KYUNGPOOK Math. J. 56(2016), 131-136
http://dx.doi.org/10.5666/KMJ.2016.56.1.131
pISSN 1225-6951 eISSN 0454-8124

Integral Formulas Involving a Product of Generalized Bessel Functions of the First Kind

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Abstract. The main object of this paper is to present two general integral formulas whose integrands are the integrand given in the integral formula (3) and a finite product of the generalized Bessel function of the first kind.

1. Introduction and Preliminaries

A remarkably large number of works on the Bessel functions have been provided

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Received February 23, 2015; revised October 2, 2015; accepted November 3, 2015.
2010 Mathematics Subject Classification: Primary 33B20, 33C20; Secondary 33B15, 33C05.
Key words and phrases: Gamma function, Hypergeometric function ${}_2F_1$, Generalized (Wright) hypergeometric functions ${}_p\Psi_q$, Generalized Bessel function of the first kind, Generalized Lauricella functions, Ramanujan Master Theorem, Garg and Mittal's integral formula.
The authors would like to express their profound gratitude to the reviewer's helpful and critical comments. The second-named author would like to express his deep thanks to NBHM (National Board of Higher Mathematics) for granting a Post-Doctoral Fellowship (sanction no. 2/40(37)/2014/R&D-II/14131). This work was supported by Dongguk University Research Fund of 2015.

by many researchers due mainly to the demonstrated applications in a wide range of research areas, for example, acoustics, radio physics, hydrodynamics, and atomic and nuclear physics (see, *e.g.*, [2],[3],[4],[5],[6],[7], [8],[15],[16],[17],[18],[22],[25]), even in analytic function theory (see, *e.g.*, [12],[23],[24]). A large number of integral formulas of a variety of special functions have been developed by many authors (see, *e.g.*, [1],[2],[7],[9],[10],[11],[14],[16],[18]). Also many integral representations for the Bessel functions have been presented (see, *e.g.*, [1],[2],[7],[8],[12]).

Motivated by the works of Ali [1], Garg and Mittal [14], Choi and Agarwal [7], Deniz *et al.* [12], and Srivastava *et al.* [22], here, in this paper, we aim at presenting two generalized integral formulas involving the generalized Bessel function $w_\nu(z)$ of the first kind, which are expressed in terms of the generalized Lauricella functions (4), by using the standard inversion of order method in a straightforward manner. Throughout this paper let $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive integers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Recall the generalized Bessel function $w_\nu(z)$ of the first kind defined by the following series (see, *e.g.*, [3, p. 10, Eq. (1.15)]; see also [4, 5, 6], [12, Eq. (1.7)] and [16, p. 2, Eq. (8)]):

$$w_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+\frac{1+b}{2})}, \tag{1}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $b, c, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$, and $\Gamma(z)$ is the familiar Gamma function (see, *e.g.*, [19, Section 1.1]). Here the multiple-valued function $\left(\frac{z}{2}\right)^{\nu+2k}$ may be assumed to take its principal branch for each $k \in \mathbb{N}_0$. It is noted that the special case of (1) when $b = 1$ and $c = 1$ reduces immediately to the Bessel function $J_\nu(z)$ of the first kind as follows:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \tag{2}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$. For more detailed special cases of (1), see also [12].

Also we need to recall the following integral formula (see, *e.g.*, [17]):

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{3}$$

provided $0 < \Re(\mu) < \Re(\lambda)$. Srivastava *et al.* [22] showed that the integral formula (3) is a change-of-variable version of a much simpler looking integral formula [22, p. 115, Eq. (14)], which Ramanujan deduced as an application of his Master Theorem.

The generalized Lauricella functions (see, *e.g.*, [21, p. 36, Eq. (19)]) which is defined by (*cf.* Srivastava and Daoust [20, p. 454]; see also [21, p. 37] and [9])

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left(\begin{array}{l} [(a):\theta^{(1)},\ldots,\theta^{(n)}]: \\ {[(c):\psi^{(1)},\ldots,\psi^{(n)}]:} \end{array}\right.$$

$$\left.\begin{array}{l} [(b)^{(1)}:\phi^{(1)}];\ldots;[(b)^{(n)}:\phi^{(n)}]; \\ {[(d)^{(1)}:\delta^{(1)}];\ldots;[(d)^{(n)}:\delta^{(n)}];} \end{array} z_1,\ldots,z_n\right) \tag{4}$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \Omega(k_1,\ldots,k_n)\,\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!},$$

where, for convenience,

$$\Omega(k_1,\ldots,k_n) = \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta_j^{(1)}+\cdots+k_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{k_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{k_1\psi_j^{(1)}+\cdots+k_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{k_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{k_n\delta_j^{(n)}}}, \tag{5}$$

the coefficients

$$\begin{cases} \theta_j^{(m)}\ (j=1,\ldots,A);\ \phi_j^{(m)}\ (j=1,\ldots,B^{(m)}); \\ \psi_j^{(m)}\ (j=1,\ldots,C);\ \delta_j^{(m)}\ (j=1,\cdots,D^{(m)});\ \forall\, m\in\{1,\ldots,n\} \end{cases} \tag{6}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters

$$b_j^{(m)} \quad (j=1,\ldots,B^{(m)}); \quad \forall\, m\in\{1,\ldots,n\},$$

with similar interpretations for (c) and $(d^{(m)})$ $(m=1,\ldots,n)$; *et cetera*.

For the details of convergence of (4), the reader may be referred (for example) to the earlier work by Srivastava and Daoust [20].

2. Main Results

We establish two (presumably) new generalized integral formulas whose integrands are a finite product of the generalized Bessel functions (1) of the first kind and the integrand in the integral formula (3), which are expressed in terms of the generalized Lauricella functions (4), asserted by the following theorems.

Theorem 1. *The following integral formula holds true: For $x > 0$, λ, μ, ν_j, b_j, $c_j \in \mathbb{C}$ with $\Re(\nu_j) > -1$ and $0 < \Re(\mu) < \Re(\lambda + \nu_j)$ $(j = 1, 2, \ldots, n)$,*

(2.1)
$$\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} \prod_{j=1}^n \omega_{\nu_j} \left(\frac{y_j}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$
$$= 2^{1-\mu-\boldsymbol{\nu_s}}\, a^{\mu-\lambda-\boldsymbol{\nu_s}}\, (\lambda + \boldsymbol{\nu_s})\, \frac{\Gamma(2\mu)\,\Gamma(\lambda - \mu + \boldsymbol{\nu_s})}{\Gamma(1 + \lambda + \mu + \boldsymbol{\nu_s})} \left\{\prod_{j=1}^n \frac{{y_j}^{\nu_j}}{\Gamma\left(\nu_j + \frac{1+b_j}{2}\right)}\right\}$$
$$\times F^{2:0,0,\ldots,0}_{2:1,1,\ldots,1}\left[\begin{array}{l} \left[1 + \lambda + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right], \left[\lambda - \mu + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right] : \\ \left[1 + \lambda + \mu + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right], \left[\lambda + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right] : \end{array}\right.$$
$$\left.\begin{array}{lll} \overline{\qquad\quad} & ; \cdots ; & \overline{\qquad\quad} \\ \left[\nu_1 + \frac{1+b_1}{2} : 1\right] & ; \cdots ; & \left[\nu_n + \frac{1+b_n}{2} : 1\right] \end{array} ; \frac{-c_1 y_1^2}{4a^2}, \ldots, \frac{-c_n y_n^2}{4a^2}\right],$$

where

$$\boldsymbol{\nu_s} := \sum_{j=1}^n \nu_j. \tag{2.2}$$

Proof. Using the series definition (1) to the integrand of (2.1) and then interchanging the order of the integral sign and the summation, and finally applying the integral formula (3) to the resulting integrals, we can get the expression as in the right-hand side of (2.1). So the detailed account of its proof is omitted. □

Theorem 2. *The following integral formula holds true: For $x > 0$, λ, μ, ν_j, b_j, $c_j \in \mathbb{C}$ with $\Re(\nu_j) > -1$ and $0 < \Re(\mu) < \Re(\lambda + \nu_j)$ $(j = 1, 2, \ldots, n)$, then following integral formula holds true:*

(2.3)
$$\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} \prod_{j=1}^n \omega_{\nu_j} \left(\frac{x y_j}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$
$$= 2^{1-\mu-2\boldsymbol{\nu_s}}\, a^{\mu-\lambda}\, (\lambda + \boldsymbol{\nu_s})\, \frac{\Gamma(\lambda - \mu)\,\Gamma(2\mu + 2\boldsymbol{\nu_s})}{\Gamma(1 + \lambda + \mu + 2\boldsymbol{\nu_s})} \left\{\prod_{j=1}^n \frac{{y_j}^{\nu_j}}{\Gamma\left(\nu_j + \frac{1+b_j}{2}\right)}\right\}$$
$$\times F^{2:0,0,\ldots,0}_{2:1,1,\ldots,1}\left[\begin{array}{r} \left[1 + \lambda + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right], \left[2\mu + 2\boldsymbol{\nu_s} : 4, 4, \ldots, 4\right] : \\ \left[1 + \lambda + \mu + 2\boldsymbol{\nu_s} : 4, 4, \ldots, 4\right], \left[\lambda + \boldsymbol{\nu_s} : 2, 2, \ldots, 2\right] : \end{array}\right.$$
$$\left.\begin{array}{lll} \overline{\qquad\quad} & ; \cdots ; & \overline{\qquad\quad} \\ \left[\nu_1 + \frac{1+b_1}{2} : 1\right] & ; \cdots ; & \left[\nu_n + \frac{1+b_n}{2} : 1\right] \end{array} ; \frac{-c_1 y_1^2}{16}, \ldots, \frac{-c_n y_n^2}{16}\right],$$

where $\boldsymbol{\nu_s}$ is given in (2.2).

Proof. A similar argument as in the proof of Theorem 1 is seen to establish the integral formula (2.3). The details of its proof are omitted. □

3. Remarks

Since the case $b = c = 1$ for the generalized Bessel function (1) of the first kind reduces to the Bessel function (2) of the first kind, further setting $n = 1$ in our main results (2.1) and (2.3) is easily found to yield, respectively, the known results Equations (2.1) and (2.2) in [7].

Special cases of (4) are established in terms of generalized hypergeometric functions of one and two variables respectively, for example, the generalized hypergeometric function ${}_pF_q$ (see, *e.g.*, [19, Section 1.5]) and the Kampé de Fériet function (see, *e.g.*, [21, p. 27]). There are certain known relationships between the generalized Bessel function $\omega_\nu(z)$ and the cosine function, the hyperbolic cosine function, the sine function, and the hyperbolic sine function, respectively (see, *e.g.*, [16]). So our main results (2.1) and (2.3) can produce many interesting and potentially useful special cases, whose detailed illustrations are omitted.

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