-a_0, a_0[\times]-a_1, a_1[\times]0, T[$ and attains its maximum in $(r_1, x_1, t_1) \in \Omega$.
This implies that

$$\forall (r, x, t) \in \Omega; \; u(r, x, t) = u(r_1, x_1, t_1) < 0.$$

In particular, for every $T > t_1$,

$$u(r_1, x_1, T) = u(r_1, x_1, t_1) < 0.$$

This contradicts the fact that

$$\lim_{T \to +\infty} u(r_1, x_1, T) = 0.$$

Theorem 4.6. *For every $f \in \mathscr{D}_e(\mathbb{R}^2)$, we have*

$$\forall (r, x, t) \in \mathbb{R}^2\times]0, +\infty[, \; \left|\nabla\big(\mathscr{U}(f)\big)(r, x, 2t)\right|^2 \leqslant \mathscr{P}^t\Big(\left|\nabla\big(\mathscr{U}(f)\big)(.,.,t)\right|^2\Big)(r, x).$$

Proof. Let $f \in \mathscr{D}_e(\mathbb{R}^2)$. As in Proposition 3.6, we put

$$v(f)(r, x, t) = \left|\nabla\big(\mathscr{U}(f)\big)(r, x, t)\right|^2,$$

and for every $s > 0$,

$$h(r, x, t) = h_s(r, x, t) = \mathscr{U}\big(v(f)(.,.,s)\big)(r, x, t) - v(f)(r, x, s + t).$$

Let us prove that the function h satisfies the hypothesis of Theorem 4.5.
. It is clear that for every $f \in \mathscr{D}_e(\mathbb{R}^2)$, the function $(r, x, t) \longmapsto \mathscr{U}(f)(r, x, t)$ is infinitely differentiable on $\mathbb{R}^2 \times [0, +\infty[$, consequently, for every $s > 0$, the function $(r, x, t) \longmapsto v(f)(r, x, s+t)$ belongs to $C^2(\mathbb{R}^2\times]0, +\infty[)$ and is even with respect to the first variable. On the other hand, from Proposition 3.6 i), for every $s > 0$, the function $v(f)(.,.,s)$ belongs to $C_{0,e}(\mathbb{R}^2)$. Applying Lemma 3.8, it follows that the function $(r, x, t) \longmapsto \mathscr{U}\big(v(f)(.,.,s)\big)(r, x, t)$ is continuous on $\mathbb{R}^2 \times [0, +\infty[$.
Now, from the relation (3.13), it folows that $\mathscr{U}\big(v(f)(.,.,s)\big)$ is infinitely differentiable on $\mathbb{R}^2\times]0, +\infty[$.

We conclude that the function $h = h_s$ belongs to $C^2(\mathbb{R}^2\times]0, +\infty[) \cap C^0(\mathbb{R}^2 \times [0, +\infty[)$ and is even with respect to the first variable.
. Applying again Lemma 3.8, we deduce that for every $(r, x) \in \mathbb{R}^2$,

$$\begin{aligned} h(r, x, 0) &= \mathscr{U}\big(v(f)(.,.,s)\big)(r, x, 0) - v(f)(r, x, s) \\ &= v(f)(r, x, s) - v(f)(r, x, s) \\ &= 0. \end{aligned}$$

. From Proposition 3.6 and Lemma 3.7, we have

$$\lim_{r^2+x^2+t^2\longmapsto+\infty} v(f)(r,x,t)=0 \text{ and } \lim_{r^2+x^2+t^2\longmapsto+\infty} \mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)=0.$$

This involves that

$$\lim_{r^2+x^2+t^2\longmapsto+\infty} h(r,x,t)=0.$$

. Using the relations (3.9), (3.13), (3.14), we deduce that for every (r,x,t) $\in\ \mathbb{R}^2\times]0,+\infty[$,

$$\Delta_\alpha\Big(\mathscr{U}\left(v(f)(.,.,s)\right)\Big)(r,x,t)=0.$$

So,

$$\begin{aligned}\Delta_\alpha h(r,x,t) &= -\Delta_\alpha\big(v(f)\big)(r,x,s+t)\\ &= -\Delta_\alpha\big(v(f)(.,.,s+.)\big)(r,x,t).\end{aligned}$$

But,

$$\begin{aligned}\Delta_\alpha\big(v(f)\big)(r,x,s+t) &= \Delta_\alpha\Big(\big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)^2\Big)\\ &+ \Delta_\alpha\Big(\big(\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)^2\Big)\\ &+ \Delta_\alpha\Big(\big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)^2\Big),\end{aligned}$$

on the other hand, for all $\varphi,\psi\in\ C^2(\mathbb{R}^2\times]0,+\infty[)$,

$$\Delta_\alpha(\varphi\psi) = \varphi\Delta_\alpha(\psi)+\psi\Delta_\alpha(\varphi)+2\big(\frac{\partial\varphi}{\partial x}\ \frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial r}\ \frac{\partial\psi}{\partial r}+\frac{\partial\varphi}{\partial t}\ \frac{\partial\psi}{\partial t}\big). \tag{4.7}$$

Using the fact that $\Delta_\alpha\big(\mathscr{U}(f)\big)=0,\ \Delta_\alpha\Big(\frac{\partial}{\partial x}\big(\mathscr{U}(f)\big)\Big)=\frac{\partial}{\partial x}\Big(\Delta_\alpha\big(\mathscr{U}(f)\big)\Big)=0$ and $\Delta_\alpha\Big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)\Big)=\frac{\partial}{\partial t}\Big(\Delta_\alpha\big(\mathscr{U}(f)\big)\Big)=0$, we deduce that

$$\begin{aligned}&\Delta_\alpha\big(v(f)\big)(r,x,s+t)\\ &=2\Big[\Big(\frac{\partial^2}{\partial r^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2+\Big(\frac{\partial^2}{\partial x^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2\\ &+\Big(\frac{\partial^2}{\partial t^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2\Big]+4\Big[\Big(\frac{\partial}{\partial x}\big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2\\ &+\Big(\frac{\partial}{\partial x}\big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2+\Big(\frac{\partial}{\partial r}\big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2\Big]\\ &+2\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\ \Delta_\alpha\Big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)\Big)(r,x,s+t),\end{aligned}$$

however,

$$\begin{aligned}&\Delta_\alpha\Big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)\Big)(r,x,s+t)\\ &= \frac{\partial}{\partial r}\Big(\Delta_\alpha\big(\mathscr{U}(f)\big)\Big)(r,x,s+t)+\frac{2\alpha+1}{r^2}\ \frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\\ &= \frac{2\alpha+1}{r^2}\ \frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t).\end{aligned}$$

Then,

$$\begin{aligned}&\Delta_\alpha\big(v(f)\big)(r,x,s+t)=\frac{4\alpha+2}{r^2}\Big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2\\ &+2\Big[\Big(\frac{\partial^2}{\partial r^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2+\Big(\frac{\partial^2}{\partial x^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2\\ &+\Big(\frac{\partial^2}{\partial t^2}\big(\mathscr{U}(f)\big)(r,x,s+t)\Big)^2\Big]+4\Big[\Big(\frac{\partial}{\partial x}\big(\frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2\\ &+\Big(\frac{\partial}{\partial x}\big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2+\Big(\frac{\partial}{\partial r}\big(\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,s+t)\big)\Big)^2\Big]\\ &\geqslant 0,\end{aligned}$$

wish means that

$$\Delta_\alpha h(r,x,t)=-\Delta_\alpha v(f)(r,x,s+t)\leqslant 0.$$

Hence, the hypothesis of Theorem 4.5 are satisfied by the function $h=h_s$. Consequently, for every $(r,x,t)\in\ \mathbb{R}^2\times]0,+\infty[$ and every $s>0$,

$$\mathscr{U}\big(v(f)(.,.,s)\big)(r,x,t)-v(f)(r,x,s+t)\geqslant 0.$$

That is

$$\mathscr{P}^t\Big(\big|\nabla\big(\mathscr{U}(f)\big)(.,.,s)\big|^2\Big)(r,x)\geqslant\big|\nabla\big(\mathscr{U}(f)\big)(r,x,s+t)\big|^2.$$

In particular, for $s=t$,

$$\big|\nabla\big(\mathscr{U}(f)\big)(r,x,2t)\big|^2\leqslant\mathscr{P}^t\Big(\big|\nabla\big(\mathscr{U}(f)\big)(.,.,t)\big|^2\Big)(r,x).$$

5. L^p-Boundedness of the Littlewood-Paley g-Function

This section contains the main result of this work.
Namely, using the results of the precedent sections, in particular, the principle of maximum for the operator Δ_α.We will prove that for every $p\in]1,+\infty[$, there exists a positive constant C_p such that for every $f\in L^p(d\nu_\alpha)$,

$$\frac{1}{C_p}||f||_{p,\nu_\alpha}\leqslant||g(f)||_{p,\nu_\alpha}\leqslant C_p||f||_{p,\nu_\alpha},$$

where $g(f)$ is the Littlewood-Paley g-function connected with the Riemann-Liouville operator defined by

Definition 5.1. The Littlewood-Paley g-function associated with the Riemann-Liouville operator is defined for $f \in \mathscr{D}_e(\mathbb{R}^2)$ by

$$g(f)(r,x) = \Big(\int_0^{+\infty} \Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2 tdt\Big)^{\frac{1}{2}}.$$

We start this section by some intermediary results.

Lemma 5.2. *For all nonnegative functions $f, h \in \mathscr{D}_e(\mathbb{R}^2)$, we have*

$$\int_o^{+\infty}\int_{\mathbb{R}} \Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x) \leqslant 4\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} t\Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2 \mathscr{U}(h)(r,x,t)d\nu_\alpha(r,x)dt.$$

Proof. By Fubini-Tonnelli theorem's, we have

$$\int_o^{+\infty}\int_{\mathbb{R}} \Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x) = \int_0^{+\infty}\Big[\int_0^{+\infty}\int_{\mathbb{R}} \Big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\Big|^2 h(r,x)d\nu_\alpha(r,x)\Big]tdt.$$

Applying Theorem 4.6, we obtain

$$\begin{aligned}\int_o^{+\infty}\int_{\mathbb{R}} &\Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x)\\ &\leqslant \int_0^{+\infty} t\Big[\int_0^{+\infty}\int_{\mathbb{R}} h(r,x)\mathscr{P}^{\frac{t}{2}}\Big(\big|\nabla\big(\mathscr{U}(f)\big)(.,.,\frac{t}{2})\big|^2\Big)(r,x)d\nu_\alpha(r,x)\Big]dt\\ &= 4\int_0^{+\infty} s\Big[\int_0^{+\infty}\int_{\mathbb{R}} h(r,x)\mathscr{P}^{s}\Big(\big|\nabla\big(\mathscr{U}(f)\big)(.,.,s)\big|^2\Big)(r,x)d\nu_\alpha(r,x)\Big]ds.\end{aligned}$$

However, for all $\varphi,\psi \in L^2(d\nu_\alpha)$,

$$\int_o^{+\infty}\int_{\mathbb{R}} \varphi(r,x)\mathscr{P}^t(\psi)(r,x)d\nu_\alpha(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \mathscr{P}^t(\varphi)(r,x)\psi(r,x)d\nu_\alpha(r,x).$$

Then,

$$\begin{aligned}\int_o^{+\infty}\int_{\mathbb{R}} &\Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x)\\ &\leqslant 4\int_0^{+\infty} s\Big(\int_0^{+\infty}\int_{\mathbb{R}} \mathscr{U}(h)(r,x,s)\ \big|\nabla\big(\mathscr{U}(f)\big)(r,x,s)\big|^2\ d\nu_\alpha(r,x)\Big)\ ds.\end{aligned}$$

Lemma 5.3. *For all nonnegative functions $f,\ h \in \mathscr{D}_e(\mathbb{R}^2)$, we have*

$$\int_0^\infty\int_0^\infty\int_{\mathbb{R}} \Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ tdt\ d\nu_\alpha(r,x) = \int_0^\infty\int_{\mathbb{R}} f^2(r,x)\ h(r,x)d\nu_\alpha(r,x).$$

Proof. Using the relation (4.7) and the fact that $\Delta_\alpha\big(\mathscr{U}(f)\big) = \Delta_\alpha\big(\mathscr{U}(h)\big) = 0$, we get

$$\Delta_\alpha\big(\mathscr{U}^2(f)\big) \quad = \quad 2\ \big|\nabla\mathscr{U}(f)\big|^2, \tag{5.1}$$

so,

$$\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big] = 2\mathscr{U}(h)\ \big|\nabla\big(\mathscr{U}(f)\big)\big|^2 + 4\mathscr{U}(f)\langle\nabla\big(\mathscr{U}(f)\big),\nabla\big(\mathscr{U}(h)\big)\rangle.$$

Then,

$$\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big] \leqslant 2\mathscr{U}(h)\big|\nabla\big(\mathscr{U}(f)\big)\big|^2 + 4\mathscr{U}(f)\ \big|\nabla\big(\mathscr{U}(f)\big)\big|\ \big|\nabla\big(\mathscr{U}(h)\big)\big|.$$

Using the fact that for every $(r,x,t)\in\mathbb{R}^2\times[0,+\infty[,\big|\mathscr{U}(f)(r,x,t)\big| \leqslant ||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha}$ and $\big|\mathscr{U}(h)(r,x,t)\big| \leqslant ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}$, we claim that

$$\Delta_\alpha\Big[\mathscr{U}^2(f)\mathscr{U}(h)\Big] \leqslant 2\big(\ ||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha} + ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}\big)\Big[\big|\nabla\big(\mathscr{U}(f)\big)\big|^2 + \big|\nabla\big(\mathscr{U}(h)\big)\big|^2\Big].$$

Consequently,

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big|\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\Big|\ t\ dt\ d\nu_\alpha(r,x)\\ &\qquad\leqslant 2\big(\ ||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha} + ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}\big)\\ &\qquad\times\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big[\big|\nabla\big(\mathscr{U}(f)(r,x,t)\big)\big|^2 + \big|\nabla\big(\mathscr{U}(h)(r,x,t)\big)\big|^2\Big]\ t\ dt\ d\nu_\alpha(r,x),\end{aligned}$$

and by the relation (5.1), we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big|\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\Big|\ t\ dt\ d\nu_\alpha(r,x)\\ &\leqslant \big(||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha} + ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}\big)\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big|\Delta_\alpha\big(\mathscr{U}^2(f)\big)(r,x,t)\Big|\ t\ dt\ d\nu_\alpha(r,x)\Big]\\ &+ \big(||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha} + ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}\big)\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big|\Delta_\alpha\big(\mathscr{U}^2(h)\big)(r,x,t)\Big|\ t\ dt\ d\nu_\alpha(r,x)\Big].\end{aligned}$$

Applying [2, Theorem 4.3], we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\Big|\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\Big|\ t\ dt\ d\nu_\alpha(r,x)\\ &\leqslant \big(||\widetilde{\mathscr{F}_\alpha}(f)||_{1,\nu_\alpha} + ||\widetilde{\mathscr{F}_\alpha}(h)||_{1,\nu_\alpha}\big)\big(||f||^2_{2,\nu_\alpha} + ||h||^2_{2,\nu_\alpha}\big) < +\infty.\end{aligned}$$

Thus, we can write

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\int_{\mathbb{R}}\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x)\\ &= \lim_{A\longmapsto+\infty}\int_0^{A}\int_0^{A}\int_{-A}^{A}\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x),\end{aligned}\tag{5.2}$$

on the other hand,

$$\int_0^{A}\int_0^{A}\int_{-A}^{A}\Delta_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x) = I_1(A)+I_2(A)+I_3(A),\tag{5.3}$$

where

$$\begin{aligned}I_1(A) &= \int_0^{A}\int_0^{A}\int_{-A}^{A}\ell_\alpha\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x),\\ I_2(A) &= \int_0^{A}\int_0^{A}\int_{-A}^{A}\frac{\partial^2}{\partial x^2}\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x),\\ I_3(A) &= \int_0^{A}\int_0^{A}\int_{-A}^{A}\frac{\partial^2}{\partial t^2}\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](r,x,t)\ t\ dt\ d\nu_\alpha(r,x),\end{aligned}$$

and ℓ_α is given by the relation (3.9).

. By Fubini's theorem,

$$I_1(A) = \frac{1}{2^\alpha\ \Gamma(\alpha+1)\sqrt{2\pi}}\ A^{2\alpha+1}\int_0^{A}\int_{-A}^{A}\frac{\partial}{\partial r}\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](A,x,t)\ t\ dt\ dx,$$

with

$$\begin{aligned}\frac{\partial}{\partial r}\Big[\mathscr{U}^2(f)\ \mathscr{U}(h)\Big](A,x,t) &= 2\mathscr{U}(f)(A,x,t)\ \frac{\partial}{\partial r}\big(\mathscr{U}(f)\big)(A,x,t)\ \mathscr{U}(h)(A,x,t)\\ &+ \big(\mathscr{U}(f)\big)^2(A,x,t)\ \frac{\partial}{\partial r}\big(\mathscr{U}(h)\big)(A,x,t).\end{aligned}$$

Suppose that $A \geqslant 2a$. From [2, Lemma 4.2], for every $(x,t)\in\ \mathbb{R}\times]0,+\infty[$,

$$\Big|\frac{\partial}{\partial r}\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)(A,x,t)\Big| \leqslant \frac{M}{(A^2+t^2+x^2)^{3\alpha+5}} \leqslant \frac{M}{A^{6\alpha+10}}.$$

We deduce that $\big|I_1(A)\big| \leqslant \dfrac{M_1}{A^{4\alpha+6}}$ and

$$\lim_{A\longmapsto+\infty} I_1(A) = 0.\tag{5.4}$$

. As the same way and using again [2, Lemma 4.2], we show that

$$\lim_{A\longmapsto+\infty} I_2(A) = 0.\tag{5.5}$$

. Let us checking $I_3(A)$. In fact,

$$I_3(A)=\int_0^A\int_{-A}^A\Big[\int_0^A\frac{\partial^2}{\partial t^2}\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)(r,x,t)\ t\ dt\Big]d\nu_\alpha(r,x),$$

but

$$\begin{aligned}\int_0^A\frac{\partial^2}{\partial t^2}\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)(r,x,t)\ t\ dt &= A\ \frac{\partial}{\partial t}\big[\mathscr{U}^2(f)\ \mathscr{U}(h)\big](r,x,A)\\ &- \big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)(r,x,A)+f^2(r,x)h(r,x),\end{aligned}$$

by [2, Lemma 4.1], for $A>0$, and $(r,x)\in\ [0,+\infty[\times\mathbb{R}$,

$$\Big|A\frac{\partial}{\partial t}\big[\mathscr{U}^2(f)\ \mathscr{U}(h)\big](r,x,A)-\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)(r,x,A)\Big|\leqslant C\big(\frac{1}{A^{6\alpha+10}}+\frac{1}{A^{6\alpha+9}}\big).$$

Consequently,

$$\begin{aligned}\lim_{A\longmapsto+\infty}I_3(A) &= \lim_{A\longmapsto+\infty}\int_0^A\int_{-A}^A f^2(r,x)h(r,x)d\nu_\alpha(r,x)\\ &= \int_0^\infty\int_{\mathbb{R}}f^2(r,x)h(r,x)d\nu_\alpha(r,x). \qquad (5.6)\end{aligned}$$

The proof is complete by combining the relations (5.2), (5.3), (5.4), (5.5) and (5.6).

Lemma 5.4. *For all nonnegative functions $f,h\in\ \mathscr{D}_e(\mathbb{R}^2)$, we have*

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}\big(g(f)(r,x)\big)^2\ h(r,x)d\nu_\alpha(r,x)\\ &\leqslant 2\int_0^\infty\int_{\mathbb{R}}\big|f(r,x)\big|^2h(r,x)d\nu_\alpha(r,x)+8\int_0^\infty\int_{\mathbb{R}}f^*(r,x)\ g(f)(r,x)\ g(h)(r,x)d\nu_\alpha(r,x)\end{aligned}$$

where f^* is defined by the relation (3.4).

Proof. From Lemma 5.2, we have

$$\begin{aligned}&\int_o^{+\infty}\int_{\mathbb{R}}\Big(g(f)(r,x)\Big)^2h(r,x)d\nu_\alpha(r,x)\\ &\qquad\leqslant 4\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}t\big|\nabla\big(\mathscr{U}(f)\big(r,x,t\big)\big|^2\mathscr{P}^t(h)(r,x)dt\ d\nu_\alpha(r,x),\end{aligned}$$

on the other hand, by the relation (5.1), we get

$$\begin{aligned}&\int_o^{+\infty}\int_{\mathbb{R}}\Big(g(f)(r,x)\Big)^2h(r,x)d\nu_\alpha(r,x)\\ &\qquad\leqslant 2\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\Delta_\alpha\big(\mathscr{U}^2(f)\big)(r,x,t)\ \mathscr{U}(h)(r,x,t)t\ dt\ d\nu_\alpha(r,x), \qquad (5.7)\end{aligned}$$

however,

$$\begin{aligned}\Delta_\alpha\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big) &= \Delta_\alpha\big(\mathscr{U}^2(f)\big)\ \mathscr{U}(h)+\mathscr{U}^2(f)\ \Delta_\alpha\big(\mathscr{U}(h)\big)\\ &+ 2\langle\ \nabla\big(\mathscr{U}^2(f)\big)\mid\nabla\big(\mathscr{U}(h)\big)\ \rangle.\end{aligned}$$

Since $\Delta_\alpha\big(\mathscr{U}(h)\big)=0$ and $\nabla\big(\mathscr{U}^2(f)\big)=2\mathscr{U}(f)\ \nabla\big(\mathscr{U}(f)\big)$, we get

$$\Delta_\alpha\big(\mathscr{U}^2(f)\big)\ \mathscr{U}(h)=\Delta_\alpha\big(\mathscr{U}^2(f)\ \mathscr{U}(h)\big)-4\mathscr{U}(f)\langle\ \nabla\big(\mathscr{U}(f)\big)\mid\nabla\big(\mathscr{U}(h)\big)\ \rangle.$$

Consequently,

$$\begin{aligned}&\int_o^{+\infty}\int_{\mathbb{R}}\Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x)\\ &\leqslant 2\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\Delta_\alpha\Big(\mathscr{U}^2(f)\ \mathscr{U}(h)\Big)(r,x,t)t\ dt\ d\nu_\alpha(r,x)\\ &-8\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\mathscr{U}(f)(r,x,t)\langle\ \nabla\big(\mathscr{U}(f)\big)(r,x,t)\mid\nabla\big(\mathscr{U}(h)\big)(r,x,t)\ \rangle t\ dt\ d\nu_\alpha(r,x),\end{aligned}$$

and from Lemma 5.3,

$$\begin{aligned}&\int_o^{+\infty}\int_{\mathbb{R}}\Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x)\\ &\leqslant 2\int_0^{+\infty}\int_{\mathbb{R}}f^2(r,x)\ h(r,x)\ d\nu_\alpha(r,x)\\ &+8\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\big|\mathscr{U}(f)(r,x,t)\big|\ \big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\big|\ \big|\nabla\big(\mathscr{U}(h)\big)(r,x,t)\big|t\ dt\ d\nu_\alpha(r,x).\end{aligned}$$

Using Fubini's theorem and the Cauchy schwartz inequality's, we get

$$\begin{aligned}&\int_o^{+\infty}\int_{\mathbb{R}}\Big(g(f)(r,x)\Big)^2 h(r,x)d\nu_\alpha(r,x)\leqslant 2\int_0^{+\infty}\int_{\mathbb{R}}f^2(r,x)\ h(r,x)\ d\nu_\alpha(r,x)\\ &+8\int_0^{+\infty}\int_{\mathbb{R}}f^*(r,x)\Big[\Big(\int_0^{+\infty}\big|\nabla\big(\mathscr{U}(f)\big)(r,x,t)\big|^2tdt\Big)^{\frac12}\Big(\int_0^{+\infty}\big|\nabla\big(\mathscr{U}(h)\big)(r,x,t)\big|^2tdt\Big)^{\frac12}\Big]d\nu_\alpha(r,x)\\ &=2\int_0^{+\infty}\int_{\mathbb{R}}f^2(r,x)\ h(r,x)\ d\nu_\alpha(r,x)+8\int_0^{+\infty}\int_{\mathbb{R}}f^*(r,x)g(f)(r,x)g(h)(r,x)d\nu_\alpha(r,x).\end{aligned}$$

Theorem 5.5. *Let f be a nonnegative function; $f\in\mathscr{D}_e(\mathbb{R}^2)$. For every $p\in[4,+\infty[$, the function $g(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$||g(f)||_{p,\nu_\alpha}\leqslant A_p\ ||f||_{p,\nu_\alpha},$$

where

$$\begin{aligned}A_p &= \sqrt{2}\Big[4(p-2)\ \big(\frac{4}{p(p-1)}\big)^{\frac{1}{p}}\ 2^{\frac{1}{2-p}}\\ &+\sqrt{1+16(p-2)^2\ \big(\frac{4}{p(p-1)}\big)^{\frac{2}{p}}\ 2^{\frac{2}{2-p}}}\ \Big].\end{aligned}\tag{5.8}$$

Proof. Let $p \in [4, +\infty[$, then $\dfrac{p}{2} \in [2, +\infty[$. Let q be the conjugate exponent of $\frac{p}{2}$. Then, q belongs to $]1, 2]$. Finally, let f, h be nonnegative functions in $\mathscr{D}_e(\mathbb{R}^2)$ such that $||h||_{q,\nu_\alpha} = 1$. From Lemma 5.4,

$$\int_0^{\infty}\int_{\mathbb{R}} \big(g(f)(r,x)\big)^2\ h(r,x)d\nu_\alpha(r,x)$$

(5.9)

$$\leqslant 2\int_0^{\infty}\int_{\mathbb{R}} \big|f(r,x)\big|^2 h(r,x)d\nu_\alpha(r,x) + 8\int_0^{\infty}\int_{\mathbb{R}} f^*(r,x)\ g(f)(r,x)\ g(h)(r,x)d\nu_\alpha(r,x).$$

From Hölder's inequality,

$$(5.10)\quad 2\int_0^{\infty}\int_{\mathbb{R}} \big(f(r,x)\big)^2 h(r,x)d\nu_\alpha(r,x) \quad\leqslant\quad 2||f||^2_{p,\nu_\alpha}\ ||h||_{q,\nu_\alpha} = 2||f||^2_{p,\nu_\alpha}.$$

Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{q} = 1$, then, from the generalized Hölder's inequality,

$$8\int_0^{\infty}\int_{\mathbb{R}} f^*(r,x)\ g(f)(r,x)\ g(h)(r,x)d\nu_\alpha(r,x)$$

$$(5.11)\qquad \leqslant 8||f^*||_{p,\nu_\alpha}\ ||g(f)||_{p,\nu_\alpha}\ ||g(h)||_{q,\nu_\alpha}.$$

Now, from [2, Relation 4.40] and the fact that $q = \dfrac{p}{p-2} \in\ ]1, 2]$ and $||h||_{q,\nu_\alpha} = 1$, we get

$$(5.12)\qquad ||g(h)||_{q,\nu_\alpha} \quad\leqslant\quad \frac{2^{\frac{2-q}{2}}}{q}\Big(\frac{q}{q-1}\Big)^{\frac{1}{q}}.$$

Applying the relations (3.5) and (5.12) and replacing q by $\dfrac{p}{p-2}$; we obtain

$$8\int_0^{\infty}\int_{\mathbb{R}} f^*(r,x)\ g(f)(r,x)\ g(h)(r,x)d\nu_\alpha(r,x)$$

$$(5.13)\qquad \leqslant 8\sqrt{2}\ (p-2)\ \Big(\frac{4}{p(p-1)}\Big)^{\frac{1}{p}}\ 2^{\frac{1}{2-p}}\ ||f||_{p,\nu_\alpha}\ ||g(f)||_{p,\nu_\alpha}.$$

Combining the relations (5.9), (5.10), (5.11) and (5.13), we deduce that for every nonnegative function $h \in \mathscr{D}_e(\mathbb{R}^2)$,

$$\int_0^{\infty}\int_{\mathbb{R}} \big(g(f)(r,x)\big)^2\ h(r,x)d\nu_\alpha(r,x)$$

$$\leqslant 2||f||^2_{p,\nu_\alpha} + 8\sqrt{2}\ (p-2)\ \Big(\frac{4}{p(p-1)}\Big)^{\frac{1}{p}}\ 2^{\frac{1}{2-p}}\ ||f||_{p,\nu_\alpha}\ ||g(f)||_{p,\nu_\alpha},$$

and by duality,

$$\begin{aligned} ||g^2(f)||_{\frac{p}{2},\nu_\alpha} &= ||g(f)||^2_{p,\nu_\alpha} \\ &\leqslant 2||f||^2_{p,\nu_\alpha} + 8\sqrt{2}(p-2)\big(\frac{4}{p(p-1)}\big)^{\frac{1}{p}}\ 2^{\frac{1}{2-p}}||f||_{p,\nu_\alpha}||g(f)||_{p,\nu_\alpha}. \end{aligned} \tag{5.14}$$

The inequality (5.14) shows that

$$||g(f)||_{p,\nu_\alpha} \leqslant A_p\ ||f||_{p,\nu_\alpha}.$$

Remark 5.6. *As the same way as the proof of [2, Proposition 4.6], we deduce that for every $f \in \mathscr{D}_e(\mathbb{R}^2)$ and every $p \in [4,+\infty[$, the function $g(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$||g(f)||_{p,\nu_\alpha} \leqslant 2A_p\ ||f||_{p,\nu_\alpha},$$

where A_p is given by the relation (5.8).

Theorem 5.7. *For every $p \in [4,+\infty[$, the mapping: $f \longmapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f \in L^p(d\nu_\alpha)$, we have*

$$||g(f)||_{p,\nu_\alpha} \leqslant 2A_p\ ||f||_{p,\nu_\alpha}.$$

Proof. The result follows from Remark 5.6, the density of $\mathscr{D}_e(\mathbb{R}^2)$ in $L^p(d\nu_\alpha)$ (see also [2, Theorem 4.7]).

Now, we are able to prove the mean result of this work.

Theorem 5.8. *For every $p \in]1,+\infty[$, the mapping: $f \longmapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f \in L^p(d\nu_\alpha)$, we have*

$$||g(f)||_{p,\nu_\alpha} \leqslant B_p\ ||f||_{p,\nu_\alpha},$$

where

$$B_p = \begin{cases} 2\ \frac{2^{\frac{2-p}{2}}}{p}\ \big(\frac{p}{p-1}\big)^{\frac{1}{p}}, & \text{if } p \in]1,2], \\ 2^{\frac{3p-4}{2p}}\ A_4^{\frac{2p-4}{p}}, & \text{if } p \in [2,4], \\ 2A_p, & \text{if } p \in [4,+\infty[, \end{cases} \tag{5.15}$$

and A_p is given by the relation (5.8).

Proof. The result follows from [2, Theorem 4.7], Theorem 5.7 and the Riesz-Thorin theorem's [26, 28] for $p \in [2,4]$.

Theorem 5.9. *For every $p \in]1,+\infty[$ and every $f \in L^p(d\nu_\alpha)$, we have*

$$||f||_{p,\nu_\alpha} \leqslant 4\ B_{\frac{p}{p-1}}\ ||g(f)||_{p,\nu_\alpha},$$

where B_p is given by the relation (5.15).

Proof. For every $f \in \mathscr{D}_e(\mathbb{R}^2)$, we put

$$g_1(f)(r,x) = \Big(\int_0^\infty \big|\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\big|^2 t dt\Big)^{\frac{1}{2}}.$$

Then, for every $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$g_1(f)(r,x) \leqslant g(f)(r,x), \tag{5.16}$$

and by the relation (3.8), we have

$$\mathscr{U}(f)(r,x,t) = \int_0^\infty \int_{\mathbb{R}} e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda),$$

so,

$$\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t) = \widetilde{\mathscr{F}}^{-1}\Big(-\sqrt{\mu^2+\lambda^2}\ e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}(f)\Big)(r,x).$$

Thus, by Fubini's theorem

$$||g_1(f)||^2_{2,\nu_\alpha} = \int_0^\infty t\Big[\int_0^\infty\int_{\mathbb{R}} \Big|\widetilde{\mathscr{F}}^{-1}\Big(-\sqrt{\mu^2+\lambda^2}\ e^{-t\sqrt{\mu^2+\lambda^2}}\ \widetilde{\mathscr{F}}(f)\Big)(r,x)\Big|^2 d\nu_\alpha(r,x)\Big]dt.$$

Applying Plancherel theorem, we obtain

$$\begin{aligned} ||g_1(f)||^2_{2,\nu_\alpha} &= \int_0^\infty t\Big[\int_0^\infty \int_{\mathbb{R}} (\mu^2+\lambda^2)\ e^{-2t\sqrt{\mu^2+\lambda^2}}\ \big|\widetilde{\mathscr{F}}(f)(\mu,\lambda)\big|^2 d\nu_\alpha(\mu,\lambda)\Big]dt \\ &= \int_0^\infty \int_{\mathbb{R}} (\mu^2+\lambda^2)\big|\widetilde{\mathscr{F}}(f)(\mu,\lambda)\big|^2\Big(\int_0^\infty e^{-2t\sqrt{\mu^2+\lambda^2}} t\ dt\Big) d\nu_\alpha(\mu,\lambda) \\ &= \frac{1}{4}\int_0^\infty \int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}(f)(\mu,\lambda)\big|^2 d\nu_\alpha(\mu,\lambda) = \frac{1}{4}\ ||f||^2_{2,\nu_\alpha}, \end{aligned}$$

wish means that

$$||g_1(f)||_{2,\nu_\alpha} = \frac{1}{2}\ ||f||_{2,\nu_\alpha}. \tag{5.17}$$

On the other hand, for every $h \in \mathscr{D}_e(\mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}} h(r,x) f(r,x) d\nu_\alpha(r,x) = \frac{1}{4}\big(||f+h||^2_{2,\nu_\alpha}\big) - \frac{1}{4}\big(||f-h||^2_{2,\nu_\alpha}\big),$$

and by the relation (5.17),

$$\int_0^\infty \int_{\mathbb{R}} h(r,x) f(r,x) d\nu_\alpha(r,x) = ||g_1(f+h)||^2_{2,\nu_\alpha} - ||g_1(f-h)||^2_{2,\nu_\alpha}. \tag{5.18}$$

By standard computation, we have

$$\begin{aligned} &||g_1(f+h)||^2_{2,\nu_\alpha} - ||g_1(f-h)||^2_{2,\nu_\alpha} \\ &= 4\int_0^\infty \int_{\mathbb{R}} \Big[\int_0^\infty \frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t) \frac{\partial}{\partial t}\big(\mathscr{U}(h)\big)(r,x,t) t\ dt\Big] d\nu_\alpha(r,x). \end{aligned} \tag{5.19}$$

Combining the relations (5.18) and (5.19), we get

$$\int_0^{\infty}\int_{\mathbb{R}} h(r,x)f(r,x)d\nu_\alpha(r,x) = 4\int_0^{\infty}\int_{\mathbb{R}}\Big[\int_0^{\infty}\frac{\partial}{\partial t}\big(\mathscr{U}(f)\big)(r,x,t)\frac{\partial}{\partial t}\big(\mathscr{U}(h)\big)(r,x,t)t\ dt\Big]d\nu_\alpha(r,x).$$

Applying Hölder's inequality with respect to the measure $t\ dt$, it follows that

$$\Big|\int_0^{\infty}\int_{\mathbb{R}} h(r,x)f(r,x)d\nu_\alpha(r,x)\Big| \leqslant 4\int_0^{\infty}\int_{\mathbb{R}} g_1(f)(r,x)\ g_1(h)(r,x)d\nu_\alpha(r,x).$$

Let $p,q \in]1,+\infty[$; $\frac{1}{p}+\frac{1}{q}=1$, again by Hölder's inequality with respect to the measure $d\nu_\alpha(r,x)$ and applying the relation (5.16), we have

$$\Big|\int_0^{\infty}\int_{\mathbb{R}} h(r,x)f(r,x)d\nu_\alpha(r,x)\Big| \leqslant 4||g(f)||_{p,\nu_\alpha}\ ||g(h)||_{q,\nu_\alpha},$$

and by means of Theorem 5.8,

$$\Big|\int_0^{\infty}\int_{\mathbb{R}} h(r,x)f(r,x)d\nu_\alpha(r,x)\Big| \leqslant 4B_q\ ||g(f)||_{p,\nu_\alpha}\ ||h||_{q,\nu_\alpha}.$$

In particular, for every $h \in \mathscr{D}_e(\mathbb{R}^2)$; $||h||_{q,\nu_\alpha} \leqslant 1$,

$$\Big|\int_0^{\infty}\int_{\mathbb{R}} h(r,x)f(r,x)d\nu_\alpha(r,x)\Big| \leqslant 4B_{\frac{p}{p-1}}\ ||g(f)||_{p,\nu_\alpha},$$

by duality,

$$||f||_{p,\nu_\alpha} \leqslant 4B_{\frac{p}{p-1}}\ ||g(f)||_{p,\nu_\alpha}.$$

The proof is complete by the fact that $\mathscr{D}_e(\mathbb{R}^2)$ is dense in $L^p(d\nu_\alpha)$.

Conclusion. By Theorem 5.8 and Theorem 5.9, we deduce that for every $p \in]1,+\infty[$, there exists a positive constant C_p such that for every $f \in L^p(d\nu_\alpha)$;

$$\frac{1}{C_p}\ ||f||_{p,\nu_\alpha} \leqslant ||g(f)||_{p,\nu_\alpha} \leqslant C_p\ ||f||_{p,\nu_\alpha}.$$

References

[1] A. Achour and K. Trimèche, *La g-fonction de Littlewood-Paley associée à un opérateur différentiel singulier sur* $]0,+\infty[$, Ann. Inst. Fourier, Grenoble, **33**(1983), 203-226.

[2] B. Amri and L. T. Rachdi, *The Littlewood-Paley g-function associated with the Riemann-Liouville operator*, Ann. Univ. Paedagog. Crac. Stud. Math., **12**(2013), 31-58.

[3] B. Amri and L. T. Rachdi, *Uncertainty principle in terms of entropy for the Riemann-Liouville operator*, Bull. Malays. Math. Sci. Soc., (Accepted papers).

[4] B. Amri and L. T. Rachdi, *Beckner Logarithmic Uncertainty principle for the Riemann-Liouville operator*, Internat. J. Math., **24(9)**(2013), 1350070 (29 pages).

[5] B. Amri and L. T. Rachdi, *Calderon-reproducing formula for singular partial differential operators*, Integral Transforms Spec. Funct., **25(8)**(2014), 597-611.

[6] H. Annabi and A. Fitouhi, *La g-fonction de Littlewood-Paley associée à une classe d'opérateurs différentiels sur* $]0,+\infty[$ *contenant l'opérateur de Bessel*, C. R. Acad. Sc. Paris, **303**(1986), 411-413.

[7] C. Baccar, N. B. Hamadi and L. T. Rachdi, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Int. J. Math. Math. Sci., **2006**(2006), pp 1-26.

[8] C. Baccar, N. B. Hamadi and L. T. Rachdi, *An analogue of Hardy's theorem and its Lp-version for the Riemann-Liouville transform associated with singular partial differential operators*, J. Math. Sci. (Dattapukur), **17(1)**(2006), 1-18.

[9] N. B. Hamadi and L. T. Rachdi, *Weyl Transforms Associated with the Riemann-Liouville Operator*, Int. J. Math. Math. Sci., 2006, Article ID 94768, 1-18.

[10] C. Baccar, N. B. Hamadi and L. T. Rachdi, *Best approximation for Weierstrass transform connected with Riemann-Liouville operator*, Commun. Math. Anal., **5**, No. 1, (2008) 65-83.

[11] C. Baccar and L. T. Rachdi, *Spaces of* D_{L^p}*-type and a convolution product associated with the Riemann-Liouville operator*, Bull. Math. Anal. Appl., Vol. 1, Iss., **3**(2009), 16-41.

[12] W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter studies in mathematics 20, walter de Gruyter, Berlin-New York 1995.

[13] A. Erdélyi et al., *Tables of integral transforms*, Mc Graw-Hill Book Compagny., **2**, New York 1954.

[14] A. Erdélyi., *Asymptotic expansions, Dover publications*, New-York 1956.

[15] J. A. Fawcett, *Inversion of n-dimensional spherical averages*, SIAM Journal on Applied Mathematics, **45(2)**(1985), 336-341.

[16] S. Helgason. *The Radon Transform.* Birkhäuser, 2nd edition, 1999.

[17] H. Hellsten and L.-E. Andersson, *An inverse method for the processing of synthetic aperture radar data*, Inverse Problems, **3(1)**(1987), 111-124.

[18] M. Herberthson, *A numerical implementation of an inverse formula for CARABAS raw data*, Internal Report D30430-3.2, National Defense Research Institude, FOA, Box 1165; S-581 11, Linköping, 1986.

[19] I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Anal. Math., **8**(1960/61), 307 -336.

[20] Kh. Hleili, S. Omri and L. T. Rachdi, *Uncertainty principle for the Riemann-Liouville operator*, Cubo, **13(3)**(2011), 91-115.

[21] N. N. Lebedev, *Special Functions and Their Applications, Dover publications*, New-York 1972.

[22] S. Omri and L. T. Rachdi, *An $L^p - L^q$ version of Morgan's theorem associated with Riemann-Liouville transform*, Int. J. Math. Anal., **1(17)**(2007), 805–824.

[23] S. Omri and L. T. Rachdi, *Heisenberg-Pauli-Weyl uncertainty principle for the Riemann-Liouville Operator*, J. Inequal. Pure and Appl. Math., **9**, Iss. 3, Art 88 (2008).

[24] L. T. Rachdi and A. Rouz, *On the range of the Fourier transform connected with Riemann-Liouville operator*, Ann. Math. Blaise Pascal, **16(2)**(2009), 355-397.

[25] F. Soltani, *Littlewood-Paley g-function in the Dunkl analysis on* $\mathbb{R}^d$, J. Ineq. Pure and Appl. Math. **6**, Issue 3, (2005), Article 84, 13 pp. (electronic).

[26] E. M. Stein, *Interpolation of linear operator*, Trans. Amer. Math. Soc., **83(2)**(1956), 482-492.

[27] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, New Jersey, **63**, 1970.

[28] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University, New Jersey 1971.

[29] K. Stempak, *La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel*, C. R. Acad. Sc. Paris, Série I, Math., **303**(1986), 15-18.

[30] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge univ. Press., 2nd ed., Cambridge 1959.