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New Generalizations of Ostrowski-Like Type Inequalities for Fractional Integrals

Çetin Yildiz*
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
e-mail : cetin@atauni.edu.tr

Muhamet Emin Özdemir
Uludağ University, Education Faculty, Department of Mathematics, 16059, Bursa, Turkey
e-mail : emos@atauni.edu.tr

Mehmet Zeki Sarikaya
Düzce University, Department of Mathematics, Faculty of Science and Arts, Düzce, Turkey
e-mail : sarikayamz@gmail.com

Abstract. In this paper, we use the Riemann-Liouville fractional integrals to establish several new inequalities for some differantiable mappings that are connected with the celebrated Ostrowski type integral inequality.

1. Introduction

In 1938, the classical integral inequality established by Ostrowski as follows:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty$$

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for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Recently, several generalizations of the Ostrowski integral inequality for mappings of bounded variation and for Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors.

In [6], M. Alomari and M. Darus proved some Ostrowski's type inequality for the class of convex(concave) functions:

Theorem 1.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I°(the interior of I) such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$, then the following inequality holds:*

(1.1)
$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{(b-a)(p+1)^{1/p}}\left[(b-x)^2\left(\frac{|f'(x)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} + (x-a)^2\left(\frac{|f'(x)|^q + |f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]$$

for each $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is concave on $[a,b]$, then, the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+x}{2}\right)\right| + \left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+x}{2}\right)\right|\right], \tag{1.2}$$

for each $x \in [a,b]$, where $p > 1$.

Theorem 1.4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, $q \geq 1$ and $|f'(x)| \leq M$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2}\left[\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{b+2x}{3}\right)\right| + \left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|\right]. \tag{1.3}$$

In recent years, such inequalities were studied extensively by many researchers and numerious generalizations, extensions and variants of them appeared in number of papers see ([1]-[8])

Now, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper, see([18]).

Definition 1.1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u} u^{\alpha-1} du$. Here is $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see ([9]-[17]).

2. Main Results

In order to prove our main theorems, we need the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then, for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\begin{aligned} &\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J_{x^+}^{\alpha} f(b) + J_{x^-}^{\alpha} f(a)\right] \\ = &\int_0^1 m(t) f'(ta + (1-t)b)dt \end{aligned}$$

for each $t \in [a,b]$, where

$$m(t) = \begin{cases} -t^{\alpha}, & t \in \left[0, \frac{b-x}{b-a}\right] \\ (1-t)^{\alpha}, & t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases},$$

for all $x \in [a,b]$.

Proof. By integration by parts, we can obtain

$$
\begin{aligned}
I &= \int_0^1 m(t) f'(ta+(1-t)b)dt \\
&= \int_0^{\frac{b-x}{b-a}} (-t^{\alpha}) f'(ta+(1-t)b)dt + \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha} f'(ta+(1-t)b)dt \\
&= \left(\frac{b-x}{b-a}\right)^{\alpha} \frac{f(x)}{b-a} - \frac{\alpha}{b-a} \int_0^{\frac{b-x}{b-a}} t^{\alpha-1} f(ta+(1-t)b)dt \\
&\quad + \left(\frac{x-a}{b-a}\right)^{\alpha} \frac{f(x)}{b-a} - \frac{\alpha}{b-a} \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha-1} f(ta+(1-t)b)dt.
\end{aligned}
$$

Using the change of the variable $u = ta + (1-t)b$ for $t \in [0,1]$, which gives

$$
\begin{aligned}
I &= \frac{(b-x)^{\alpha}}{(b-a)^{\alpha+1}} f(x) - \frac{\alpha}{(b-a)^{\alpha+1}} \int_x^b (b-u)^{\alpha-1} f(u)du \\
&\quad + \frac{(x-a)^{\alpha}}{(b-a)^{\alpha+1}} f(x) - \frac{\alpha}{(b-a)^{\alpha+1}} \int_a^x (u-a)^{\alpha-1} f(u)du \\
&= \left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J_{x^+}^{\alpha} f(b) + J_{x^-}^{\alpha} f(a)\right].
\end{aligned}
$$

This is completed the proof. □

The main results may be stated as follows:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|$ is convex on $[a,b]$ and $x \in [a,b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$
\begin{aligned}
& \left| \left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J_{x^+}^{\alpha} f(b) + J_{x^-}^{\alpha} f(a)\right] \right| \\
\leq\; & \frac{1}{\alpha+2} \left\{ \left(\frac{(b-x)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}} \left[\frac{1}{\alpha+1} + \frac{b-x}{b-a}\right] \right) |f'(a)| \right. \\
& \left. + \left(\frac{(x-a)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}} \left[\frac{1}{\alpha+1} + \frac{x-a}{b-a}\right] \right) |f'(b)| \right\}
\end{aligned} \tag{2.1}
$$

where Γ is Euler Gamma function.

Proof. From Lemma and since $|f'|$ is convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^{\alpha}f(b)+J_{x^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \int_0^{\frac{b-x}{b-a}}|-t|^{\alpha}\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^{1}\left|(1-t)^{\alpha}\right|\left|f'(ta+(1-t)b)\right|dt \\
\leq\ & \int_0^{\frac{b-x}{b-a}}t^{\alpha}\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt+\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
=\ & \left[\frac{1}{\alpha+2}\frac{(b-x)^{\alpha+2}-(x-a)^{\alpha+2}}{(b-a)^{\alpha+2}}+\frac{1}{\alpha+1}\frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}}\right]\left|f'(a)\right| \\
&+\left[\frac{1}{\alpha+2}\frac{(x-a)^{\alpha+2}-(b-x)^{\alpha+2}}{(b-a)^{\alpha+2}}+\frac{1}{\alpha+1}\frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}}\right]\left|f'(b)\right| \\
=\ & \frac{1}{\alpha+2}\left(\frac{(b-x)^{\alpha+2}}{(b-a)^{\alpha+2}}+\frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1}+\frac{b-x}{b-a}\right]\right)\left|f'(a)\right| \\
&+\frac{1}{\alpha+2}\left(\frac{(x-a)^{\alpha+2}}{(b-a)^{\alpha+2}}+\frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1}+\frac{x-a}{b-a}\right]\right)\left|f'(b)\right|
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *If we take $x=\frac{a+b}{2}$ in (2.1), we get*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{b-a}{2(\alpha+1)}\left(\frac{|f'(a)|+|f'(b)|}{2}\right).
\end{aligned} \tag{2.2}$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is convex on $[a,b]$, $q>1$ and $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^{\alpha}f(b)+J_{x^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{1}{(b-a)^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[(b-x)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+(x-a)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{2.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$.

Proof. From Lemma and using the well known Hölder inequality, we have

$$\begin{aligned}
&\left|\left[\frac{(x-a)^\alpha+(b-x)^\alpha}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^\alpha f(b)+J_{x^-}^\alpha f(a)\right]\right| \\
\le& \int_0^{\frac{b-x}{b-a}}\left|-t\right|^\alpha\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^1\left|(1-t)^\alpha\right|\left|f'(ta+(1-t)b)\right|dt \\
\le& \left(\int_0^{\frac{b-x}{b-a}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{b-x}{b-a}}^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|$ is convex, by Hermite-Hadamard inequality we have,

$$\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\le\frac{b-x}{b-a}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right),$$

$$\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\le\frac{x-a}{b-a}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)$$

and by simple computation

$$\int_0^{\frac{b-x}{b-a}}t^{\alpha p}dt=\frac{1}{\alpha p+1}\left(\frac{b-x}{b-a}\right)^{\alpha p+1},$$

$$\int_{\frac{b-x}{b-a}}^1(1-t)^{\alpha p}dt=\frac{1}{\alpha p+1}\left(\frac{x-a}{b-a}\right)^{\alpha p+1}.$$

Therefore

$$\begin{aligned}
&\left|\left[\frac{(x-a)^\alpha+(b-x)^\alpha}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^\alpha f(b)+J_{x^-}^\alpha f(a)\right]\right| \\
\le& \frac{1}{(b-a)^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[(b-x)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+(x-a)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence, using the formula $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ $(\alpha>0)$ for Euler Gamma function, the proof is complete. □

Remark 2.1. In Theorem , if we choose $\alpha=1$, then we obtain inequality (1.1).

Corollary 2.2. *If we take* $x=\frac{a+b}{2}$ *in* (2.3), *we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right]\right|\\ \le\ &\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(\frac{a+b}{2})\right|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(\frac{a+b}{2})\right|^{q}+|f'(a)|^{q}}{2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 2.3. *Let* $f:[a,b]\to\mathbb{R}$, *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f'\in L[a,b]$. *If* $|f'|^{q}$ *is convex on* $[a,b]$, $q\ge 1$ *and* $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}(2.4)\qquad &\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^{\alpha}_{x^{+}}f(b)+J^{\alpha}_{x^{-}}f(a)\right]\right|\\ \le\ &\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+2}\right)^{\frac{1}{q}}\\ &\times\left\{\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left[\left(\frac{b-x}{b-a}\right)|f'(a)|^{q}+\left(\frac{1}{\alpha+1}+\frac{x-a}{b-a}\right)|f'(b)|^{q}\right]^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left[\left(\frac{1}{\alpha+1}+\frac{b-x}{b-a}\right)|f'(a)|^{q}+\left(\frac{x-a}{b-a}\right)|f'(b)|^{q}\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $\alpha>0$ *and* Γ *is Euler Gamma function.*

Proof. From Lemma and using the well known power mean inequality, we have

$$\begin{aligned}(2.5)\qquad &\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^{\alpha}_{x^{+}}f(b)+J^{\alpha}_{x^{-}}f(a)\right]\right|\\ \le\ &\int_{0}^{\frac{b-x}{b-a}}|-t|^{\alpha}\,|f'(ta+(1-t)b)|\,dt+\int_{\frac{b-x}{b-a}}^{1}|(1-t)^{\alpha}|\,|f'(ta+(1-t)b)|\,dt\\ \le\ &\left(\int_{0}^{\frac{b-x}{b-a}}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{b-x}{b-a}}t^{\alpha}\,|f'(ta+(1-t)b)|^{q}\,dt\right)^{\frac{1}{q}}\\ &+\left(\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}\,|f'(ta+(1-t)b)|^{q}\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is convex, we have

$$
\begin{aligned}
(2.6)\qquad & \int_0^{\frac{b-x}{b-a}} t^{\alpha} |f'(ta+(1-t)b)|^q \, dt \\
\leq & \int_0^{\frac{b-x}{b-a}} t^{\alpha} \left[t |f'(a)|^q + (1-t) |f'(b)|^q \right] dt \\
= & \frac{1}{\alpha+2} \left(\frac{b-x}{b-a} \right)^{\alpha+2} |f'(a)|^q \\
& + \left(\frac{1}{\alpha+1} \left(\frac{b-x}{b-a} \right)^{\alpha+1} - \frac{1}{\alpha+2} \left(\frac{b-x}{b-a} \right)^{\alpha+2} \right) |f'(b)|^q
\end{aligned}
$$

and

(2.7)

$$
\begin{aligned}
& \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha} |f'(ta+(1-t)b)|^q \, dt \\
\leq & \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha} \left[t |f'(a)|^q + (1-t) |f'(b)|^q \right] dt \\
= & \left(\frac{1}{\alpha+1} \left(\frac{x-a}{b-a} \right)^{\alpha+1} - \frac{1}{\alpha+2} \left(\frac{x-a}{b-a} \right)^{\alpha+2} \right) |f'(a)|^q \\
& + \frac{1}{\alpha+2} \left(\frac{x-a}{b-a} \right)^{\alpha+2} |f'(b)|^q .
\end{aligned}
$$

Therefore, if we write (2.6) and (2.7) in (2.5), then we get (2.4) which is required. □

Corollary 2.3. *If we take $x = \frac{a+b}{2}$ in (2.4), we have*

$$
\begin{aligned}
& \left| f\left(\frac{a+b}{2} \right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \right| \\
\leq & \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+2} \right)^{\frac{1}{q}} \left(\frac{1}{2} \right)^{\frac{\alpha+1}{q}+1} \\
& \times \left\{ \left(|f'(a)|^q + \frac{\alpha+3}{(\alpha+1)} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{\alpha+3}{(\alpha+1)} |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

Theorem 2.4. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is concave on $[a,b]$ and $x \in [a,b]$, then the* *following inequality for fractional integrals holds:*

$$\left|\left[\frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^{\alpha}f(b)+J_{x^-}^{\alpha}f(a)\right]\right| \tag{2.8}$$
$$\leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}(b-a)^{\alpha+1}}\left[(b-x)^{\alpha+1}\left|f'\left(\frac{b+x}{2}\right)\right|+(x-a)^{\alpha+1}\left|f'\left(\frac{a+x}{2}\right)\right|\right]$$

for each $x\in[a,b]$, where $p>1$.

Proof. From Lemma and using the Hölder inequality, we have

$$\left|\left[\frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^{\alpha}f(b)+J_{x^-}^{\alpha}f(a)\right]\right| \tag{2.9}$$
$$\leq \int_0^{\frac{b-x}{b-a}}|-t|^{\alpha}\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^1\left|(1-t)^{\alpha}\right|\left|f'(ta+(1-t)b)\right|dt$$
$$\leq \left(\int_0^{\frac{b-x}{b-a}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{b-x}{b-a}}^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave on $[a,b]$, by Hermite-Hadamard's inequality we get

$$\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\leq\frac{b-x}{b-a}\left|f'\left(\frac{b+x}{2}\right)\right|^q \tag{2.10}$$

and

$$\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\leq\frac{x-a}{b-a}\left|f'\left(\frac{a+x}{2}\right)\right|^q. \tag{2.11}$$

Therefore, if we write (2.10) and (2.11) in (2.9), we get (2.8). This completes the proof. □

Remark 2.2. In Theorem , if we choose $\alpha=1$, then the inequality (2.8) reduces the inequality (1.2) of Theorem .

Corollary 2.4. *In Theorem if we take $x=\frac{a+b}{2}$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{b-a}{4}\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right].$$

Theorem 2.5. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is concave on $[a,b]$, $q\geq 1$ and $x\in[a,b]$, then the following inequality for fractional integrals holds:*

(2.12)
$$\left|\left[\frac{(x-a)^\alpha+(x-b)^\alpha}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^\alpha f(b)+J_{x^-}^\alpha f(a)\right]\right|$$
$$\leq\frac{1}{\alpha+1}\left[\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{b+(\alpha+1)x}{\alpha+2}\right)\right|+\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{a+(\alpha+1)x}{\alpha+2}\right)\right|\right].$$

Proof. We note that by concavity of $|f'|^q$ and power-mean inequality, we have

$$\begin{aligned}
&\left|\left[\frac{(x-a)^\alpha+(x-b)^\alpha}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^+}^\alpha f(b)+J_{x^-}^\alpha f(a)\right]\right|\\
\leq\;&\int_0^{\frac{b-x}{b-a}}|-t|^\alpha\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^1\left|(1-t)^\alpha\right|\left|f'(ta+(1-t)b)\right|dt\\
\leq\;&\left(\int_0^{\frac{b-x}{b-a}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}}t^\alpha\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Accordingly, by Lemma and the Jensen integral inequality, we obtain

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}}t^\alpha\left|f'(ta+(1-t)b)\right|^q dt\;&\leq\;\left(\int_0^{\frac{b-x}{b-a}}t^\alpha dt\right)\left|f'\left(\frac{\int_0^{\frac{b-x}{b-a}}t^\alpha(ta+(1-t)b)dt}{\int_0^{\frac{b-x}{b-a}}t^\alpha dt}\right)\right|^q\\
&=\;\frac{1}{\alpha+1}\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{b+(\alpha+1)x}{\alpha+2}\right)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha\left|f'(ta+(1-t)b)\right|^q dt\\
\leq\;&\left(\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha dt\right)\left|f'\left(\frac{\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha(ta+(1-t)b)dt}{\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha dt}\right)\right|^q\\
=\;&\frac{1}{\alpha+1}\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{a+(\alpha+1)x}{\alpha+2}\right)\right|^q.
\end{aligned}$$

Therefore

$$\left|\left[\frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{x^{+}}^{\alpha}f(b)+J_{x^{-}}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{1}{\alpha+1}\left[\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{b+(\alpha+1)x}{\alpha+2}\right)\right|+\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left|f'\left(\frac{a+(\alpha+1)x}{\alpha+2}\right)\right|\right]$$

which completes the proof. □

Remark 2.3. In Theorem , if we choose $\alpha = 1$, then the inequality (2.12) reduces the inequality (1.3) of Theorem .

Corollary 2.5. *If we take* $x = \frac{a+b}{2}$ *in* (2.12)*, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{b-a}{4(\alpha+1)}\left\{\left|f'\left(\frac{a(\alpha+1)+b(\alpha+3)}{2(\alpha+2)}\right)\right|+\left|f'\left(\frac{a(\alpha+3)+b(\alpha+1)}{2(\alpha+2)}\right)\right|\right\}.$$

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