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Some Properties for Certain Subclasses of Starlike Functions Defined by Convolution

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Abstract. In this paper, we obtained some properties for subclasses of starlike functions defined by convolution such as partial sums, integral means, square root and integral transform for these classes.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively, starlike and convex of order α $(0 \leq \alpha < 1)$. Thus,

$$S^*(\alpha) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \; (0 \leq \alpha < 1; z \in U \right\} \tag{1.2}$$

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and

$$K(\alpha) = \left\{ f \in S : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ (0 \leq \alpha < 1; z \in U) \right\}. \tag{1.3}$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Rebertson [18]. From (1.2) and (1.3) it follows that

$$f(z) \in K(\alpha) \Longleftrightarrow zf'(z) \in S^*(\alpha). \tag{1.4}$$

We note that

$$S^*(0) = S^* \text{ and } K(0) = K,$$

which are, respectivly, starlike and convex functions.
Let $f \in S$ be given by (1.1) and $g \in S$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (b_k > 0). \tag{1.5}$$

We define the Hadmard product (or convolution) of f and g as follows:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.6}$$

We denote by $S_{A,B}(f, g, \alpha, \beta, \gamma)$ $(-1 \leq A < B \leq 1,\ 0 < B \leq 1)$ the subclass of S, where f and g are given by (1.1) and (1.2), respectively and satisfies

$$\left| \frac{\frac{z(f*g)'(z)}{(f*g)(z)} - 1}{2(B-A)\gamma\left(\frac{z(f*g)'(z)}{(f*g)(z)} - \alpha\right) - B\left(\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right)} \right| < \beta$$

$$(z \in U; 0 \leq \alpha < 1; 0 < \beta \leq 1) \tag{1.7}$$

where $(f * g)(z)$ is given by (1.6) and $\frac{B}{2(B-A)} < \gamma \leq \begin{cases} \frac{B}{2(B-A)\alpha} & , \alpha \neq 0 \\ 1 & , \alpha = 0. \end{cases}$

We also let

$T_{A,B}(f, g, \alpha, \beta, \gamma) = S_{A,B}(f, g, \alpha, \beta, \gamma) \cap T$, where

$$T = \left\{ f \in S : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \ ; z \in U \right\}. \tag{1.8}$$

We note that:

(i) $T_{A,B}(f, S_\delta, \alpha, \beta, \gamma)$

$$= \left\{ f \in T : \left| \frac{\frac{z\left(f*S_\delta\right)'(z)}{(f*S_\delta)(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{z\left(f*S_\delta\right)'(z)}{\left(f*S_\delta\right)(z)} - \alpha\right) - B\left(\frac{z\left(f*S_\delta\right)'(z)}{(f*S_\delta)(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\},$$

for $S_\delta = \dfrac{z}{(1-z)^{2(1-\delta)}}$, $0 \le \delta < 1$ (see Magesh et al. [10 with $m = 0$]);

(ii) $T_{A,B}\left(f, z + \sum\limits_{k=2}^{\infty} \left| \left(\dfrac{1+b}{k+b}\right)^{\mu} \cdot \dfrac{\lambda!\,(k+n-2)!}{(n-1)!\,(k+\lambda-1)!} \right| z^k, \alpha, \beta, \gamma\right)$

$$= \left\{ f \in T : \left| \frac{\frac{z\left(J_{\mu,b}^{\lambda,n} f(z)\right)'}{J_{\mu,b}^{\lambda,n} f(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{z\left(J_{\mu,b}^{\lambda,n} f(z)\right)'}{J_{\mu,b}^{\lambda,n} f(z)} - \alpha\right) - B\left(\frac{z\left(J_{\mu,b}^{\lambda,n} f(z)\right)'}{J_{\mu,b}^{\lambda,n} f(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\},$$

for $n \ge 2$, $\lambda > -1$, $\mu \in \mathbb{C}$, $b \in \mathbb{C}\backslash\{\mathbb{Z}_0^- = 0, -1, -2, ...\}$ (see Owa et al. [16]);

(iii) $T_{A,B}\left(f, z + \sum\limits_{k=2}^{\infty} \dfrac{\left(\Gamma\left(n+1\right)\right)^2 \Gamma\left(2+\eta-\mu\right)\Gamma\left(2-\eta\right)}{\Gamma\left(n+\eta-\mu+1\right)\Gamma\left(n-\eta+1\right)} z^k, \alpha, \beta, \gamma\right)$

$$= \left\{ f \in T : \left| \frac{\frac{z\left(\Im_\mu^\eta f(z)\right)'}{\Im_\mu^\eta f(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{z(\Im_\mu^\eta f(z))'}{\Im_\mu^\eta f(z)} - \alpha\right) - B\left(\frac{z(\Im_\mu^\eta f(z))'}{\Im_\mu^\eta f(z)} - 1\right)} \right| < \beta \left(z \in U\right) \right\},$$

for $\eta - 1 < \mu < \eta < 2$ (see Murugusndramoorthy and Thilagvathi [13]);

(iv) $T_{A,B}\left(f, z + \sum\limits_{k=2}^{\infty} \dfrac{\Omega\Gamma\left(p_1 + A_1\left(n-1\right)\right)...\Gamma\left(p_\ell + A_\ell\left(n-1\right)\right)}{(n-1)!\Gamma(q_1 + B_1\left(n-1\right))...\Gamma(q_m + B_m\left(n-1\right))} z^k, \alpha, \beta, \gamma\right) =$

$$\left\{ f \in T : \left| \frac{\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - \alpha\right) - B\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\},$$

for $\ell \le m+1$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, ...\}$, $\Omega = \{\prod_{t=0}^{l} \Gamma(p_t)\}^{-1}\{\prod_{t=0}^{m} \Gamma(q_t)\}$ (see Murugusndramoorthy and Magesh [14]).

Also we note that:

(i) $T_{A,B}(f, \frac{z}{1-z}, \alpha, \beta, \gamma) = S_{A,B}\left(\alpha, \beta, \gamma\right)$

$$= \left\{ f \in T : \left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{zf'(z)}{f(z)} - \alpha\right) - B\left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\};$$

(ii) $T_{A,B}\left(f, \frac{z}{(1-z)^2}, \alpha, \beta, \gamma\right) = K_{A,B}\left(\alpha, \beta, \gamma\right)$

$$= \left\{ f \in T : \left| \frac{\frac{zf''(z)}{f'(z)}}{2\left(B-A\right)\gamma\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) - B\frac{zf''(z)}{f'(z)}} \right| < \beta \ \ (z \in U) \right\};$$

(iii) $T_{A,B}\left(f, z + \sum\limits_{k=2}^{\infty} k^n z^k, \alpha, \beta, \gamma\right) = T_{A,B}\left(n, \alpha, \beta, \gamma\right)$

$$= \left\{ f \in T : \left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{2\left(B-A\right)\gamma\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right) - B\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\},$$

for $n \in \mathbb{N}_0$ and where D^n is the Salagean operator (see [20]);
(iv) $T_{A,B}\left(f, z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n z^k, \alpha, \beta, \gamma\right)=T_{A,B}(n, \lambda, \alpha, \beta, \gamma)$
$=\left\{f \in T:\left|\frac{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1}{2(B-A)\gamma\left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-\alpha\right)-B\left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1\right)}\right|<\beta\ (z \in U)\right\}$, for $\lambda>0$, $n \in \mathbb{N}_0$ and where D_λ^n is the Al-Oboudi operator (see [2]);
(v) $T_{A,B}\left(f, z+\sum_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda} z^k, \alpha, \beta, \gamma\right)=S_{A,B}(\lambda, \alpha, \beta, \gamma)$
$=\left\{f \in T:\left|\frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1}{2(B-A)\gamma\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-\alpha\right)-B\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1\right)}\right|<\beta\ (z \in U)\right\}$,
for $\lambda>-1$ and where D^λ is the λ-th order Ruscheweyh derivative of $f(z) \in S$ (see [1], [19]);
(vi) $T_{A,B}\left(f, z+\sum_{k=2}^{\infty}\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k, \alpha, \beta, \gamma\right)=$
$\left\{f \in T:\left|\frac{\frac{z(J^m(\lambda,\ell) f(z))'}{J^m(\lambda,\ell) f(z)}-1}{2(B-A)\gamma\left(\frac{z(J^m(\lambda,\ell) f(z))'}{J^m(\lambda,\ell) f(z)}-\alpha\right)-B\left(\frac{z(J^m(\lambda,\ell) f(z))'}{J^m(\lambda,\ell) f(z)}-1\right)}\right|<\beta\,(z \in U)\right\}$,
for $\lambda \geq 0$, $\ell>-1$, $m \in \mathbb{Z}=\{0, \pm 1, \ldots\}$ and where $J^m(\lambda, \ell)$ is the Prajapat operator (see [17], [4], [6], with $p=1$);
(vii) $T_{A,B}\left(f, z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}\cdot\frac{1}{(k-1)!} z^k, \alpha, \beta, \gamma\right)=$
$\left\{f \in T:\left|\frac{\frac{z(H_{q,s}(\alpha_1) f(z))'}{H_{q,s}(\alpha_1) f(z)}-1}{2(B-A)\gamma\left(\frac{z(H_{q,s}(\alpha_1) f(z))'}{H_{q,s}(\alpha_1) f(z)}-\alpha\right)-B\left(\frac{z(H_{q,s}(\alpha_1) f(z))'}{H_{q,s}(\alpha_1) f(z)}-1\right)}\right|<\beta\,(z \in U)\right\}$,
for $\alpha_i>0$, $i=1,\ldots,q$, $\beta_j>0$, $j=1,\ldots,s$, $q \leq s+1$, $q,s \in \mathbb{N}_0$ and where $H_{q,s}(\alpha_1) f(z)$ is the Dzoik-Srivastava operator (see [5]).

Now we recall the following lemma and definition which are very needed for our study.
Lemma 1.([7]) *Let the function $f(z)$ be defined by (1.8). Then $f(z)$ is in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \Psi(\alpha, \beta, \gamma, k, A, B)\left|a_k\right| \leq 2(B-A)\beta\gamma(1-\alpha), \tag{1.9}$$

where

$$\Psi(\alpha, \beta, \gamma, k, A, B)=[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\, b_k\ . \tag{1.10}$$

Definition 1.([12])(Subordination) For analytic functions f and g with

$f(0) = g(0)$, f is said to be subordinate to g, denote by $f \prec g$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$, for all $z \in U$.

2. Partial Sums

Unless otherwise mentioned, we assume in the reminder of this paper that $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $z \in U$, g is given by (1.5) and $\Psi(\alpha, \beta, \gamma, k, A, B)$ is given by (1.10).
Following the earlier works by Silverman [21] and Siliva [22] on partial sums of analytic functions, we consider in this section partial sums of functions in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$, $f_n(z)$ to $f(z)$, $f'(z)$ to $f'_n(z)$ and $f'_n(z)$ to $f'(z)$, respectively.

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in the class $S_{A,B}(f, g, \alpha, \beta, \gamma)$. Define the partial sums $f_1(z)$ and $f_n(z)$, by*

$$f_1(z) = z \text{ and } f_n(z) = z + \sum_{k=2}^{n} a_k z^k \qquad (n \in \mathbb{N}/\{1\}). \tag{2.1}$$

Suppose also that

$$\sum_{k=2}^{\infty} d_k |a_k| \leq 1,$$

where

$$d_k = \frac{\Psi(\alpha, \beta, \gamma, 2, A, B)}{2(B - A)\beta\gamma(1 - \alpha)}. \tag{2.2}$$

Then $f \in T_{A,B}(f, g, \alpha, \beta, \gamma)$. Furthermore,

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{d_{n+1}}, \;\; z \in U, \;\; n \in \mathbb{N}, \tag{2.3}$$

and

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{d_{n+1}}{1 + d_{n+1}}. \tag{2.4}$$

The result is sharp for the extremal function is given by

$$f(z) = z + \frac{z^{n+1}}{d_{n+1}}, \tag{2.5}$$

Proof. For d_k given by (2.2) it is easily to show that $d_{k+1} > d_k > 1$. Therefore we have

$$\sum_{k=2}^{n} |a_k| + d_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k |a_k| \leq 1, \tag{2.6}$$

by using (2.2). By setting

$$k_1(z) = d_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{d_{n+1}}\right)\right\}$$

$$= 1 + \frac{d_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}}, \tag{2.7}$$

and using (2.6), we have

$$\left|\frac{k_1(z) - 1}{k_1(z) + 1}\right| \le \frac{d_{n+1}\sum_{k=n+1}^{\infty}|a_k|}{2 - 2\sum_{k=2}^{n}|a_k| - d_{n+1}\sum_{k=n+1}^{\infty}|a_k|} \le 1 \ (z \in U) \tag{2.8}$$

which yields the assertion (2.3) of Theorem 1. For $z = re^{\frac{i\pi}{n}}$ that

$$\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{d_{n+1}} \to 1 - \frac{1}{d_{n+1}}$$

as $r \to 1^-$. Similarly, if we take

$$k_2(z) = (1 + d_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{d_{n+1}}{1 + d_{n+1}}\right\}$$

$$= 1 - \frac{(1 + d_{n+1})\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}, \tag{2.9}$$

and using (2.6), we have

$$\left|\frac{k_2(z) - 1}{k_2(z) + 1}\right| \le \frac{(1 + d_{n+1})\sum_{k=n+1}^{\infty}|a_k|}{2 - 2\sum_{k=2}^{\infty}|a_k| - (1 - d_{n+1})\sum_{k=n+1}^{\infty}|a_k|} \tag{2.10}$$

which leads us immediately to the assertion (2.4). This completes the proof of Theorem 1.

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_{A,B}(f, g, \alpha, \beta, \gamma)$ *satisfies the condition* (1.9), *then*

$$\Re\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge 1 - \frac{n+1}{d_{n+1}}. \tag{2.11}$$

The result is sharp for the extremal function is given by (2.5).

Proof. Let

$$\begin{aligned} k(z) &= d_{n+1}\left\{\frac{f'(z)}{f_n'(z)} - \left(1 - \frac{n+1}{d_{n+1}}\right)\right\} \\ &= \frac{1 + \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}} \\ &= 1 + \frac{\frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}}, \end{aligned}$$

using (2.6), we have

$$\left|\frac{k(z)-1}{k(z)+1}\right| \le \frac{\frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k|}{2-2\sum_{k=2}^{\infty} k\,|a_k| - \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k|} \le 1, \tag{2.12}$$

if

$$\sum_{k=2}^{n} k\,|a_k| + \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k\,|a_k| \le 1 \tag{2.13}$$

Since the left hand of (2.13) is bounded above by $\sum_{k=2}^{\infty} d_k\,|a_k|$ if

$$\sum_{k=2}^{n}(d_k-k)\,|a_k| + \sum_{k=n+1}^{\infty}(d_k - \frac{d_{n+1}}{n+1}k)\,|a_k| \ge 0. \tag{2.14}$$

This completes the proof of Theorem 2.

Theorem 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_{A,B}(f,g,\alpha,\beta,\gamma)$ *satisfis the condition* (1.9), *then*

$$\Re\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge 1 - \frac{d_{n+1}}{n+1+d_{n+1}}. \tag{2.15}$$

Proof. Let

$$\begin{aligned} k(z) &= [(k+1)+d_{n+1}]\left\{\frac{f_n'(z)}{f'(z)} - \frac{d_{n+1}}{n+1+d_{n+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1+\sum_{k=2}^{\infty} k a_k z^{k-1}}, \end{aligned}$$

and using (2.14), we have

$$\left|\frac{k(z)-1}{k(z)+1}\right| \le \frac{(1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k\,|a_k|}{2-2\sum_{k=2}^{\infty} k\,|a_k| - (1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k\,|a_k|} \le 1,$$

This completes the proof of Theorem 3.

3. Integral Means

In 1925, Littlewood [9] proved the following subordination theorem.

Lemma 2. *If the functions* f *and* g *are analytic in* U *with* $g \prec f$, *then for* $\delta > 0$, *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\phi})\right|^{\delta} d\phi \le \int_0^{2\pi} \left|f(re^{i\phi})\right|^{\delta} d\phi. \tag{3.1}$$

Using Lemma 1 and Lemma 2, we prove the following result.

Theorem 4. *Let* $f \in T_{A,B}(f, g, \alpha, \beta, \gamma)$, $\delta > 0$, $0 \le \alpha < 1$, $0 \le \gamma < 1$, $n \ge 0$ *and* $f_2(z)$ *is given by*

$$f_2(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z^2,$$

where $\Psi(\alpha, \beta, \gamma, 2, A, B)$ *is defined by (1.10). Then for* $z = re^{i\phi}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\delta \, d\phi \le \int_0^{2\pi} |f_2(z)|^\delta \, d\phi. \tag{3.2}$$

Proof. For $f(z)$ is given by (1.8), (3.2) is equivalent proving that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} |a_k| \, z^{k-1} \right|^\delta d\phi \le \int_0^{2\pi} \left| 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} \right|^\delta d\phi$$

By Lemma 2, it suffices to show that

$$1 - \sum_{k=2}^{\infty} |a_k| \, z^{k-1} \prec 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} |a_k| \, z^{k-1} = 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} w(z). \tag{3.3}$$

From (3.3) and (1.9), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=2}^{\infty} \frac{\Psi(\alpha, \beta, \gamma, k, A, B)}{2(B-A)\beta\gamma(1-\alpha)} a_k z^{k-1} \right| \\ &\le |z| \sum_{k=2}^{\infty} \frac{\Psi(\alpha, \beta, \gamma, k, A, B)}{2(B-A)\beta\gamma(1-\alpha)} |a_k| \le |z| \end{aligned}$$

This completes the proof of Theorem 4.

4. Square Root Transformation

Defintion 2. Let $f \in S$ and $h(z) = \sqrt{f(z^2)}$, then $h \in S$ and $h(z) = z + \sum_{k=2}^{\infty} c_{2k-1} z^{2k-1}$ for $|z| < 1$, the function m is called a square root transformation of f.

Theorem 5. *Let the function* $f(z)$ *defined by (1.8) be in the class* $T_{A,B}(f, g, \alpha, \beta, \gamma)$, $2(B-A)\beta\gamma(1-\alpha) \le \Psi(\alpha, \beta, \gamma, 2, A, B)$ *and* h *be the square root transformation* *of f, then*

$$r\sqrt{1-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2}\leq|h(z)|$$

and

$$|h(z)|\leq r\sqrt{1+\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2}, \tag{4.1}$$

where

$$f(z)=z-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}z^2\ \ (|z|=\pm r). \tag{4.2}$$

Proof. In the view of Lemma 1, we have

$$r^2-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^4\leq\left|f(z^2)\right|\leq r^2+\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^4 \tag{4.3}$$

Using (4.3) in the defintion 2 we find

$$\begin{aligned}|h(z)| &= \sqrt{|f(z^2)|}\\ &\leq \sqrt{r^2+\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^4}\end{aligned}$$

$$=r\sqrt{1+\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2}. \tag{4.4}$$

Since, $2\,(B-A)\,\beta\gamma\,(1-\alpha)\leq\Psi(\alpha,\beta,\gamma,2,A,B)$ and $r=|z|<1$, we have

$$1+\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2\geq1-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2 \tag{4.5}$$

and hence,

$$\begin{aligned}|h(z)| &= \sqrt{|f(z^2)|}\\ &\geq \sqrt{r^2-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^4}\end{aligned}$$

$$=r\sqrt{1-\frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\Psi(\alpha,\beta,\gamma,2,A,B)}r^2}. \tag{4.6}$$

This completes the proof of Theorem 5.

5. Integral Transform of the Class $T_{A,B}(f,g,\alpha,\beta,\gamma)$

For $f \in S$ we define the integral transform

$$V_\mu(f(z)) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt, \tag{5.1}$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator [3], and

$$\mu(t) = \frac{(c+1)^\eta}{\Gamma(\eta)} t^c \left(\log\frac{1}{t}\right)^{\eta-1}, \ c > -1, \ \eta \geq 0, \tag{5.2}$$

which gives the Komatu operator [8], see also [15].

Now we show that the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$ is closed under $V_\mu(f(z))$.

Theorem 6. *Let the function $f(z)$ defined by* (1.8) *be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$, then $V_\mu(f(z)) \in T_{A,B}(f,g,\alpha,\beta,\gamma)$.*

Proof. From (5.1), we have

$$\begin{aligned} V_\mu(f(z)) &= \frac{(c+1)^\eta}{\Gamma(\eta)}\int_0^1 (-1)^{\eta-1}t^c(\log t)^{\eta-1}\left(z - \sum_{k=2}^{\infty} a_k z^k t^{k-1}\right)dt \\ &= \frac{(-1)^{\eta-1}(c+1)^\eta}{\Gamma(\eta)} \lim_{r\to 0^+}\left\{\int_r^1 t^c(\log t)^{\eta-1}\left(z - \sum_{k=2}^{\infty} a_k z^k t^{k-1}\right)dt\right\} \\ &= z - \sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^\eta a_k z^k. \end{aligned}$$

We need to prove that

$$\sum_{k=2}^{\infty}\frac{\Psi(\alpha,\beta,\gamma,k,A,B)}{2(B-A)\beta\gamma(1-\alpha)}\left(\frac{c+1}{c+k}\right)^\eta a_k \leq 1. \tag{5.3}$$

On the other hand by Lemma 1, $f(z) \in T_{A,B}(f,g,\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}\frac{\Psi(\alpha,\beta,\gamma,k,A,B)}{2(B-A)\beta\gamma(1-\alpha)} a_k \leq 1.$$

Since $\frac{c+1}{c+k} < 1$, therefore (5.3) holds and the proof of Theorem 6 is completed.

Theorem 7. *Let the function $f(z)$ defined by (1.8) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $V_\mu(f(z))$ is starlike of order ξ $(0 \leq \xi < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{k\geq 2}\left[\left(\frac{c+k}{c+1}\right)^\eta \frac{(1-\xi)\Psi(\alpha,\beta,\gamma,k,A,B)}{(k-\xi)[2(B-A)\beta\gamma(1-\alpha)]}\right]^{\frac{1}{k-1}}. \tag{5.4}$$

Proof. It is sufficient to show that

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right| \le 1-\xi \text{ for } |z|<r_1, \tag{5.5}$$

where r_1 is given by (5.4). Indeed we find, again from the definition (1.8) that

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right| \le \frac{\sum_{k=2}^{\infty}(1-k)\left(\frac{c+1}{c+k}\right)^{\eta} a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\eta} a_k |z|^{k-1}}.$$

Thus

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right| \le 1-\xi,$$

if

$$\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\eta}\left(\frac{k-\xi}{1-\xi}\right) a_k |z|^{k-1} \le 1. \tag{5.6}$$

But, by Lemma 1, (5.6) will be true if

$$\left(\frac{c+1}{c+k}\right)^{\eta}\left(\frac{k-\xi}{1-\xi}\right)|z|^{k-1} \le \frac{\Psi(\alpha,\beta,\gamma,k,A,B)}{2(B-A)\beta\gamma(1-\alpha)},$$

that is, if

$$r_1=|z| \le \left[\left(\frac{c+k}{c+1}\right)^{\eta}\frac{(1-\xi)\Psi(\alpha,\beta,\gamma,k,A,B)}{(k-\xi)[2(B-A)\beta\gamma(1-\alpha)]}\right]^{\frac{1}{k-1}}. \tag{5.7}$$

Theorem 7 follwos easily from (5.7).

Theorem 8. *Let the function $f(z)$ defined by (1.8) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $V_\mu(f(z))$ is convex of order ξ $(0\le\xi<1)$ in the disc $|z|<r_2$, where*

$$r_2=\inf_{k\ge2}\left[\left(\frac{c+k}{c+1}\right)^{\eta}\frac{(1-\xi)\Psi(\alpha,\beta,\gamma,k,A,B)}{k(k-\xi)[2(B-A)\beta\gamma(1-\alpha)]}\right]^{\frac{1}{k-1}}. \tag{5.8}$$

Remark. (i) Putting $g=\dfrac{z}{(1-z)^{2(1-\delta)}}$ $(0\le\delta<1)$, in the above results, we obtain the corresponding results obtained by Magesh et al. [10, with $m=0$]);
(ii) Specializing the function $g(z)$, we obtain different results corresponding to the classes mentioned in the introduction.

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