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A Difference of Two Composition Operators on L^2 and H^2

Takahiko Nakazi
Professor Emeritus Hokkaido University Sapporo 060-0810, Japan
e-mail : tnakazi70@gmail.com

Abstract. A finite rank difference of two composition operators is studied on a Hilbert Lebesgue space or a Hilbert Hardy space.

1. Introduction

Let $(X, \mathcal{B}, m)$ be a finite complete Borel measure space and let $\phi : X \to X$ be a measurable transformation, that is, $\phi^{-1}(E) \in \mathcal{B}$ for any $E \in \mathcal{B}$. As a typical example of $(X, \mathcal{B}, m)$, (1) X is a unit circle ∂D or a closed unit disc $\partial D \cup D$, (2) $\mathcal{B}$ is a Borel σ-algebra, (3) m is a normalized Lebesgue measure on ∂D or a normalized area measure on $\partial D \cup D$.

$L^2 = L^2(\partial D)$ denotes the usual Lebesgue space and $H^2 = H^2(D)$ denotes the usual Hardy space. Let $C = C(\partial D)$ be the set of all continuous functions on ∂D and $A = A(D)$ the disc algebra on $\bar{D}$.

For a measurable function f on X, $(C_\phi f)(z) = f(\phi(z))$ $(z \in X)$. If C_ϕ is an operator defined on L^2, C, H^2 or A then ϕ belongs to L^2, C, H^2 or A, respectively. For we can choose z as f. It is known that C_ϕ is bounded on L^2 if and only if $m(\phi^{-1}(E)) \leq \gamma m(E)$ $(E \in \mathcal{B})$ where $\gamma = \gamma_\phi$ is possitive constant and$\gamma \geq 1$. Clearly, C_ϕ is bounded on C. It is well-known that C_ϕ is bounded on H^2 and it is easy to see that C_ϕ is bounded on A. Many mathematicians have been interested in composition operators. For example, see [6], [7] on L^2, [2], [6], [8] on $C(X)$ and [3] on H^2. A difference of two composition operators has been studied on H^2 or A (see [1], [5], [3]).

In this paper, we study when $C_\phi - C_\psi$ is of finite rank in very general setting.

2. Case of L^2 and C

$(X, \mathcal{B}, m)$ denotes a finite complete Borel measure space in Theorem 1 and X

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is a compact Hausdorff space in Theorem 2. $L^2 = L^2(X, m)$ is a set of square summable functions with respect to m and $C = C(X)$ is a set of all continuous functions on X.

Lemma 1. *Suppose a measurable subset E in X with $m(E) > 0$ and m does not have a point mass. If $C_\phi - C_\psi$ is of finite rank n on L^2 and it is zero on $\chi_E L^2$ then $C_\phi - C_\psi$ is of finite rank n on $(1 - \chi_E)L^2$.*

Proof. We may assume $n \neq 0$. Hence there exist $\{f_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ in L^2 such that $(C_\phi - C_\psi)(f) = \sum_{j=1}^n \langle f, f_j\rangle g_j$ where $\langle f, f_j\rangle = \displaystyle\int f\bar{f}_j dm$ $(f \in L^2)$. Since $(C_\phi - C_\psi)(\chi_E f) = \sum_{j=1}^n \langle \chi_E f, f_j\rangle g_j = 0$ $(f \in L^2)$, $f_j = 0$ on E for any $1 \leq j \leq n$. Hence $C_\phi - C_\psi$ is of finite rank n on $(1 - \chi_E)L^2$. □

Theorem 1. *Suppose m does not have a point mass.* (1) *C_ϕ can not be of finite rank.* (2) *If $C_\phi - C_\psi$ is of finite rank on L^2 then $C_\phi = C_\psi$.*

Proof. (1) Put $Y = \phi(X)$. Since $\chi_Y L^2 = \{f \circ \phi : f \in L^2\}$ and m does not have a point mass, $\chi_Y L^2 = \{0\}$ when $\dim(\chi_Y L^2) < \infty$. Hence we may assume $f \circ \phi = 0$ a.e. for any $f \in L^2$. This shows the rank is zero.

(2) We may assume $m(\{z \in X : \phi(z) \neq \psi(z)\}) > 0$. Since m does not have a point mass, $\{z \in X : \phi(z) \neq \psi(z)\}$ is an uncountable set. Hence there exists a point $\zeta \in X$ such that $m(\phi^{-1}(\zeta)) = 0$ and $m(\psi^{-1}(\zeta)) = 0$. Then $m(\phi^{-1}(\zeta) \cup \psi^{-1}(\zeta)) = 0$. If $C_\phi - C_\psi$ is of finite rank $n \neq 0$ then there exist $\{f_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ in L^2 such that

$$(C_\phi - C_\psi)(f) = \sum_{j=1}^n \langle f, f_j\rangle g_j \quad (f \in L^2)$$

where $\langle f, f_j\rangle = \int_X f\bar{f}_j dm$. By Lemma 1, there exists a Borel subset E_1 such that $\zeta \in E_1$ and $\langle \chi_{E_1}, f_1\rangle \neq 0$. Again, by Lemma 1, there exists a Borel subset E_2 such that $\zeta \in E_2$ and $\langle \chi_{E_2}, f_1\rangle \neq 0$ and $\chi_{E_1} \gneqq \chi_{E_2}$. Repeating this process, we get a Borel sequence subset $\{E_n\}$ such that $E_n \supsetneqq E_{n+1}$ and $\bigcap_n E_n = \{\zeta\}$ and

$$\langle \chi_{E_\ell}, f_1\rangle \neq 0 \quad (\ell = 1, 2, \cdots).$$

Then

$$\bigcap_{n=1}^\infty \phi^{-1}(E_n) = \{\phi^{-1}(\zeta)\} \text{ and } \bigcap_{n=1}^\infty \psi^{-1}(E_n) = \{\psi^{-1}(\zeta)\}.$$

Since $(C_\phi - C_\psi)(\chi_{E_\ell}) = \chi_{\phi^{-1}(E_\ell)} - \chi_{\psi^{-1}(E_\ell)}$,

$$\sum_{j=1}^n \langle \chi_{E_\ell}, f_j\rangle g_j(z) = 0 \quad (z \in \phi^{-1}(E_\ell)^c \cap \psi^{-1}(E_\ell)^c).$$

Put $F_\ell = \phi^{-1}(E_\ell)^c \cap \psi^{-1}(E_\ell)^c$. Since $m(\phi^{-1}(\zeta) \cup \psi^{-1}(\zeta)) = 0$, $m(F_\ell) \to 1$ as $\ell \to \infty$. Put $a_{\ell j} = \langle \chi_{E_\ell}, f_j\rangle$ for $1 \leq j \leq n$ and $\ell = 1, 2, \cdots$. Then for any ℓ

$$a_{\ell 1} g_1(z) + a_{\ell 2} g_2(z) + \cdots + a_{\ell n} g_n(z) = 0 \quad (z \in F_\ell)$$

and $a_{\ell 1} \neq 0$. Put $|a_{\ell k(\ell)}| = \max(|a_{\ell 1}|, \cdots, |a_{\ell n}|)$ for each ℓ. Then there exists $k_0(\ell_0)$ such that $k_0(\ell_0) = k_0(\ell)$ for infinitely many ℓ. Hence by choosing a subsequence, we may assume $|a_{\ell 1}| = 1$ and $|a_{\ell j}| \leq 1$ $(1 \leq j \leq n, 1 \leq \ell < \infty)$. Again by choosing a subsequence, we may assume $\lim_{\ell\to\infty} a_{\ell j} = a_j$ $(1 \leq j \leq n)$. Then $a_1 g_1(z) + a_2 g_2(z) + \cdots + a_n g_n(z) = 0$ a.e.z because $\lim_{\ell\to\infty} m(F_\ell) = 1$. Since $a_1 \neq 0$ and $\{g_j\}_{j=1}^n$ is an independent set, it is a contradiction. Therefore $n = 0$ and $C_\phi = C_\psi$. □

Theorem 2. *Suppose X is a compact Hausdorff space and an infinite set. If $C_\phi - C_\psi$ is of finite rank on $C(X)$ then $\phi = \psi$ except some finite set.*

Proof. Suppose $C_\phi - C_\psi$ is of finite rank $n \neq 0$. Then there exist measures $\{\mu_j\}_{j=1}^n$ on X and functions $\{e_j\}_{j=1}^n$ in $C(X)$ such that

$$(C_\phi - C_\psi)(f) = \sum_{j=1}^n \left(\int f d\mu_j\right) e_j \quad (f \in C(X))$$

where $\{\mu_j\}_{j=1}^n$ is an independent set of $C(X)^*$ and $\{e_j\}_{j=1}^n$ is an independent set of $C(X)$. Hence for any $z \in X$,

$$f(\phi(z)) - f(\psi(z)) = \sum_{j=1}^n \alpha_j(f)(e_j(z))$$

where $f \in C(X)$ and $\alpha_j(f) = \int f d\mu_j$ $(1 \leq j \leq n)$. For fixed $z \in X$, put

$$Y(z) = \{(\alpha_1(f), \cdots, \alpha_n(f)) : f \in C(X) \text{ and } f(\phi(z)) = f(\psi(z))\}.$$

Since $\{\mu_j\}_{j=1}^n$ is an independent set in $C(X)^*$, it is easy to see that $Y(z) = \mathbb{C}^n$ for each z except a finite set in X. In fact, if $F = \{z \in X : \phi(z) = \psi(z)\}$ then $\{\delta_{\phi(z)} - \delta_{\psi(z)}\}_{z\in X\setminus F}$ is an independent set in $C(X)^*$ where $\delta_{\phi(z)}$ and $\delta_{\psi(z)}$ are Dirac measures. When $X\setminus F$ is a finite subset, we need not to prove it. Suppose $X\setminus F$ is an infinite subset of X. For $z \in X\setminus F$, suppose $\delta_{\phi(z)} - \delta_{\psi(z)}$ is not in the linear span of $\{\mu_1, \cdots, \mu_n\}$. Then, for each $1 \leq j \leq n$ there exists f_j in $C(X)$ such that $f_j = 0$ on $\{\delta_{\phi(z)} - \delta_{\psi(z)}, \mu_1, \cdots, \mu_n\}\setminus\{\mu_j\}$ and $\int f d\mu_j = 1$. This shows for $z \in X\setminus F$ $Y(z) = \mathbb{C}^n$. Hence $Y(z) = \mathbb{C}^n$ for each z except a finite set E in $X\setminus F$. Hence for $z \in X\setminus E$

$$\sum_{j=1}^n \alpha_j(f)\ e_j(z) = 0 \text{ and } Y(z) = \mathbb{C}^n$$

and so $e_j(z) = 0 \quad (j = 1, \cdots, n)$. Thus for any f in $C(X) f \circ \phi(z) \equiv f \circ \psi(z)$ $(z \in X\setminus E)$. Therefore $\phi(z) = \psi(z)$ $(z \in X\setminus E)$.

3. Case of H^2 and A

Let X be a domain $\mathcal{D}$ in $\mathbb{C}$ or $\partial\mathcal{D} \cup \mathcal{D}$. We assume $A(\mathcal{D})$ and $H^2(\mathcal{D})$ contain all polynomials.

Lemma 2. *If $C_\phi - C_\psi$ is of finite rank $n \neq 0$ on $H^2(\mathcal{D})$ then for any $\ell \geq 1$ $\phi^\ell - \psi^\ell = \sum_{j=1}^n \langle z^\ell, x_j\rangle y_j$ where $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ are independent sets in $H^2(\mathcal{D})$.*

Proof. Since $C_\phi - C_\psi$ is of finite rank $n \neq 0$, there exist $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ in $H^2(\mathcal{D})$ such that $C_\phi f - C_\psi f = \sum_{j=1}^n \langle f, x_j\rangle y_j$ $(f \in H^2)$. Suppose $f = z^\ell$. □

Theorem 3. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $H^2(\mathcal{D})$ a Hilbert space of holomorphic functions on $\mathcal{D}$. Let $X = \mathcal{D}$. If $C_\phi - C_\psi$ is of finite rank n on $H^2(\mathcal{D})$ then there exists a nonzero polynomial f which is of degree $\leq n+1$, $f(0) = 0$ and $f \circ \phi = f \circ \psi$.*

Proof. Since $z \in H^2(\mathcal{D})$, ϕ and ψ are holomorphic on $\mathcal{D}$. Since C_ϕ and C_ψ are defined on $H^2(\mathcal{D})$, $\phi(\mathcal{D}) \subseteq \mathcal{D}$ and $\psi(\mathcal{D}) \subseteq \mathcal{D}$. Suppose $C_\phi - C_\psi$ is of finite rank n. If $n = 0$ then the conclusion is clear and so we may assume $n \geq 1$. Then by Lemma 2

$$\phi^i - \psi^i = \sum_{j=1}^n \langle z^i, x_j\rangle y_j = \sum_{j=1}^n a_{ij} y_j.$$

Let $\mathbf{a} = [a_{ij}]_{n\times n}$ be the matrix defined by a_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq n)$ and $\mathbf{a}_i = (a_{i1}, \cdots, a_{in})$ $(1 \leq i \leq n)$. If det $\mathbf{a} = 0$ then we may assume $\mathbf{a}_1 = \sum_{j=2}^n \lambda_j \mathbf{a}_j$. Hence $a_{1j} = \sum_{i=2}^n \lambda_i a_{ij}$ $(1 \leq j \leq n)$ and so

$$\begin{aligned}\phi - \psi &= \sum_{j=1}^n a_{1j} y_j = \sum_{j=1}^n (\sum_{i=2}^n \lambda_i a_{ij}) y_j \\ &= \sum_{i=2}^n \lambda_i (\sum_{j=1}^n a_{ij} y_j) = \sum_{i=2}^n \lambda_i (\phi^i - \psi^i).\end{aligned}$$

Therefore the polynomial $f = z - \sum_{i=2}^n \lambda_i z^i$ is the requested one.

If det $\mathbf{a} \neq 0$ then y_j can be written as $\sum_{k=1}^n b_{jk}(\phi^k - \psi^k)$ where $b_{jk} \in \mathbb{C}$ and $1 \leq j \leq n$. Since $\phi^{n+1} - \psi^{n+1} = \sum_{j=1}^n \langle z^{n+1}, x_j\rangle y_j$, $f = z^{n+1} - \sum_{j=1}^n \sum_{k=1}^n \langle z^{n+1}, x_j\rangle b_{jk} z^j$ is the requested one. □

Corollary 1. *Let ϕ and ψ be self-maps of $\mathcal{D}$. $C_\phi - C_\psi$ is of rank 0 if and only if $\phi \equiv \psi$. $C_\phi - C_\psi$ is of rank 1 if and only if ϕ and ψ are constants, and $\phi \not\equiv \psi$.*

Proof. The first statement is clear. We will show the second statement. The 'if' part is clear. We will show the 'only if' part. Suppose $C_\phi - C_\psi = x \otimes y$. If $\langle z, x\rangle = 0$ then $\phi \equiv \psi$. Hence we may assume $\langle z, x\rangle \neq 0$ and so $y = (\phi - \psi)/\langle z, x\rangle$. If $\langle z^2, x\rangle = 0$ then $\phi^2 - \psi^2 \equiv 0$ and so $\psi \equiv -\phi$. Since $f \circ \phi - f \circ (-\phi) = 2\phi\langle f, x\rangle/\langle z, x\rangle$, $2\phi^3 = 2\phi\langle z^3, x\rangle/\langle z, x\rangle$ and so ϕ is constant. If $\langle z^2, x\rangle \neq 0$, $\phi^2 - \psi^2 = \dfrac{\langle z^2, x\rangle}{\langle z, x\rangle}(\phi - \psi)$ and so $\phi + \psi \equiv a$ for some complex constant a. When $\langle z^3, x\rangle = 0$, $\phi^3 - \psi^3 \equiv 0$ and so $\phi^2 + \phi\psi + \psi^2 \equiv 0$. When $\langle z^3, x\rangle \neq 0$,

$$\phi^3 - \psi^3 = \frac{\langle z^3, x\rangle}{\langle z, x\rangle}(\phi - \psi)$$

and so $\phi^2+\phi\psi+\psi^2 \equiv b$ for some complex constant b. Hence when $\langle z^2, x\rangle \neq 0$ then $\phi+\psi \equiv a$ and $\phi^2+\phi\psi+\psi^2 \equiv b$. Therefore $\phi^2 - a\phi + a^2 - b = 0$. This shows ϕ and ψ are constant. □

When $\mathcal{D}$ is the open unit disc, an inner function q in $H^2(\mathcal{D})$ means a unimodular function in $\partial\mathcal{D}$ and sing q denotes the subset of $\partial\mathcal{D}$ on which q can not be analytically extended.

Corollary 2. *Let $\mathcal{D}$ be the open unit disc. Suppose $C_\phi - C_\psi$ is of finite rank. If ϕ and ψ are inner then sing $\phi =$ sing ψ.*

Corollary 3. *Let $\mathcal{D}$ be the open unit disc. Suppose $C_\phi - C_\psi$ is of finite rank n. When ϕ and ψ are inner, if ϕ is a finite Blaschke product of degree n then $\phi \equiv \psi$.*

Proof. Let f be a polynomial in Theorem 3. By Corollary 2, ψ is also a finite Blaschke product. If ϕ has a pole at z_0 with multiplicity ℓ then so does ψ. This shows $\phi \equiv \alpha\psi$ for some constant α.

Corollary 4. *Let $\mathcal{D}$ be the open unit disc. Suppose $C_\phi - C_\psi$ is of finite rank n. When ϕ and ψ be inner, if ϕ is a Blaschke product then $\psi = \phi s$ and s is a singular inner with sing $s \subseteq$ sing ϕ.*

Proof. Since f is a polynomial with $f(0) = 0$, if ϕ has a pole at z_0 with multiplicity ℓ then so does the Blaschke part of ψ. This and Corollary 2 show the corollary. □

Theorem 4. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $H^2(\mathcal{D})$ a Hilbert space of holomorphic functions on $\mathcal{D}$. Suppose $H^2(\mathcal{D})$ contains all polynomials. Let $X = \mathcal{D}$. If $C_\phi - C_\psi$ is of finite rank $n \neq 0$ on $H^2(\mathcal{D})$ then for any enough large ℓ*

$$\phi^\ell - \psi^\ell = \textstyle\sum_{j=1}^n b_{\ell j}(\phi^{S_0(j)} - \psi^{S_0(j)})$$

where $\{S_0(j)\}_{j=1}^n$ is a fixed subset of natural numbers.

Proof. For $t \geq 1$, put $a_{tj} = \langle z^t, x_j\rangle$ $(1 \leq j \leq n)$. Then by Lemma 2 $\phi^t - \psi^t = \sum_{j=1}^n a_{tj} y_j$. When $S = \{S(i)\}_{i=1}^n$ is a subset of natural numbers and $S(i) \leq S(i+1)$, we write $\mathbf{a}_S = [a_{S(i)j}]_{n\times n}$. Put $r = \max_S r(\mathbf{a}_S)$ where $r(\mathbf{a}_s)$ denotes the rank of $\mathbf{a}_s$ and $r = r(\mathbf{a}_{S_0})$. If $\ell > S_0(n)$, then there exist $b_{1\ell}, \cdots, b_{n\ell}$ in $\mathbb{C}$ such that

$$\phi^\ell - \psi^\ell = \sum_{j=1}^n b_{\ell j}(\phi^{S_0(j)} - \psi^{S_0(j)}).$$ □

Theorem 5. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}$ and $A(\mathcal{D})$ a set of holomorphic functions on $\mathcal{D}$ which are continuous on $\mathcal{D} \cup \partial\mathcal{D}$. Let $X = \mathcal{D} \cup \partial\mathcal{D}$. If $C_\phi - C_\psi$ is of finite rank n on $A(\mathcal{D})$ then there exists a polynomial f which is of degree $\leq n$ and $f \circ \phi = f \circ \psi$.*

Proof. Since $z \in A(\mathcal{D})$, ϕ and ψ belong to $A(\mathcal{D})$, $\phi(\mathcal{D}) \subseteq \mathcal{D}$ and $\psi(\mathcal{D}) \subseteq \mathcal{D}$. Suppose $C_\phi - C_\psi$ is of finite rank n. If $n = 0$ then the conclusion is clear and so we may assume $n \geq 1$. Then there exist $\{\mu_j\}_{j=1}^n$ in $C(X)^*$ and $\{y_j\}_{j=1}^n$ in $A(\mathcal{D})$ such that

$$(C_\phi - C_\psi)(g) = \sum_{j=1}^n (\int_X g d\mu_j) y_j \quad (g \in A(\mathcal{D}))$$

where $\{\mu_j + A(\mathcal{D})^\perp \cap C(X)^*\}_{j=1}^n$ is independent in $C(X)^*/A(\mathcal{D})$. Now we can prove as in the proof of Theorem 3. □

Corollary 5. *Let ϕ and ψ be self-maps of $\mathcal{D}$. $C_\phi - C_\psi$ is of rank 0 if and only if $\phi \equiv \psi$. $C_\phi - C_\psi$ is of rank 1 if and only if ϕ and ψ are constants, and $\phi \neq \psi$.*

Proof. The proof is similar to that of Corollary 1. □

Corollary 6. *Let $\mathcal{D}$ be an open unit disc. Suppose $C_\phi - C_\psi$ is of finite rank. If ϕ and ψ are inner then ϕ and ψ are Blaschke products and $\phi \equiv \alpha\psi$ for some some constant α.*

Proof. Since ϕ and ψ belong to $A(\mathcal{D})$, both ϕ and ψ are finite Blaschke products. By Theorem 5 $\psi \equiv \alpha\psi$ for some constant α. □

Theorem 6. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}$ and $A(\mathcal{D})$ a set of holomorphic functions on $\mathcal{D}$ which are continuous on $\mathcal{D} \cup \partial\mathcal{D}$. Let $X = \mathcal{D} \cup \partial\mathcal{D}$. If $C_\phi - C_\psi$ is of finite rank n on $A(\mathcal{D})$ then for any enough large ℓ*

$$\phi^\ell - \psi^\ell = \sum_{j=1}^n b_{\ell j}(\phi^{S_0(j)} - \psi^{S_0(j)})$$

where $\{S_0(j)\}_{j=1}^n$ is a fixed subset of natural number.

Proof. The proofs of Theorem 4 and 5 show the theorem. □

References

[1] E. Berkson, *Composition operators isolated in the uniform operator topology*, Proc. Amer. Math. Soc., **81**(1981), 230–232.

[2] H. Kamowitz, *Compact weighted endomorphisms of $C(X)$*, Proc. Amer. Math. Soc., **83**(1981), 517–521.

[3] B. D. MacCluer, S. Ohno and R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integr. Eq. Op. Th., **40**(2001), 481–494.

[4] E. A. Nordgren, *Composition operators on Hilbert spaces, Hilbert Space Operators*, Lecture Notes in Math., **693**, Springer-Verlag, Berlin, Heidelberg, and New York, (1978), 37–63.

[5] J. H. Shapino and C. Sundberg, *Isolation amongst the composition operators*, Pacific. J. Math., **145**(1990), 117–152.

[6] R. K. Singh and R. D. C. Kumar, *Compact weighted composition operators on $L^2(\lambda)$*, Acta Sci. Math., **49**(1985), 339–344.

[7] H. Takagi, *Compact weighted composition operators on L^p*, Proc. Amer. Math. Soc., **116**(1992), 505–511.

[8] H. Takagi, *Fredholm weighted composition operators*, Integr. Equat. Oper. Th., **16**(1993), 267–276.