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Oscillation Results for Second Order Nonlinear Differential Equation with Delay and Advanced Arguments

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Abstract. In this paper we study the oscillation criteria for the second order nonlinear differential equation with delay and advanced arguments of the form

$$([x(t)+a(t)x(t-\sigma_1)+b(t)x(t+\sigma_2)]^{\alpha})''+q(t)x^{\beta}(t-\tau_1)+p(t)x^{\gamma}(t+\tau_2)=0,\ t\geq t_0$$

where $\sigma_1,\ \sigma_2,\ \tau_1$ and τ_2 are nonnegative constants and $\alpha,\ \beta$ and γ are the ratios of odd positive integers. Examples are provided to illustrate the main results.

1. Introduction

In this paper, we consider the following second order nonlinear differential equation with delay and advanced arguments of the form

$$([x(t)+a(t)x(t-\sigma_1)+b(t)x(t+\sigma_2)]^{\alpha})''+q(t)x^{\beta}(t-\tau_1)+p(t)x^{\gamma}(t+\tau_2)=0 \tag{1.1}$$

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for all $t \geq t_0$, subject to the following conditions:

(A_1) $a(t)$ and $b(t)$ are non negative and twice continuously differentiable functions on $[t_0, \infty)$ and there exist constants a and b such that $a(t) \leq a < \infty$ and $b(t) \leq b < \infty$;

(A_2) $q(t)$ and $p(t)$ are nonnegative continuous functions on $[t_0, \infty)$ and are not identically zero for infinitely many values of t;

(A_3) $\sigma_1,\ \sigma_2,\ \tau_1$ and τ_2 are nonnegative constants and $\alpha,\ \beta$ and γ are the ratios of odd positive integers.

By a solution of equation (1.1), we mean a function $x(t) \in C[T_x, \infty)$ defined for all $t \geq t_0 - \max(\sigma_1, \tau_1)$ and satisfying the equation (1.1) for all $t \geq T_x \geq t_0$. A nontrivial solution of equation (1.1) is said to be oscillatory if it has infinitely many zeros on $[t_0, \infty)$, otherwise it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its nontrivial solutions are oscillatory.

In recent years, many results have been obtained on the oscillation of solutions of different types of differential equations, see [2, 3, 7, 8, 9, 14, 17, 18, 19, 22] .

In 1987 the authors in [15] and in 1992 the authors in [9] obtained some oscillation criteria for the second order nonlinear differential equation of the form

(1.2)
$$\big(r(t)|((x(t)+p(t)x\tau(t))')^{\gamma-1}|(x(t)+p(t)x\tau(t))'\big)' + q(t)|x(\sigma(t))|^{\gamma-1}x(\sigma(t)) = 0.$$

In 2003 the authors in [7] found some sufficient conditions for the oscillation of the second order half-linear differential equation of the form

$$\big(r(t)|x'(t)|^{\gamma-1}x'(t)\big)' + q(t)|x(\tau(t))|^{\gamma-1}x(\tau(t)) = 0,\ t \geq t_0 \tag{1.3}$$

by using Riccati transformation.

In [3, 8, 18] the authors obtained some oscillation criteria for the following differential equation with mixed arguments

$$(x(t)+p(t)x(t-\tau_1)+q(t)x(t+\tau_2))'' = q_1(t)x(t-\sigma_1)+q_2(t)x(t+\sigma_2),\ t \geq t_0. \tag{1.4}$$

In [12, 23], the authors established some oscillation results for the following higher order neutral functional differential equation of the form

$$(x(t)+ax(t-h)+Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0,\ t \geq 0 \tag{1.5}$$

where q and Q are nonnegative real constants.

In [21], the authors studied the oscillation of equation (1.1) for the case $0 < \gamma = \beta < 1,\ \gamma = \beta = 1,\ 1 \leq \gamma = \beta > \alpha,\ 1 \leq \gamma = \beta < \alpha$. Motivated by this we study the oscillation of equation (1.1) for the cases $0 < \beta \leq 1,\ \gamma \geq 1$ and $\beta \geq 1,\ 0 \leq \gamma \leq 1$ and different values of a and b.

In the sequel when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large values of t.

2. Oscillation Theorems

In this section, we establish some sufficient conditions for the oscillation of all the solutions of equation (1.1). For simplicity, we use the following notations throughout this paper without further mention.

$$\begin{aligned} z(t) &= [x(t) + a(t)x(t-\sigma_1) + b(t)x(t+\sigma_2)]^{\alpha}; \\ Q(t) &= \min(q(t), q(t-\sigma_1), q(t+\sigma_2)); \end{aligned}$$

and

$$P(t) = \min(p(t), p(t-\sigma_1), p(t+\sigma_2)).$$

We begin with the following lemmas, which will be useful in proving our main theorems.

Lemma 2.1. *If* $A \geq 0,\ B \geq 0$ *and* $\delta \geq 1$, *then*

$$A^{\delta} + B^{\delta} \geq \frac{1}{2^{\delta-1}}(A+B)^{\delta}. \tag{2.1}$$

Lemma 2.2. *If* $A \geq 0,\ B \geq 0$ *and* $0 < \delta \leq 1$, *then*

$$A^{\delta} + B^{\delta} \geq (A+B)^{\delta}. \tag{2.2}$$

The proofs of above two lemmas can be found in [14].

Lemma 2.3. *If* $x(t)$ *is a positive solution of equation* (1.1), *then* $z(t) > 0, z'(t) > 0$ *and* $z''(t) \leq 0$ *eventually.*

Lemma 2.4. *If* $y(t) > 0,\ y^{'}(t) > 0$ *and* $y^{"}(t) \leq 0$ *for all* $t \geq t_0$ *then* $y(t) \geq \frac{t}{2}y^{'}(t)$. *for all* $t \geq t_1 \geq t_0$

The proofs of last two lemmas are elementary and hence omitted.

Lemma 2.5. *If*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} Q(s)ds > \frac{1}{e},$$

then the differential inequality

$$y^{'}(t) + Q(t)y(t-\sigma) < 0 \text{ for all } t \geq t_0$$

has no positive solution.

Proof. The proof can be found in [13]. □

Theorem 2.6. *Assume that $\beta > 1$, $0 \le \gamma < 1$, $a \le 1$, $b \le 1$ and $\beta > \alpha > \gamma$. If the differential inequality*

$$(2.3)\quad \liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_2}(s)Q^{\eta_1}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\beta-1})^{\eta_1}\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\gamma+b^\gamma)}{e}$$

where $\eta_1 = \dfrac{\alpha-\gamma}{\beta-\gamma}$, $\eta_2 = \dfrac{\beta-\alpha}{\beta-\gamma}$ *and* $\tau = \max(\tau_1, \tau_2)$ *holds, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t)$ is a positive solution. Then there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$ and $x(t-\sigma_1) > 0$ for all $t \ge t_1$. Then $z(t) > 0$ for all $t \ge t_1$.

Define a function $y(t)$ by

$$y(t) = z(t) + a^\gamma z(t-\sigma_1) + b^\gamma z(t+\sigma_2)$$

for all $t \ge t_1$. Now

$$\begin{aligned} 0 &= y''(t) + q(t)x^\beta(t-\tau_1) + p(t)x^\gamma(t+\tau_2) + a^\gamma q(t-\sigma_1)x^\beta(t-\sigma_1-\tau_1) \\ &\quad + a^\gamma p(t-\sigma_1)x^\gamma(t-\sigma_1+\tau_2) + b^\gamma q(t+\sigma_2)x^\beta(t+\sigma_2-\tau_1) \\ &\quad + b^\gamma p(t+\sigma_2)x^\gamma(t+\sigma_2+\tau_2) \\ &\ge y''(t) + Q(t)\left[x^\beta(t-\tau_1) + a^\gamma x^\beta(t-\sigma_1-\tau_1) + b^\gamma x^\beta(t+\sigma_2-\tau_1)\right] + \\ &\quad P(t)\left[x^\gamma(t+\tau_2) + a^\gamma x^\gamma(t-\sigma_1+\tau_2) + b^\gamma x^\gamma(t+\sigma_2+\tau_2)\right] \text{ for all } t \ge t_1. \end{aligned}$$

Using the fact $a \le 1$, $b \le 1$, $\beta > 1$ and $0 < \gamma < 1$, the last inequality becomes

$$\begin{aligned} 0 &\ge y''(t) + Q(t)\left[x^\beta(t-\tau_1) + a^\beta x^\beta(t-\sigma_1-\tau_1) + \frac{b^\beta}{2^{\beta-1}}x^\beta(t+\sigma_2-\tau_1)\right] + \\ &\quad P(t)\left[x^\gamma(t+\tau_2) + a^\gamma x^\gamma(t-\sigma_1+\tau_2) + b^\gamma x^\gamma(t+\sigma_2+\tau_2)\right] \text{ for all } t \ge t_1. \end{aligned}$$

Now using the Lemma 2.2 and Lemma 2.1 twice on the first and second part of the right hand side of the last inequality, respectively, we have

$$(2.4)\qquad 0 \ge y''(t) + \frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\tau_1) + P(t)z^{\gamma/\alpha}(t+\tau_2) \text{ for all } t \ge t_1.$$

From Lemma 2.3, we have $z(t) > 0$ and $z'(t) > 0$ and therefore $y(t) > 0$ and $y'(t) \ge 0$. Now using the monotonicity of $z(t)$ in (2.4), we obtain

$$(2.5)\qquad 0 \ge y''(t) + \frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\tau) + P(t)z^{\gamma/\alpha}(t-\tau)$$

for all $t \ge t_1$.

Let $u_1\eta_1 = \dfrac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\tau)$ and $u_2\eta_2 = P(t)z^{\gamma/\alpha}(t-\tau)$. Using the arithmetic-geometric mean inequality $\dfrac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \geq (u_1^{\eta_1}u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}}$, the last inequality becomes

$$0 \geq y''(t) + \left(\frac{Q(t)}{4^{\beta-1}}\right)^{\eta_1} \eta_1^{-\eta_1} P^{\eta_2}(t)\eta_2^{-\eta_2} z(t-\tau) \text{ for all } t \geq t_1. \tag{2.6}$$

From the definition of $y(t)$, we have

$$\begin{aligned} y(t) &= z(t) + a^{\gamma}z(t-\sigma_1) + b^{\gamma}z(t+\sigma_2) && (2.7)\\ &\leq (1+a^{\gamma}+b^{\gamma})z(t+\sigma_2) \text{ for all } t \geq t_1. && (2.8)\end{aligned}$$

Using (2.8) in (2.6), we see that

$$0 \geq y''(t) + \left(\frac{Q(t)}{4^{\beta-1}}\right)^{\eta_1} \frac{P^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^{\gamma}+b^{\gamma})} y(t-\tau-\sigma_2) \text{ for all } t \geq t_1. \tag{2.9}$$

Using Lemma 2.4, the last inequality becomes

$$0 \geq y''(t) + \left(\frac{Q(t)}{4^{\beta-1}}\right)^{\eta_1} \frac{P^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^{\gamma}+b^{\gamma})} \frac{(t-\tau-\sigma_2)}{2} y^{'}(t-\tau-\sigma_2) \tag{2.10}$$

for all $t \geq t_1$. By taking $w(t) = y^{'}(t)$, we see that $w(t)$ is a positive solution of the inequality

(2.11)

$$0 \geq w^{'}t) + \left(\frac{Q(t)}{4^{\beta-1}}\right)^{\eta_1} \frac{P^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^{\gamma}+b^{\gamma})} \frac{(t-\tau-\sigma_2)}{2} w(t-\tau-\sigma_2) \text{ for all } t \geq t_1.$$

Then by Lemma 2.5, we see that the last inequality has no positive solution. This contradiction completes the proof. □

Theorem 2.7. *Assume that* $\gamma > 1,\ 0 < \beta < 1,\ a \geq 1,\ b \geq 1$ *and* $\gamma > \alpha > \beta$. *If the differential inequality*

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_1}(s)Q^{\eta_2}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\gamma-1})^{\eta_1}\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^{\beta}+b^{\beta})}{e} \tag{2.12}$$

where $\eta_1 = \dfrac{\alpha-\beta}{\gamma-\beta}$, $\eta_2 = \dfrac{\gamma-\alpha}{\gamma-\beta}$ *and* $t = \max(\tau_1, \tau_2)$ *holds, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a positive solution of equation (1.1) (since the proof of other case $x(t)$ negative is similar). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$ and $x(t-\sigma_1) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ and from equation (1.1), we have $z'(t) > 0$ for all $t \geq t_1$. Define a function $y(t)$ by

$$y(t) = z(t) + a^{\beta}z(t-\sigma_1) + b^{\beta}z(t+\sigma_2) \tag{2.13}$$

for all $t \geq t_1$. Then $y(t) > 0$ and $y'(t) > 0$ for all $t \geq t_1$. Now

$$\begin{aligned} 0 &= y''(t) + q(t)x^{\beta}(t-\tau_1) + p(t)x^{\gamma}(t+\tau_2) + a^{\beta}q(t-\sigma_1)x^{\beta}(t-\sigma_1-\tau_1) + \\ &\quad a^{\beta}p(t-\sigma_1)x^{\gamma}(t-\sigma_1+\tau_2) + b^{\beta}q(t+\sigma_2)x^{\beta}(t+\sigma_2-\tau_1) + b^{\beta}x^{\beta}(t+\sigma_2+\tau_2) \\ &\geq y''(t) + Q(t)\left[x^{\beta}(t-\tau_1) + a^{\beta}x^{\beta}(t-\tau_1-\sigma_1) + b^{\beta}x^{\beta}(t-\tau_1+\sigma_2)\right] + \\ &\quad P(t)\left[x^{\gamma}(t+\tau_2) + a^{\beta}x^{\gamma}(t+\tau_2-\sigma_1) + b^{\beta}x^{\gamma}(t+\tau_2+\sigma_2)\right] \end{aligned}$$

for all $t \geq t_1$. Since $a \geq 1$, $b \geq 1$, $\beta < 1$, and $\gamma \geq 1$ the last inequality becomes

$$\begin{aligned} 0 &\geq y''(t) + Q(t)\left[x^{\beta}(t-\tau_1) + a^{\beta}x^{\beta}(t-\tau_1-\sigma_1)\right. \\ &\quad \left. + b^{\beta}x^{\beta}(t-\tau_1-\sigma_1) + b^{\beta}x^{\beta}(t-\tau_1+\sigma_2)\right] \\ &\quad + P(t)\left[x^{\gamma}(t+\tau_2) + a^{\gamma}x^{\gamma}(t+\tau_2-\sigma_1) + \frac{b^{\gamma}}{2^{\gamma-1}}x^{\gamma}(t+\tau_2+\sigma_2)\right] \end{aligned} \tag{2.14}$$

for all $t \geq t_1$. Now using the Lemmas 2.1 and 2.2 twice on the first and second part of right hand side of the last inequality, respectively, we have

$$0 \geq y''(t) + Q(t)z^{\beta/\alpha}(t-\tau_1) + \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\tau_2). \tag{2.15}$$

Since $z(t)$ is nondecreasing the inequality (2.15) becomes

$$0 \geq y''(t) + Q(t)z^{\beta/\alpha}(t-\tau) + \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t-\tau) \tag{2.16}$$

for all $t \geq t_1$.

Let $u_2\eta_2 = Q(t)z^{\beta/\alpha}(t-\tau)$ and $u_1\eta_1 = \dfrac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t-\tau)$. Then using arithmetic and geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \geq (u_1^{\eta_1}u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}},$$

the last inequality becomes

$$0 \geq y''(t) + Q^{\eta_2}(t)\left(\frac{P(t)}{4^{\gamma-1}}\right)^{\eta_1}\eta_1^{-\eta_1}\eta_2^{-\eta_2}z(t-\tau) \text{ for all } t \geq t_1 \tag{2.17}$$

Now from the monotonicity of $z(t)$, we have

$$y(t) = z(t) + a^{\beta}z(t-\sigma_1) + b^{\beta}z(t+\sigma_2) \leq (1+a^{\beta}+b^{\beta})z(t+\sigma_2) \text{ for all } t \geq t_1. \tag{2.18}$$

Using the inequality (2.18) in the inequality (2.17), we see that

$$0 \geq y''(t) + \left(\frac{P(t)}{4^{\gamma-1}}\right)^{\eta_1}\frac{Q^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^{\beta}+b^{\beta})}y(t-\tau-\sigma_2) \text{ for all } t \geq t_1. \tag{2.19}$$

Using Lemma 2.4, the inequality (2.19) becomes

$$0 \geq y''(t) + \left(\frac{P(t)}{4^{\gamma-1}}\right)^{\eta_1} \frac{Q^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^\beta+b^\beta)} \frac{(t-\tau-\sigma_2)}{2} y'(t-\tau-\sigma_2) \tag{2.20}$$

for all $t \geq t_1$. By taking $w(t) = y'(t)$, we see that $w(t)$ is a positive solution of the inequality

$$0 \geq w' t) + \left(\frac{P(t)}{4^{\gamma-1}}\right)^{\eta_1} \frac{Q^{\eta_2}(t)\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+a^\beta+b^\beta)} \frac{(t-\tau-\sigma_2)}{2} w(t-\tau-\sigma_2) \tag{2.21}$$

for all $t \geq t_1$. But by Lemma 2.5, we see that the inequality (2.21) has no positive solution. This contradiction completes the proof. □

Theorem 2.8. *Assume that* $\beta \geq 1$, $0 \leq \gamma < 1$, $a \geq 1$, $b < 1$ *and* $\beta > \alpha > \gamma$. *If the differential inequality*

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_2}(s)Q^{\eta_1}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\beta-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\beta+b^\gamma)}{e} \tag{2.22}$$

where $\eta_1 = \dfrac{\alpha-\gamma}{\beta-\gamma}$, $\eta_2 = \dfrac{\beta-\alpha}{\beta-\gamma}$ *and* $\tau = \max(\tau_1, \tau_2)$ *holds, then every solution of equation* (1.1) *is oscillatory.*

Theorem 2.9. *Assume that* $\beta \geq 1$, $0 \leq \gamma < 1$, $a < 1$, $b \geq 1$ *and* $\beta > \alpha > \gamma$.

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_2}(s)Q^{\eta_1}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\beta-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\gamma+b^\beta)}{e} \tag{2.23}$$

where $\eta_1 = \dfrac{\alpha-\gamma}{\beta-\gamma}$, $\eta_2 = \dfrac{\beta-\alpha}{\beta-\gamma}$ *and* $\tau = \max(\tau_1, \tau_2)$ *holds , then every solution of equation* (1.1) *is oscillatory.*

The proofs of Theorem 2.8 and Theorem 2.9 are similar to that of Theorem 2.6, and hence the details are omitted.

Theorem 2.10. *Assume that* $\gamma \geq 1$, $0 < \beta < 1$, $a < 1$, $b \geq 1$ *and* $\beta > \alpha > \gamma$.

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_1}(s)Q^{\eta_2}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\gamma-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\beta+b^\gamma)}{e} \tag{2.24}$$

where $\eta_1 = \dfrac{\beta-\alpha}{\beta-\gamma}$, $\eta_2 = \dfrac{\alpha-\gamma}{\beta-\gamma}$ *and* $\tau = \max(\tau_1, \tau_2)$ *holds, then every solution of equation* (1.1) *is oscillatory.*

Theorem 2.11. *Assume that $\gamma \geq 1$, $0 < \beta < 1$, $a \geq 1$, $b < 1$ and $\beta > \alpha > \gamma$. If the differential inequality*

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_1}(s)Q^{\eta_2}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\gamma-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\gamma+b^\beta)}{e} \tag{2.25}$$

where $\eta_1 = \dfrac{\beta-\alpha}{\beta-\gamma}$, $\eta_2 = \dfrac{\alpha-\gamma}{\beta-\gamma}$ and $\tau = \max(\tau_1,\tau_2)$ holds, then every solution of equation (1.1) is oscillatory.

The proofs of Theorem 2.10 and Theorem 2.11 are similar to that of Theorem 2.7, and hence the details are omitted.

Example 2.12. Consider the differential equation

$$\left(x(t)+\frac{1}{27}x(t-1)+x(t+2)\right)'' + \frac{q}{t}x^3(t-2)+\frac{p}{t}x^{1/3}(t+1) = 0, \text{ for all } t \geq 2,$$

where q and p are positive constants. Here $a = \dfrac{1}{27}$, $b = 1$, $q(t) = \dfrac{q}{t}$, $p(t) = \dfrac{p}{t}$, $\alpha = 1$, $\beta = 3$, $\gamma = \frac{1}{3}, \sigma_1 = 1, \sigma_2 = 2, \tau_1 = 2$ and $\tau_2 = 1$.

Then $\eta_1 = \dfrac{\alpha-\gamma}{\beta-\gamma} = \dfrac{1}{4}$, $\eta_2 = \dfrac{\beta-\alpha}{\beta-\gamma} = \dfrac{3}{4}$ and $\tau = \max(\tau_1,\tau_2) = 2$,

$$\begin{aligned} Q(t) &= \min\left(\frac{q}{t},\frac{q}{t-1},\frac{q}{t+2}\right) = \frac{q}{t+2} \\ P(t) &= \min\left(\frac{p}{t},\frac{p}{t-1},\frac{p}{t+2}\right) = \frac{p}{t+2} \end{aligned}$$

By Theorem 2.9 if

$$\liminf_{t\to\infty} \int_{t-\tau-\sigma_2}^{t} P^{\eta_2}(s)Q^{\eta_1}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\beta-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^\gamma+b^\beta)}{e}, \tag{2.26}$$

then every solution of equation (2.27) is oscillatory. That is, if $3^{\frac{1}{4}}p^{\frac{3}{4}}q^{\frac{1}{4}} > \dfrac{14}{e}$, then every solution of equation (1.1) is oscillatory.

Example 2.13. Consider the differential equation

$$\left(x(t)+x(t-1)+\frac{1}{27}x(t+2)\right)'' + \frac{q}{t}x^{\frac{1}{3}}(t-2)+\frac{p}{t}x(t+1) = 0, \text{ for all } t \geq 2,$$

where q and p are positive constants. Here $a = 1$, $b = \dfrac{1}{27}$, $q(t) = \dfrac{q}{t}$, $p(t) = \dfrac{p}{t}$, $\alpha = 1$, $\beta = \dfrac{1}{3}$, $\gamma = 3, \sigma_1 = 1, \sigma_2 = 2, \tau_1 = 2$ and $\tau_2 = 1$.

Then $\eta_1 = \dfrac{\alpha-\beta}{\gamma-\beta} = \dfrac{1}{4}$, $\eta_2 = \dfrac{\gamma-\alpha}{\gamma-\beta} = \dfrac{3}{4}$ and $\tau = \max(\tau_1, \tau_2) = 2$,

$$\begin{aligned} Q(t) &= \min\left(\frac{q}{t}, \frac{q}{t-1}, \frac{q}{t+2}\right) = \frac{q}{t+2} \\ P(t) &= \min\left(\frac{p}{t}, \frac{p}{t-1}, \frac{p}{t+2}\right) = \frac{p}{t+2} \end{aligned}$$

By Theorem 2.11 if

$$\liminf_{t\to\infty} \int\limits_{t-\tau-\sigma_2}^{t} P^{\eta_1}(s)Q^{\eta_2}(s)(s-\tau-\sigma_2)ds > \frac{2(4^{\gamma-1})\eta_1^{\eta_1}\eta_2^{\eta_2}(1+a^{\gamma}+b^{\beta})}{e}, \tag{2.27}$$

then every solution of equation (1.1) is oscillatory. That is, if $3^{\frac{1}{4}}p^{\frac{1}{4}}q^{\frac{3}{4}} > \dfrac{14}{e}$, then every solution of equation (1.1) is oscillatory.

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