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On 2-Absorbing and Weakly 2-Absorbing Primary Ideals of a Commutative Semiring

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Abstract. Let R be a commutative semiring. The purpose of this note is to investigate the concept of 2-absorbing (resp., weakly 2-absorbing) primary ideals generalizing of 2-absorbing (resp., weakly 2-absorbing) ideals of semirings. A proper ideal I of R said to be a 2-absorbing (resp., weakly 2-absorbing) primary ideal if whenever $a, b, c \in R$ such that $abc \in I$ (resp., $0 \neq abc \in I$), then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Moreover, when I is a Q-ideal and P is a k-ideal of R/I with $I \subseteq P$, it is shown that if P is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of R, then P/I is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of R/I and it is also proved that if I and P/I are weakly 2-absorbing primary ideals, then P is a weakly 2-absorbing primary ideal of R.

1. Introduction

We assume that all rings are commutative semiring with non-zero identity. The concept of semiring was studied by Vandive [17] in 1934. A none-empty set R with two binary operations addition and multiplication is called *semiring* if:

(1) $(R, +)$ is a commutative monoid with identity element 0.

(2) $(R, .)$ is a monoid with identity element $1 \neq 0$.

(3) The multiplication both from left and right is distributes over addition.

(4) $0.a = a.0 = 0$ for every $a \in R$.

If $(R, .)$ is a commutative semigroup, so R is a commutative semiring. The set $\mathbb{Z}_0^+$, which denotes the set of all non-negative integer, is a semiring under usual addition and multiplication of non-negative integer but it is not a ring. Semirings have got important structure in rings theory. A non-empty set I is called an *ideal* if for every $a, b \in I$ and $r \in R$, then $a + b \in I$ and $ra \in I$. The ideal I is called a *k-ideal* (*subtractive ideal*) if $a, a + b \in I$, then $b \in I$. By definition, every ideal of semiring

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R is a k-ideal of R. An ideal I of semiring R is called *strongly k-ideal*, whenever $a+b \in I$ for some $a, b \in R$, then $a \in I$ and $b \in I$. Clearly, every strongly k-ideal is a k-ideal. Let I be an ideal of semiring R. I is also called a *Q-ideal* (*partitioning ideal*) if there exists a subset Q of R such that

(1) $R = \bigcup\{q+I | q \in Q\}$ and

(2) If $q_1, q_2 \in Q$, then $(q_1+I) \cap (q_2+I) \neq \emptyset$ if and only if $q_1 = q_2$.

Let I be a Q-ideal of R and $R/I_{(Q)} = \{q+I | q \in Q\}$. Then $R/I_{(Q)}$ forms a semiring under the binary operations " $\oplus$ " and " $\odot$ " define as follows:

$$(q_1+I) \oplus (q_2+I) = q_3+I$$

where $q_3 \in Q$ is unique such that $q_1+q_2+I \subseteq q_3+I$.

$$r \odot (q_1+I) = q_4+I$$

where $q_4 \in Q$ is unique such that $rq_1+I \subseteq q_4+I$. This semiring $R/I_{(Q)}$ is said to be *the quotient semiring* of R by I. By definition of Q-ideal, there exists a unique element $q\prime$ such that $0+I \subseteq q\prime+I$, so $q\prime+I$ is a zero element of R/I. Let R be a semiring, I be a Q-ideal and P be a k-ideal of R with $I \subseteq P$. Then $P/I = \{q+I | q \in P \cap Q\}$ is a k-ideal of R/I. If I is a Q-ideal of R and L a k-ideal of R/I, then $L = J/I$ where $J = \{r \in R : q_1+I \in L\}$ is a k-ideal of R, [3]. If R and S are semirings, then a function $\gamma : R \rightarrow S$ is a morphism of semiring if and only if (1) $\gamma(0_R) = 0_S$;

(2) $\gamma(1_R) = 1_S$ and

(3) $\gamma(r+s) = \gamma(r)+\gamma(s)$ and $\gamma(rs) = \gamma(r)\gamma(s)$ for all $r, s \in R$.

A morphism of semirings which is both monomorphism and epimorphism is called isomorphism. In this case, we write $R \cong S$. If $\gamma : R \rightarrow S$ is a morphism of semirings and ρ is a *congruence relation* on S, then the relation ρ' on R defines by $r\rho' r'$ if and only if $\gamma(r)\rho g(r')$, is a congruence relation on R. In particular, each morphism of semirings $\gamma : R \rightarrow S$ defines a congruence relation $\equiv_\gamma$ on R by setting $r \equiv_\gamma s$ if and only if $\gamma(r) = \gamma(s)$. Let $\gamma : R \rightarrow S$ be a morphism of semirings. If J is an ideal of S, then $\gamma^{-1}(J)$ is an ideal of R. Moreover, if J is k-ideal, then so is $\gamma^{-1}(J)$. If γ is an epimorphism and I is an ideal of R, then $\gamma(I)$ is an ideal of S, [12, Proposition 9.46]. If $\gamma : R \rightarrow S$ is a morphism of semirings, then $\gamma^{-1}(0)$ is an ideal of R. So it said to be the $Kernel$ of γ and denoted by $ker(\gamma)$. Therefor another congruence relation defined on R by γ is the relation $\equiv_{ker(\gamma)}$. It is obviously true that $r \equiv_\gamma s$ whenever $r \equiv_{ker(\gamma)} s$. Notice that the converse is not necessary true. When the relation $\equiv_\gamma$ and $\equiv_{ker(\gamma)}$ coincide, then the morphism γ is called *steady.* A steady morphism $\gamma : R \rightarrow S$ is monomorphism if and only if $ker(\gamma) = \{0\}$, [12, Proposition 9.45].

Let R be a commutative semiring. Recall that an ideal I of semiring R is called proper if $I \subset R$ and a proper ideal I of R is called prime (resp., weakly prime) ideal if whenever $a, b \in R$ such that $ab \in I$ (resp., $0 \neq ab \in I$), then either $a \in I$ or $b \in I$. A proper ideal I of R is called primary (resp., weakly primary) ideal if whenever $a, b \in R$ such that $ab \in I$ (resp., $0 \neq ab \in I$), then either $a \in I$ or $b^n \in I$

for some positive integer n. In this case, if I is a primary ideal of R and $P := \sqrt{I}$ is a prime ideal of R, we call that I is a P-primary ideal of R. The radical of an ideal I denoted by $\sqrt{I}$ and defined as the set of all elements $a \in R$ such that $a^n \in I$ for some positive integer n, that is, $\sqrt{I} = \{a \in R | a^n \in I$ for some positive integer $n\}$. It is an ideal of R containing I, and is the intersection of all prime ideals of R containing I. It is easy to show that if an ideal I is k-ideal, then $\sqrt{I}$ is a k-ideal. Furthermore, an element $a \in R$ said to be nilpotent whenever there exists positive integer n such that $a^n = 0$. The set $\{a \in R | a^n = 0$ for some positive integer $n\}$ denoted by $Nil(R)$.

A. Badawi in [6] introduced a new generalization of prime ideals over a commutative ring. A proper ideal I of a commutative ring R with $1 \neq 0$ is said to be a 2-absorbing ideal if whenever $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Clearly, every prime ideal is a 2-absorbing ideal. A 2-absorbing (resp,. weakly 2-absorbing) ideal of a semiring was introduced by A. Yousefian Darani in [18]. He defined that a proper ideal I of semiring R said to be a 2-absorbing (resp,. weakly 2-absorbing) ideal if whenever $a, b, c \in R$ such that $abc \in I$ (resp, $0 \neq abc \in I$), then either $ab \in I$ or $ac \in I$ or $bc \in I$. Recently, A. Badawi, U. Tekir and E. Yetkin in [8] have introduced the concept of 2-*absorbing primary* ideals over a commutative ring which is a generalization of primary ideals. A proper ideal I of R said to be a 2-absorbing primary ideal if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

In this paper, we will define the concept of 2-absorbing (resp., weakly 2-absorbing) primary ideal of a semiring. Let R be a semiring and I be an ideal of R. I is a 2-*absorbing* (*resp., weakly* 2-*absorbing*) *primary ideal* if whenever $a, b, c \in R$ with $abc \in I$ (resp., $0 \neq abc \in I$), then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. We generalize the concept of strongly 2-absorbing primary ideal. Then a proper ideal I of semiring R calls *strongly* 2-*absorbing primary ideal* if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$. In fact, among the other things we prove that the radical of a 2-absorbing primary ideal of a semiring is a 2-absorbing ideal (Theorem 2.4). It is shown that if I_1 is a P_1-primary ideal of R and I_2 is a P_2-primary ideal of R, then I_1I_2, $I_1 \cap I_2$ and P_1P_2 are 2-absorbing primary ideals of R (Theorem 2.6). It is shown that if $\sqrt{I}$ is a proper ideal of semiring R such that I is a prime ideal, then I is a 2-absorbing primary ideal of R (Theorem 2.8). It is shown that if I is a Q-ideal and P a 2-absorbing primary k-ideal of R/I with $I \subseteq P$, then P/I is a 2-absorbing primary ideal of R/I (Theorem 2.11). Let $R = R_1 \times R_2$ where R_1, R_2 be commutative semirings. It is shown that I_1 (resp., I_2) is a 2-absorbing primary ideal of R_1 (resp., R_2) if and only if $I_1 \times R_2$ (resp., $R_1 \times I_2$) is a 2-absorbing primary ideal of R and $I = I_1 \times I_2$ is a 2-absorbing primary ideal of R if and only if $I = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2 or $I = I_1 \times I_2$ for some primary ideal I_1 of R_1 and for some primary ideal I_2 of R_2 (Theorems 2.16 and 2.17). It is shown that if I is a proper strongly k-ideal of R, then I is a 2-absorbing primary ideal if and only if $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of R, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$ (Theorem 2.20). In section 3, we study the concept of weakly 2-absorbing primary ideal of commutative semirings. Indeed it is shown that if I is a weakly 2-absorbing primary k-ideal of R, then either I is 2-absorbing primary or $I^3 = 0$ (Theorem 3.6). In the section 4, is got some characterizations in the semirings $(\mathbb{Z}_0^+, \gcd, lcm)$ and $(\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$.

2. 2-Absorbing Primary Ideals in Commutative Semirings

Definition 2.1. Let R be a semiring and I be a proper ideal. The ideal I said to be a *2-absorbing primary ideal* if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$.

Lemma 2.2. *Let R be a semiring. Then the following statements hold:*
(1) *Every primary ideal is* 2*-absorbing primary;*
(2) *Every* 2*-absorbing ideal is* 2*-absorbing primary.*

Proposition 2.3. *Let I and K be ideals of semiring R. If I is a* 2*-absorbing primary strongly k-ideal of R and $abK \subseteq I$ for some $a, b \in R$, then $ab \in I$ or $aK \subseteq \sqrt{I}$ or $bK \subseteq \sqrt{I}$.*

Proof. Assume that $ab \notin I$, $aK \nsubseteq \sqrt{I}$ and $bK \nsubseteq \sqrt{I}$. So there exists $k_1, k_2 \in K$ such that $ak_1 \notin \sqrt{I}$ and $bk_2 \notin \sqrt{I}$. Since $abk_1, abk_2 \in I$ and I is a 2-absorbing primary ideal of R, we conclude that $bk_1 \in \sqrt{I}$ and $ak_2 \in \sqrt{I}$. Now since $ab(k_1 + k_2) \in I$, $ab \notin I$ and I is a 2-absorbing primary ideal of R, we have $a(k_1 + k_2) \in \sqrt{I}$ or $b(k_1 + k_2) \in \sqrt{I}$. If $a(k_1 + k_2) \in \sqrt{I}$, since I is a strongly k-ideal and $ak_2 \in \sqrt{I}$, we have $ak_1 \in \sqrt{I}$, which is a contradiction. If $b(k_1 + k_2) \in \sqrt{I}$ by previous sense and as $bk_1 \in \sqrt{I}$, we conclude that $bk_2 \in \sqrt{I}$, which is a contradiction. Therefore the result is true. □

Theorem 2.4. *Let R be a semiring and I be an ideal of R. If I is a* 2*-absorbing primary ideal of R, then $\sqrt{I}$ is a* 2*-absorbing ideal of R.*

Proof. Let $abc \in \sqrt{I}$ for some $a, b, c \in R$ but $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Then there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in I$. Since I is a 2-absorbing primary ideal of R and $ac, bc \notin \sqrt{I}$, we have $a^n b^n \in I$ and so $ab \in \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of R. □

Lemma 2.5. *Let R be a commutative semiring. Then the following statements hold:*
(1) *If I and J are ideals of R, then $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$;*
(2) *If P is a prime ideal of R, then $\sqrt{P} = P$. Moreover, $\sqrt{P^n} = P$ for some positive integer n.*

Theorem 2.6. *Let R be a commutative semiring, I_1, I_2 be ideals of R and P_1, P_2 be prime ideals of R. Suppose that I_1 is a P_1-primary ideal of R and I_2 is a P_2-primary ideal of R. Then the following statements hold:*
(1) *$I_1 I_2$ is a* 2*-absorbing primary ideal of R;*
(2) *$I_1 \cap I_2$ is a* 2*-absorbing primary ideal of R;*
(3) *$P_1 P_2$ is a* 2*-absorbing primary ideal of R.*

Proof. (1) Assume that $a, b, c \in R$ with $abc \in I_1 I_2$ but $ac \notin \sqrt{I_1 I_2}$ and $bc \notin \sqrt{I_1 I_2}$. Since $\sqrt{I_1 I_2} = P_1 \cap P_2$, we get $a, b, c \notin \sqrt{I_1 I_2} = P_1 \cap P_2$ and $\sqrt{I_1 I_2} = P_1 \cap P_2$ is a 2-absorbing ideal of R, by [18, Theorem 2.3]. Then $ab \in \sqrt{I_1 I_2}$. Now it is enough that is shown $ab \in I_1 I_2$. Since $ab \in \sqrt{I_1 I_2} = P_1 \cap P_2 \subseteq P_1$, we can conclude $a \in P_1$. On the other hand, $a \notin \sqrt{I_1 I_2}$ and $ab \in \sqrt{I_1 I_2} = P_1 \cap P_2 \subseteq P_2$ and $a \in P_1$, then we conclude $a \notin P_2$ and $b \in P_2$. Furthermore, $b \in P_2$ and $b \notin \sqrt{I_1 I_2}$, so we have $b \notin P_1$. Hence $a \in P_1$ and $b \in P_2$. Now if $a \in I_1$ and $b \in I_2$, then $ab \in I_1 I_2$. So we can assume that $a \notin I_1$. Then as I_1 is a P_1-primary ideal of R, we have $bc \in P_1 = \sqrt{I_1}$. Since $b \in P_2$, we conclude that $bc \in \sqrt{I_1 I_2}$, which is a contradiction. Hence $a \in I_1$. By similar way, we get that $b \in I_2$. Therefore $ab \in I_1 I_2$. Consequently, $I_1 I_2$ is a 2-absorbing primary ideal of R.
(2) Assume that $a, b, c \in R$ with $abc \in I_1 \cap I_2$ but $ac \notin \sqrt{I_1 \cap I_2}$ and $bc \notin \sqrt{I_1 \cap I_2}$. Since I_1 is a P_1-primary ideal and I_2 is a P_2-primary ideal of R, we have $\sqrt{P} := \sqrt{I_1 \cap I_2} = P_1 \cap P_2$. Then $a, b, c \notin \sqrt{P} = P_1 \cap P_2$. Now the proof is completely similar to that of part (1).
(3) This part is similar (1). □

Corollary 2.7. *Let R be a commutative semiring and P_1, P_2 be prime ideals of R. If P_1^n is a P_1-primary ideal and P_2^m is a P_2-primary ideal for every $n, m \geq 1$, then $P_1^n P_2^m$ and $P_1^n \cap P_2^m$ are 2-absorbing primary ideals of R.*

Let R be a commutative semiring and I be a proper ideal of R. If $\sqrt{I}$ is maximal in R, then I is primary [2, Theorem 40]. Recall that a proper ideal M is said to be maximal if there is no ideal I of R satisfying $M \subset I \subset R$. Moreover, every maximal ideal of a semiring is a prime ideal. In the following result, we show that if $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R.

Theorem 2.8. *Let R be a commutative semiring and I be an ideal of R. If $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R.*

Proof. Let $a, b, c \in R$ with $abc \in I$ and $ab \notin I$. Since $(ac)(bc) = abc^2 \in I \subseteq \sqrt{I}$ and $\sqrt{I}$ is a prime ideal, we can conclude $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence I is a 2-absorbing primary ideal of R. □

Corollary 2.9. *If P is a prime ideal of semiring R, then P^n is a 2-absorbing primary ideal of R for each positive integer n.*

In the following we show that an example of 2-absorbing primary principal ideal in semiring $(\mathbb{Z}_0^+, +, .)$.

Example 2.10. Let I be a 2-absorbing primary principal ideal in semiring $(\mathbb{Z}_0^+, +, .)$. Then $I = \{0\}$ or $I =< p^n >$ where p is a prime number and positive integer $n > 1$ or $I =< p_1^n p_2^m >=< d >$ where $d = p_1^n p_2^m$ is the power factorization of d and some positive integer $n, m > 1$.

Theorem 2.11. *Let R be a commutative semiring, I be a Q-ideal and P be a k-ideal of R/I with $I \subseteq P$. If P is a 2-absorbing primary ideal of R, then P/I is a 2-absorbing primary ideal of R/I.*

Proof. Let P be a 2-absorbing primary ideal of R. Assume that $q_1+I, q_2+I, q_3+I \in R/I$ such that $(q_1+I)\odot(q_2+I)\odot(q_3+I) \in P/I$ where $q_1, q_2, q_3 \in Q$. So there exists a unique element $q_4 \in P\cap Q$ such that $q_1q_2q_3 + I \subseteq q_4 + I \in P/I$, then $q_1q_2q_3 \in P$. Since P is a 2-absorbing primary ideal, we have $q_1q_2 \in P$ or $q_2q_3 \in \sqrt{P}$ or $q_1q_3 \in \sqrt{P}$. If $q_1q_2 \in P$, then $(q_1 + I) \odot (q_2 + I) = q_5 + I$ where q_5 is the unique element with $q_1q_2 + I \subseteq q_5 + I$. Hence $q_1q_2 + r = q_1q_2 + s$ for some $r, s \in I$, as P is a k-ideal and $q_5 \in P\cap Q$. So $(q_1+I)\odot(q_2+I) \in P/I$. Now we assume that $q_1q_3 \in \sqrt{P}$. Then there exists positive integer n such that $(q_1q_3)^n = q_1^nq_3^n \in P$. Since $q_1q_3 \subseteq q_1q_3 + I$, we can conclude that $(q_1q_3)^n \subseteq (q_1q_3 + I)^n$, thus $(q_1q_3)^n \subseteq (q_1q_3 + I)^n \cap q_1^nq_3^n + I$, and it follows that $(q_1q_3+I)^n = q_1^nq_3^n+I \in P/I$, that is, $(q_1+I)^n \odot(q_3+I)^n \in P/I$. By the similar way, we can show that $(q_2 + I)^n \odot (q_3 + I)^n \in P/I$, hence P/I is a 2-absorbing primary ideal of R/I. □

In the following we get some characterizations of 2-absorbing primary ideals in the morphisms of semirings.

Theorem 2.12. *Let $\gamma : R \to S$ be a morphism of commutative semirings. Then the following statements hold:*
(1) *If J is a 2-absorbing primary ideal of S, then $\gamma^{-1}(J)$ is a 2-absorbing primary ideal of R;*
(2) *If I is a 2-absorbing primary k-ideal of R with $ker(\gamma) \subseteq I$ and γ is onto steady morphism, then $\gamma(I)$ is a 2-absorbing primary k-ideal of S.*

Proof. (1) Assume that $a, b, c \in R$ with $abc \in \gamma^{-1}(J)$. Then $\gamma(abc) = \gamma(a)\gamma(b)\gamma(c) \in J$. Since J is a 2-absorbing primary ideal of S, we have $\gamma(a)\gamma(b) \in J$ or $\gamma(b)\gamma(c) \in \sqrt{J}$ or $\gamma(a)\gamma(c) \in \sqrt{J}$. Hence $ab \in \gamma^{-1}(J)$ or $bc \in \gamma^{-1}(\sqrt{J})$ or $ac \in \gamma^{-1}(\sqrt{J})$. Obviously $\gamma^{-1}(\sqrt{J}) = \sqrt{\gamma^{-1}(J)}$. Therefore $\gamma^{-1}(J)$ is a 2-absorbing primary ideal of R.
(2) Assume that I is a 2-absorbing primary ideal of R and $ker(\gamma) \subseteq I$. Clearly, $\gamma(I)$ is a k-ideal of S. Let $abc \in \gamma(I)$ for some $a, b, c \in S$. There exists $x, y, z \in R$ such that $\gamma(x) = a$, $\gamma(y) = b$ and $\gamma(z) = c$. Then $abc = \gamma(x)\gamma(y)\gamma(z) = \gamma(xyz) \in \gamma(I)$ and so $\gamma(xyz) = \gamma(r)$ for some $r \in I$. Since γ is steady, $xyz + s = r + t$ for some $s, t \in I$. Hence $xyz \in I$, as I is a k-ideal of R and $ker(\gamma) \subseteq I$. Since I is 2-absorbing primary, we have $xy \in I$ or $yz \in \sqrt{I}$ or $xz \in \sqrt{I}$. Thus $ab \in \gamma(I)$ or $bc \in \gamma(\sqrt{I}) \subseteq \sqrt{\gamma(I)}$ or $ac \in \gamma(\sqrt{I}) \subseteq \sqrt{\gamma(I)}$. Therefore $\gamma(I)$ is 2-absorbing primary. □

Let I and J be ideals of semiring R with $I \subseteq J$. Then $J/I = \{a + I | a \in J\}$ is an ideal of R. Moreover, if J is a k-ideal of R, then J/I is a k-ideal of R/J, [5, Lemma 2]. In the following we can use it to show next result.

Corollary 2.13 *Let R be a commutative semiring and J be an ideal of R. If I is a 2-absorbing primary k-ideal of R with $J \subseteq I$, then I/J is a 2-absorbing primary ideal of R/J.*

A non-empty subset S of a semiring R said to be multiplicatively closed subset whenever $a, b \in S$ implies that $ab \in S$. Let S be a multiplicatively closed subset of a semiring R. The relation is defined on the set $R\times S$ by $(r,s)\sim(t,y)\Leftrightarrow ury=uts$ for some $u\in S$ is an equivalence relation and the equivalence class of $(r,s)\in R\times S$ denoted by r/s. The set of all equivalence classes of $R\times S$ under " $\sim$ " denoted by $S^{-1}R$. The addition and multiplication are defined $r/s+t/y=(ry+ts)/sy$ and $(r/s)(t/y)=rt/sy$. The semiring $S^{-1}R$ is called *quotient semiring R by S*. Suppose that R is a commutative semiring, S be a multiplicatively closed subset and I be an ideal. The set $S^{-1}I=\{a/b|a\in I, b\in S\}$ is an ideal of $S^{-1}R$. It is easy to show that if I is a k-ideal, then $S^{-1}I$ is a k-ideal of $S^{-1}R$, (see [12, 14, 15]). Clearly, we get some results that follow by $(r/s)=(t/y)\Leftrightarrow ury=uts$ for some $u\in S$ and $r/s=ar/as$ for all $a\in R$ and $r,s\in S$; its zero element is $0/1$ and its multiplicative identity element is $1/1$.

Theorem 2.14. *Let R be a commutative semiring and S be a multiplicatively closed subset and I be a k-ideal of R. If I is a 2-absorbing primary ideal of R with $I\cap S=\emptyset$, then $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$.*

Proof. Assume that $a,b,c\in R$ and $s,t,r\in S$ with $(a/s)(b/t)(c/r)\in S^{-1}I$. Then there exists $u\in S$ such that $(ua)bc\in I$. As I is a 2-absorbing primary ideal of R, we conclude that $(ua)b\in I$ or $bc\in\sqrt{I}$ or $(ua)c\in\sqrt{I}$. Firstly, if $(ua)b\in I$, then $(a/s)(b/t)=uab/ust\in S^{-1}I$. If $bc\in\sqrt{I}$, then $(b/t)(c/r)\in S^{-1}(\sqrt{I})=\sqrt{S^{-1}I}$. Finally, if $(ua)c\in\sqrt{I}$, then $(a/s)(c/r)=uac/usr\in\sqrt{S^{-1}I}$. Hence $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$. □

Proposition 2.15. *Let R be a commutative Semiring and P be a 2-absorbing primary ideal of $S^{-1}R$. Then $P\cap R$ is a 2-absorbing primary ideal of R.*

Proof. Assume that $a,b,c\in R$ with $abc\in P\cap R$. Then $(a/1)(b/1)(c/1)\in P\cap R$. Since P is a 2-absorbing primary ideal of $S^{-1}R$, we have $(a/1)(b/1)\in P$ or $(b/1)(c/1)\in\sqrt{P}$ or $(a/1)(c/1)\in\sqrt{P}$. Hence $ab\in P\cap R$ or $bc\in\sqrt{P\cap R}$ or $ac\in\sqrt{P\cap R}$. Therefore $P\cap R$ is a 2-absorbing primary ideal of R. □

Theorem 2.16. *Let $R=R_1\times R_2$ where R_1,R_2 be commutative semirings. Then the following statements hold:*
(1) I_1 is a 2-absorbing primary ideal of R_1 if and only if $I_1\times R_2$ is a 2-absorbing primary ideal of R;
(2) I_2 is a 2-absorbing primary ideal of R_2 if and only if $R_1\times I_2$ is a 2-absorbing primary ideal of R.

Proof. (1) Let I_1 be a 2-absorbing primary ideal of R_1. Assume that $(a,1)(b,1)(c,1)=(abc,1)\in I_1\times R_2$ such that $a,b,c\in R_1$. Then $abc\in I_1$ and so we conclude that $ab\in I_1$ or $bc\in\sqrt{I_1}$ or $ac\in\sqrt{I_1}$. Hence $(ab,1)\in I_1\times R_2$ or $(bc,1)\in\sqrt{I_1\times R_2}=\sqrt{I_1}\times R_2$ or $(ac,1)\in\sqrt{I_1\times R_2}=\sqrt{I_1}\times R_2$. Therefore $I_1\times R_2$ is a 2-absorbing primary ideal of R. Conversely, the proof is trivial.
(2) The proof is similar (1). □

Theorem 2.17. *Let $R=R_1\times R_2$ where R_1,R_2 be commutative semirings and $I=I_1\times I_2$ be an ideal of R such that I_1 and I_2 are ideals of R_1 and R_2 respectively. Then the following statements are equivalent:*

(1) *I is a 2-absorbing primary ideal of R;*
(2) *$I = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2 or $I = I_1 \times I_2$ for some primary ideal I_1 of R_1 and for some primary ideal I_2 of R_2.*

Proof. (1) $\Rightarrow$ (2) Assume that I is a 2-absorbing primary ideal of R. If $I_2 = R_2$, then I is 2-absorbing primary, by Theorem 2.16. If $I_1 = R_1$, then I is 2-absorbing primary, by Theorem 2.16. Suppose that $I_2 \neq R_2$ and $I_1 \neq R_1$. On the other hand $\sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$. Assume that I_1 is not a primary ideal of R_1. So there exists $a, b \in R_1$ such that $ab \in I_1$ but $a \notin I_1$ and $b \notin \sqrt{I_1}$. Let $x = (a, 1), y = (1, 0)$ and $z = (b, 1)$. Hence $xyz = (a, 1)(1, 0)(b, 1) = (ab, 0) \in I$ but neither $(a, 1)(1, 0) \in I$ nor $(1, 0)(b, 1) \in \sqrt{I}$ nor $(a, 1)(b, 1) \in \sqrt{I}$, which is a contradiction. Then I_1 is a primary ideal of R_1. Now assume that I_2 is not a primary ideal of R_2. Then there are $c, d \in R_2$ such that $cd \in I_2$ but $c \notin I_2$ and $d \notin \sqrt{I_2}$. Let $x = (1, c), y = (0, 1)$ and $z = (1, d)$. Hence $xyz = (1, c)(0, 1)(1, d) = (0, cd) \in I$ but neither $(1, c)(0, 1) \in I$ nor $(0, 1)(1, d) \in \sqrt{I}$ nor $(1, c)(1, d) \in \sqrt{I}$, which is a contradiction. Hence I_2 is a primary ideal of R_2.
(2) $\Rightarrow$ (1) If $I_2 = R_2$ and I_1 is a 2-absorbing primary ideal of R_1, then $I = I_1 \times R_2$ is a 2-absorbing primary ideal of R, by Theorem 2.16. Similarly, if $I_1 = R_1$ and I_2 is a 2-absorbing primary ideal of R_2, then $R_1 \times I_2$ is a 2-absorbing primary ideal of R. Now assume that I_1 and I_2 are primary ideals of R_1 and R_2 respectively. Suppose that $(a_1, b_1)(a_2, b_2)(a_3, b_3) \in I_1 \times I_2$ for some $a_1, a_2, a_3 \in R_1$ and $b_1, b_2, b_3 \in R_2$. Since I_1 and I_2 are primary ideals, we may assume that one of a_i's is in I_1, say a_1 and one of b_i's is in I_2, say b_2. Hence $(a_1, b_1)(a_2, b_2) \in I_1 \times I_2$. Consequently, $I_1 \times I_2$ is a 2-absorbing primary ideal of R. $\square$

Example 2.18. Let $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ be a semiring.
(1) We consider $I_1 = 12\mathbb{Z}$ and $I_2 = 6\mathbb{Z}$ which are 2-absorbing primary ideals of $\mathbb{Z}_0^+$. Then $I = 12\mathbb{Z} \times 6\mathbb{Z}$ is a 2-absorbing primary ideal. However, they are not primary ideals.
(2) Assume that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is an ideal of R. As we know that $4\mathbb{Z}$ is a primary ideal and $6\mathbb{Z}$ is not a primary ideal. Although it is a 2-absorbing primary ideal. Then it is easy to see that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is a 2-absorbing primary ideal of R.

Definition 2.19. Let R be a commutative semiring and I be a proper ideal of R. The ideal I is said to be a *strongly 2-absorbing primary ideal* if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.

Theorem 2.20. *Let R be a commutative semiring and I be a proper strongly k-ideal of R. Then the following statements are equivalent:*
(1) *I is a 2-absorbing primary ideal;*
(2) *If $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.*

Proof. (1) $\Rightarrow$ (2) Assume that I is a 2-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of R. Let $I_1I_2 \nsubseteq I$, $I_2I_3 \nsubseteq \sqrt{I}$ and $I_1I_3 \nsubseteq \sqrt{I}$. Then there exists $i_1 \in I_1$ and $i_2 \in I_2$ such that $i_1i_2I_3 \subseteq I$ with $i_1I_3 \nsubseteq \sqrt{I}$ and $i_2I_3 \nsubseteq \sqrt{I}$.

Hence $i_1 i_2 \in I$, by Proposition 2.3. Since $I_1 I_2 \nsubseteq I$, there exists $a \in I_1$ and $b \in I_2$ such that $ab \notin I$. By Proposition 2.3 and since $abI_3 \subseteq I$ and I is 2-absorbing primary, we have $aI_3 \subseteq \sqrt{I}$ or $bI_3 \subseteq \sqrt{I}$. Now we have three cases:
Case I: We assume that $aI_3 \subseteq \sqrt{I}$ but $bI_3 \nsubseteq \sqrt{I}$. Since $i_1 bI_3 \subseteq I$ but $bI_3 \nsubseteq \sqrt{I}$ and $i_1 I_3 \nsubseteq \sqrt{I}$, we have $i_1 b \in I$, by Proposition 2.3. We have $aI_3 \subseteq \sqrt{I}$ but $i_1 I_3 \nsubseteq \sqrt{I}$, then $(a+i_1)I_3 \nsubseteq \sqrt{I}$. Since I is a strongly k-ideal. On the other hand, $(a+i)bI_3 \subseteq I$, $bI_3 \nsubseteq \sqrt{I}$ and $(a+i_1)I_3 \nsubseteq \sqrt{I}$, we conclude that $(a+i)b \in I$, by Proposition 2.3. Then $ab \in I$ as I is a strongly k-ideal, which is a contradiction.
Case II: We assume that $aI_3 \nsubseteq \sqrt{I}$ but $bI_3 \subseteq \sqrt{I}$. Hence the complete proof is the same way by Case I.
Case III: We assume that $aI_3 \subseteq \sqrt{I}$ and $bI_3 \subseteq \sqrt{I}$. At the first we consider $bI_3 \subseteq \sqrt{I}$. Since $i_2 I_3 \nsubseteq \sqrt{I}$ and I is a strongly k-ideal, we can conclude that $(b+i_2)I_3 \nsubseteq \sqrt{I}$. Since $i_1(b+i_2)I_3 \subseteq I$ but $i_1 I_3 \nsubseteq \sqrt{I}$ and $(b+i_2)I_3 \nsubseteq \sqrt{I}$, we have $i_1(b+i_2) \in I$. Then $i_1 b \in I$ and $i_1 i_2 \in I$. Since I is a strongly k-ideal. Now we consider $aI_3 \subseteq \sqrt{I}$ but $i_1 I_3 \nsubseteq \sqrt{I}$, so $(a+i_1)I_3 \nsubseteq \sqrt{I}$. As $(a+i_1)i_2 I_3 \subseteq I$ but $(a+i_1)I_3 \nsubseteq \sqrt{I}$ and $i_2 I_3 \nsubseteq \sqrt{I}$, we conclude that $(a+i_1)i_2 \in I$. Then $ai_2 \in I$ and $i_1 i_2 \in I$. Now as $(a+i_1)(b+i_2)I_3 \subseteq I$ but $(a+i_1)I_3 \nsubseteq \sqrt{I}$ and $(b+i_2)I_3 \nsubseteq \sqrt{I}$, we can conclude that $(a+i_1)(b+i_2) = ab+c \in I$ and so $ab \in I$, which is a contradiction. Therefore $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq \sqrt{I}$ or $I_1 I_3 \subseteq \sqrt{I}$.
(2) $\Rightarrow$ (1) The proof is straightforward. □

One of the main sense, that is generalized for semirings, is the concept of primary decomposition. Let R be a commutative semiring and I be a proper ideal of R. A *primary decomposition* of I is an epithet for I as an intersection of finitely many primary ideals of R. On the other words a primary decomposition of I is $I = I_1 \cap \cdots \cap I_r$ where each I_i is P_i-primary ideal in semiring R. It is easy to show that if R is a Noetherian semiring, then every proper k-ideal is a finite intersection of primary k-ideals. Since every primary ideal of a semiring R is a 2-absorbing primary ideal, we claim that every proper ideal of R has a 2-absorbing primary decomposition. In the next, we define the concept of P-2-absorbing primary ideal that is generalization of the concept of P-primary ideal in semirings.

Definition 2.21. Let R be a semiring and I be a 2-absorbing primary ideal of R. If $\sqrt{I} = P$ is a 2-absorbing ideal of R, then I is called *P-2-absorbing primary ideal* of R.

The following theorem gives a characterization of P-2-absorbing primary ideals of semiring R.

Theorem 2.22. *Let $I_1, \cdots, I_r$ be P-2-absorbing primary ideals of semiring R where P is a 2-absorbing ideal of R. Then $I = \bigcap_{i=1}^{n} I_i$ is a P-2-absorbing primary ideal of R.*

Proof. Assume that $abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Then $ab \notin I_i$ for some $1 \leq i \leq n$. Since every I_i is P-2-absorbing primary ideal, we can conclude that $ac \in \sqrt{I_i} = P$ or $bc \in \sqrt{I_i} = P$. Therefore I is a P-2-absorbing primary ideal of

R. □

3. Weakly 2-Absorbing Primary Ideals in Commutative Semirings

In this section we define the concept of weakly 2-absorbing primary ideal of a commutative semiring and generalize some basic results in semirings.

Definition 3.1. Let R be a semiring and I be a proper ideal. The ideal I is said to be a *weakly 2-absorbing primary ideal* if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$.

Lemma 3.2. *Let R be a semiring. Then the following statements hold:*
(1) *Every weakly primary ideal is weakly 2-absorbing primary;*
(2) *Every 2-absorbing primary ideal is weakly 2-absorbing primary.*

Theorem 3.3. *Let R be a commutative semiring, I be a Q-ideal and P be a k-ideal of R/I with $I \subseteq P$. Then the following statements hold:*
(1) *If P is a weakly 2-absorbing primary ideal of R, then P/I is a weakly 2-absorbing primary ideal of R/I;*
(2) *If I and P/I are weakly 2-absorbing primary ideals, then P is a weakly 2-absorbing primary ideal of R.*

Proof. (1) Assume that $q_1 + I, q_2 + I, q_3 + I \in R/I$ such that $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I$ where $q_1, q_2, q_3 \in Q$ and $0 \neq q_1q_2q_3 \in I$. Now this part proves completely similar Theorem 2.11.
(2) Assume that I and P/I are weakly 2-absorbing primary ideals. Let $0 \neq abc \in P$ for some $a, b, c \in R$. If $abc \in I$, then $ab \in I \subseteq P$ or $(bc)^n \in I \subseteq P$ or $(ac)^n \in I \subseteq P$. Since I is a weakly 2-absorbing primary ideal. So we can assume that $abc \notin I$. Then there exists $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Hence $a = q_1 + e$, $b = q_2 + f$ and $c = q_3 + g$ for some $e, f, g \in I$. Since $abc = (q_1 + e)(q_2 + f)(q_3 + g) = q_1q_2q_3 + q_1q_3f + q_2q_3e + q_3ef + q_1q_2g + q_1fg + q_2eg + efg$ and P is a k-ideal, we have $q_1q_2q_3 \in P$. Assume that q is the unique element in Q such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I$ where $q_1q_2q_3 + I \subset q + I$. Then $q_1q_2q_3 + i = q + h$ for some $i, h \in I$ and so $q \in P \cap Q$ and $q + I \in P/I$. Now suppose that $q' \in Q$ is the unique element such that $q' + I$ is the zero element in R/I. If $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q' + I$, then $q_1q_2q_3 + j = q' + l$ for some $j, l \in I$. As I is a Q-ideal of R, it is a k-ideal by [16, Corollary 2]. Thus $q_1q_2q_3 \in I$ and so $abc \in I$, which is a contradiction. Hence $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I$. Since P/I is a weakly 2-absorbing primary ideal, we conclude $q_1q_2 + I \in P/I$ or $(q_2q_3 + I)^n \in P/I$ or $(q_1q_3 + I)^n \in P/I$ for some n. If $q_1q_2 + I \in P/I$, then $ab = q_1q_2 + ef \in P$. If $(q_1q_3 + I)^n = q_1^n q_3^n + I \in P/I$, then it follows that $(ac)^n \in P$. In a similar way, we can show that $(bc)^n \in P$. Then it follows that either $q_1q_2 \in P$ or $q_2q_3 \in \sqrt{P}$ or $q_1q_3 \in \sqrt{P}$. Hence $ab \in P$ or $bc \in \sqrt{P}$ or $ac \in \sqrt{P}$. Therefore P is a weakly 2-absorbing primary ideal of semiring R. □

Example 3.4. Let $R = \mathbb{Z}_{12}$ be a commutative semiring and $I = \{0\}$ be an ideal of R. Then I is a weakly 2-absorbing primary ideal of R, by definition. Now we consider $2.2.3 \in I$ but neither $2.2 \in I$ nor $2.3 \in \sqrt{I}$. So I is not a 2-absorbing primary ideal. It is noticeable that every 2-absorbing primary ideal is a weakly 2-absorbing ideal by Lemma 3.2, but a weakly 2-absorbing primary ideal need not to be a 2-absorbing primary ideal. In the following result we show that provided which conditions it can be possible.

Lemma 3.5. *Let R be a commutative semiring and I be a k-ideal of R. If $a \in I$ and $a + b \in \sqrt{I}$ for some $a, b \in R$, then $b \in \sqrt{I}$.*

Theorem 3.6. *Let R be a commutative semiring and I be an ideal of R. If I is a weakly 2-absorbing primary k-ideal of R, then either I is 2-absorbing primary or $I^3 = 0$.*

Proof. Assume that $I^3 \neq 0$. We show that I is a 2-absorbing primary ideal of R. Let $a, b, c \in R$ such that $abc \in I$. If $abc \neq 0$, then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, that is, I is a 2-absorbing primary ideal of R. So we suppose that $abc = 0$. At the first, we assume that $abI \neq 0$ and say $abr_0 \neq 0$ for some $r_0 \in I$. Then $0 \neq abr_0 = ab(c + r_0) \in I$. As I is weakly 2-absorbing primary, we get that $ab \in I$ or $b(c+r_0) \in \sqrt{I}$ or $a(c+r_0) \in \sqrt{I}$. Then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, by Lemma 3.5. So we can suppose that $abI = 0$. Likewise we can assume that $acI = 0$ and $bcI = 0$. Since $I^3 \neq 0$, there exists $a_0, b_0, c_0 \in I$ with $a_0b_0c_0 \neq 0$. If $ab_0c_0 \neq 0$, then $a(b + b_0)(c + c_0) \in I$, so it implies that $a(b + b_0) \in I$ or $(b + b_0)(c + c_0) \in \sqrt{I}$ or $a(c + c_0) \in \sqrt{I}$. Hence $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, by Lemma 3.5. So we can assume that $ab_0c_0 = 0$. Likewise we can consider $a_0bc_0 = 0$ and $a_0b_0c = 0$. Now we can conclude that $0 \neq a_0b_0c_0 = (a + a_0)(b + b_0)(c + c_0) \in I$, so we get $(a + a_0)(b + b_0) \in I$ or $(b + b_0)(c + c_0) \in \sqrt{I}$ or $(a + a_0)(c + c_0) \in \sqrt{I}$. By Lemma 3.5, $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence I is a 2-absorbing primary ideal of R. □

We can now use Theorem 3.6 to characterize weakly 2-absorbing primary ideals in semirings.

Corollary 3.7. *Let R be a commutative semiring and I be a weakly 2-absorbing primary k-ideal of R. If I is not a 2-absorbing primary ideal, then $\sqrt{I} = \sqrt{0}$.*

Proof. Clearly $\sqrt{0} \subseteq \sqrt{I}$. By Theorem 3.6, $I^3 = 0$. So we get $I \subseteq \sqrt{0}$, then $\sqrt{I} \subseteq \sqrt{0}$. Hence $\sqrt{I} = \sqrt{0}$. □

Corollary 3.8. *Let R be a commutative semiring. If I is a weakly 2-absorbing primary k-ideal of R that is not 2-absorbing primary ideal, then I is nilpotent.*

Proposition 3.9. *Let R be a commutative Semiring and P be a weakly 2-absorbing primary ideal of $S^{-1}R$. Then $P \cap R$ is a weakly 2-absorbing primary ideal of R.*

Proof. This following from Proposition 2.15. □

Theorem 3.10. *Let $R = R_1 \times R_2$ where R_1, R_2 be commutative semirings and $I = I_1 \times I_2$ be an ideal of R such that I_1 and I_2 are ideals of R_1 and R_2 respectively. If I is a weakly 2-absorbing primary ideal of R, then either $I = 0$ or I is 2-absorbing primary.*

Proof. Assume that $I = I_1 \times I_2$ is a weakly 2-absorbing primary and $I \neq 0$. We show that I is 2-absorbing primary. Let $(a,b) \in I = I_1 \times I_2$ such that $(a,b) \neq (0,0)$. Then $(0,0) \neq (a,1)(1,1)(1,b) \in I$. So either $(a,1)(1,1) \in I$ or $(1,1)(1,b) \in \sqrt{I}$ or $(a,1)(1,b) \in \sqrt{I}$. If $(a,1) \in I$, then $(a,1) \in I_1 \times R_2$. We show that I_1 is a 2-absorbing primary ideal of R_1. Let $x,y,z \in R_1$ such that $xyz \in I_1$. Then $(0,0) \neq (x,1)(y,1)(z,1) \in I$. Since I is weakly 2-absorbing primary, we have $(x,1)(y,1) \in I_1 \times R_2$ or $(y,1)(z,1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$ or $(x,1)(z,1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$ and so $xy \in I_1$ or $yz \in \sqrt{I_1}$ or $xz \in \sqrt{I_1}$. Then $I_1 \times R_2$ is a 2-absorbing primary ideal of R, by Theorem 2.16. If $(1,b) \in \sqrt{I_1 \times I_2}$, then $(1,b^n) \in I_1 \times I_2$ for some positive integer n and so $I = R_1 \times I_2$. By similar way, $R_1 \times I_2$ is a 2-absorbing primary ideal. Now if $(a,1)(1,b) \in \sqrt{I_1 \times I_2}$, we have $(a^n,b^n) \in I_1 \times I_2$ for some positive integer n. We show that I_1 and I_2 are primary ideals. Suppose that $R_2 \neq I_2$. Let $a,b \in R_2$ such that $ab \in I_2$ and $0 \neq i_1 \in I_1$. Then $(0,0) \neq (i_1,1)(1,a)(1,b) = (i_1,ab) \in I_1 \times I_2$. Since $(1,a)(1,b) \notin \sqrt{I_1 \times I_2}$, we can conclude that $(i,1)(1,a) \in I_1 \times I_2$ or $(i,1)(1,b) \in \sqrt{I_1 \times I_2}$. Then $a \in I_2$ or $b \in \sqrt{I_2}$, that is, I_2 is a primary ideal. Similarly, we assume that $c,d \in R_1$ such that $cd \in I_1$ and let $0 \neq i_2 \in I_2$. Hence $(0,0) \neq (1,i_2)(c,1)(d,1) = (cd,i_2) \in I_1 \times I_2$ and as $R_1 \neq I_1$, we have$(c,1)(d,1) \notin \sqrt{I_1 \times I_2}$. Then we can conclude that $(1,i_2)(c,1) \in I_1 \times I_2$ or $(1,i_2)(d,1) \in \sqrt{I_1 \times I_2}$. Hence either $c \in I_1$ or $d \in \sqrt{I_1}$ and so I_1 is a primary ideal. Therefore $I_1 \times I_2$ is a 2-absorbing primary ideal of R, by Theorem 2.17. □

4. Properties of 2-Absorbing Primary Ideals in Semiring $\mathbb{Z}_0^+$

In this section we give characterizations of 2-absorbing primary ideal in semiring $\mathbb{Z}_0^+$. In the following theorems we get that some results in semiring $(\mathbb{Z}_0^+, \gcd, lcm)$ where $a \oplus b = \gcd\{a,b\}$ and $a \otimes b = lcm\{a,b\}$ for $a,b \in \mathbb{N}$. $a \oplus 0 = a$ and $a \otimes 0 = 0$ for all $a \in \mathbb{Z}_0^+$.

Theorem 4.1. *A non-zero ideal I of semiring $(\mathbb{Z}_0^+, \gcd, lcm)$ is 2-absorbing primary ideal if and only if I is a 2-absorbing ideal.*

Proof. Assume that I is a 2-absorbing primary ideal. Let $a \otimes b \otimes c \in I$ for some $a,b,c \in \mathbb{N}$. Then $a \otimes b \in I$ or $b \otimes c \in \sqrt{I}$ or $a \otimes c \in \sqrt{I}$. So there exists $n \in \mathbb{Z}_0^+$ such that $(a \otimes c)^n = a^n \otimes c^n = a \otimes a \otimes \cdots \otimes a \otimes c \otimes c \otimes \cdots \otimes c \in I$ and so $a \otimes c \in I$. By similar way, we can conclude that $b \otimes c \in I$. Hence I is a 2-absorbing ideal. Converse is clear. □

Theorem 4.2. *A non-zero ideal I of semiring $(\mathbb{Z}_0^+, \gcd, lcm)$ is 2-absorbing primary ideal if and only if $I =< p^n >$ for some positive integer $n > 1$ or $I =< p_1^n p_2^m >$ for some pairwise distinct prime numbers p_1, p_2 and some positive integer $n, m > 1$.*

Proof. Assume that I is a 2-absorbing primary ideal. Then by Theorem 2.22, I is a 2-absorbing ideal. By [10, Lemma 2.2], $I =< d >$ such that $d \in \mathbb{Z}_0^+ \setminus \{0,1\}$. Set $d = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \cdots, p_k$ are pairwise distinct prime. Now we consider $k > 2$. Then $p_1^{r_1} \otimes p_2^{r_2} \otimes (p_3^{r_3} \otimes \cdots \otimes p_k^{r_k}) = d \in I$ so neither $p_1^{r_1} \otimes p_2^{r_2} \in I$ nor

$p_2^{r_2} \otimes (p_3^{r_3} \otimes \cdots \otimes p_k^{r_k}) \in I$ nor $p_1^{r_1} \otimes (p_3^{r_3} \otimes \cdots \otimes p_k^{r_k}) \in I$, that is a contradiction. Hence $k \leq 2$. Therefore $I =< p^n >$ for some positive integer $n > 1$ or $I =< p_1^n p_2^m >$ for some pairwise distinct prime numbers p_1, p_2 and some positive integer $n, m > 1$. Conversely, if $I =< p^n >$ for some positive integer $n > 1$, we are done. So we assume that $I =< p_1^n p_2^m >$ for some pairwise distinct prime numbers p_1, p_2 and some positive integer $n, m > 1$. Then $I =< p_1^n > \cap < p_2^m >$. Since p_1, p_2 are prime numbers, $< p_1^n >$ and $< p_2^m >$ are prime ideals, by [10, Theorem 2.7]. Hence I is a 2-absorbing primary ideal, by Theorem 2.6. □

Let $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$ be a semiring with identity ∞, where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Moreover, if I is an ideal of R, then $I = \{0, 1, 2, \cdots, t\}$ for some $t \in \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+$ or $I = R$, [13, Theorem 5]. In the next theorem we show that a characterization in semiring $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$.

Theorem 4.3. *Every ideal in* $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$ *is a 2-absorbing primary ideal.*

Proof. Let I be a proper ideal of R. Then $I = \{0, 1, 2, \cdots, t\}$ for some $t \in \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+$. Assume that $a \wedge b \wedge c \in I$ for some $a, b, c \in R$. Hence a or b or $c = \min\{a, b, c\} = a \wedge b \wedge c \in I$. So we can conclude that $a \wedge b \in I$ or $b \wedge c \in \sqrt{I}$ or $a \wedge c \in \sqrt{I}$. Therefore I is a 2-absorbing primary ideal of R. □

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