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The k-Rainbow Domination and Domatic Numbers of Digraphs

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Abstract. For a positive integer k, a *k-rainbow dominating function* of a digraph D is a function f from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled, where $N^-(v)$ is the set of in-neighbors of v. A set $\{f_1, f_2, \ldots, f_d\}$ of k-rainbow dominating functions on D with the property that $\sum_{i=1}^{d} |f_i(v)| \leq k$ for each $v \in V(D)$, is called a *k-rainbow dominating family* (of functions) on D. The maximum number of functions in a k-rainbow dominating family on D is the *k-rainbow domatic number* of D, denoted by $d_{rk}(D)$. In this paper we initiate the study of the k-rainbow domatic number in digraphs, and we present some bounds for $d_{rk}(D)$.

1. Introduction

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $d^+(v) = d_D^+(v)$ for the outdegree of a vertex v and $d^-(v) = d_D^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then v is an *out-neighbor* of u and u is an *in-neighbor* of v, we also write $u \to v$ and say that u *dominates* v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and

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$N^+(v) = N_D^+(v)$, respectively. Let $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. For $S \subseteq V(D)$, we define $N^+[S] = \bigcup_{v \in S} N^+[v]$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from X to v. The underlying graph of a digraph D is the graph G obtained by replacing each arc of a digraph by a corresponding (undirected) edge. A digraph is weakly connected if its underlying graph is connected. The weakly connected components of a digraph are its maximal weakly connected subdigraphs. Consult [12] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$.

A vertex v dominates all vertices in $N^+[v]$. A subset S of vertices of D is a *dominating set* if S dominates $V(D)$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. Domination in digraphs have been studied, for example, in [6, 11, 14, 15, 19, 20].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a digraph D is a function f from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V(D)} |f(v)|$. The *$k$-rainbow domination number* of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. A *$\gamma_{rk}(D)$-function* is a k-rainbow dominating function of D with weight $\gamma_{rk}(D)$. Note that $\gamma_{r1}(D)$ is the classical domination number $\gamma(D)$. The k-rainbow domination number of a digraph was introduced by Amjadi, Bahremandpour, Sheikholeslami and Volkmann [1] and has been studied in [2].

The definition of the k-rainbow domination number for undirected graphs was introduced by Brešar, Henning and Rall [3] and has been studied by several authors (see for example, [4, 5, 7, 8, 9, 18]).

A set $\{f_1, f_2, \ldots, f_d\}$ of k-rainbow dominating functions of D such that $\sum_{i=1}^d |f_i(v)| \le k$ for each $v \in V(D)$, is called a *k-rainbow dominating family* (of functions) on D. The maximum number of functions in a k-rainbow dominating family (kRD family) on D is the *k-rainbow domatic number* of D, denoted by $d_{rk}(D)$. The case $k = 1$ was defined and investigated by Zelinka [20] in 1984 as the *outside-semidomatic number* $d^+(D) = d_{r1}(D)$.

The k-rainbow domatic number is well-defined and

$$d_{rk}(D) \ge k \tag{1.1}$$

for all digraphs D, since the set consisting of the function $f_i : V(D) \rightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $f_i(v) = \{i\}$ for each $v \in V(D)$ and each $i \in \{1, 2, \ldots, k\}$, forms a kRD family on D.

The definition of the k-rainbow domatic number for undirected graphs was given by Sheikholeslami and Volkmann [17] and has been studied by several authors [10, 16].

Our purpose in this paper is to initiate the study of the k-rainbow domatic number in digraphs. We start with some bounds on the k-rainbow domination number, and then we study basic properties for the k-rainbow domatic number of a digraph. In addition, we present some Nordhaus-Gaddum type results on the k-rainbow domatic number.

2. Bounds on the k-Rainbow Domination Number

In [1] the following bounds on the k-rainbow domination number were proved.

Proposition A. ([1]) *Let $k \geq 1$ be an integer. If D is a digraph of order n, then*

$$\min\{k, n\} \leq \gamma_{rk}(D) \leq n.$$

Proposition B. ([1]) *If $k \geq 1$ is an integer, and D is a digraph of order n, then*

$$\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1.$$

Proposition 1. *Let k be a positive integer. If D is a digraph of order n with the property that* $\max\{\Delta^+(D), \Delta^-(D)\} \geq k$, *then $\gamma_{rk}(D) \leq n - 1$.*

Proof. If $\Delta^+(D) \geq k$, then Proposition B implies that $\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1 \leq n - 1$.

Assume next that $\Delta^-(D) \geq k$. Let $d^-(v) = \Delta^-(D)$, and let $w_1, w_1, \ldots, w_k$ be k in-neighbors of v. Define the function $f : V(D) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(w_i) = \{i\}$ for $1 \leq i \leq k$, $f(v) = \emptyset$ and $f(x) = \{1\}$ otherwise. Then f is a k-rainbow dominating function of weight $\omega(f) = n - 1$ and thus $\gamma_{rk}(D) \leq n - 1$. □

Corollary 2. *Let $k \geq 1$ be an integer. If D is a digraph of order n such that $\gamma_{rk}(D) = n$, then* $\max\{\Delta^+(D), \Delta^-(D)\} \leq k - 1$.

For $1 \leq k \leq 2$, we show that the converse of Corollary 2 is valid.

Proposition 3. *Let $k \geq 1$ be an integer such that $k \leq 2$, and let D be a digraph of order n. If* $\max\{\Delta^+(D), \Delta^-(D)\} \leq k - 1$, *then $\gamma_{rk}(D) = n$.*

Proof. If $k = 1$ and $\max\{\Delta^+(D), \Delta^-(D)\} \leq k - 1 = 0$, then D is the empty digraph and hence $\gamma_{r1}(D) = \gamma(D) = n$.

Now let $k = 2$. If $\max\{\Delta^+(D), \Delta^-(D)\} \leq k - 1 = 1$, then the weakly components of D are directed paths or directed cycles and therefore $\gamma_{r2}(D) = n$. □

The following example will demonstrate that Proposition 3 is not valid for $k \geq 3$ in general.

Example 4. Let $k \geq 3$ be an integer. Define the digraph H by the vertex set u, v and $x_1, x_2, \ldots, x_{k-1}$ such that u and v dominate x_i for $1 \leq i \leq k - 1$. Then $\Delta^+(H) = k - 1$ and $\Delta^-(H) = 2$ and therefore $\max\{\Delta^+(H), \Delta^-(H)\} \leq k - 1$. Now define the function $f : V(H) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(u) = \{1, 2, \ldots, k - 1\}$, $f(v) = \{k\}$ and $f(x_i) = \emptyset$ for $1 \leq i \leq k - 1$. Then f is a k-rainbow dominating function on H of weight $\omega(f) = k$ and thus $\gamma_{rk}(H) \leq k = n(H) - 1$.

Theorem 5. *Let $k \geq 1$ be an integer, and let D be a digraph of order $n \geq k$. Then $\gamma_{rk}(D) = k$ if and only if $n = k$ or $n > k$ and there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subset V(D)$ with $t \leq k$ such $V(D) - A \subseteq N^+(v_i)$ for $1 \leq i \leq t$.*

Proof. According to Proposition A, we note that $\gamma_{rk}(D) \geq k$. If $n = k$, then obviously $\gamma_{rk}(D) = k$. Now let $n > k$. Define the function $f : V(D) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v_i) = \{i\}$ for $1 \leq i \leq t-1$, $f(v_t) = \{t, t+1, \ldots, k\}$ and $f(x) = \emptyset$ otherwise. Then f is a k-rainbow dominating function on D of weight $\omega(f) = k$ and thus $\gamma_{rk}(D) \leq k$ and so $\gamma_{rk}(D) = k$.

Conversely, assume that $\gamma_{rk}(D) = k$. Let f be a $\gamma_{rk}(D)$-function, and let $V_0 = \{v : |f(v)| = 0\}$. If $V_0 = \emptyset$, then $n = k$. If $V_0 \neq \emptyset$, then let $v \in V_0$. By definition, we have $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$. Now let $v_1, v_2, \ldots, v_t \in N^-(v)$ all vertices in $N^-(v)$ with the property that $|f(v_i)| \neq 0$ for $1 \leq i \leq t$. Then the condition $\gamma_{rk}(D) = k$ implies that $\sum_{i=1}^{t} |f(v_i)| = k$, $t \leq k$ and $V(D) - \{v_1, v_2, \ldots, v_t\} \subseteq N^+(v_i)$ for each $i \in \{1, 2, \ldots, t\}$. □

Now we prove a lower bound on the k-rainbow domination number in terms of order and maximum outdegree.

Theorem 6. *Let $k \geq 1$ be an integer. If D is a digraph of order n, then*

$$\gamma_{rk}(D) \geq \left\lceil \frac{kn}{\Delta^+(D) + k} \right\rceil.$$

Proof. Let f be a $\gamma_{rk}(D)$-function, and let $V_i = \{v : |f(v)| = i\}$ for $i = 0, 1, \ldots, k$. Then $\gamma_{rk}(D) = |V_1| + 2|V_2| + \ldots + k|V_k|$ and $n = |V_0| + |V_1| + \ldots + |V_k|$. Let $A_0 = (V(D) - V_0, V_0)$ be the set of arcs from $V(D) - V_0$ to V_0. Since f is a $\gamma_{rk}(D)$-function, we obtain

(2.1)
$$k|V_0| \leq \sum_{xy \in A_0,\ x \in V(D) - V_0} |f(x)| \leq \Delta^+(D)(|V_1| + 2|V_2| + \ldots + k|V_k|) = \gamma_{rk}(D)\Delta^+(D).$$

Now it follows from (2.1) that

$$\begin{aligned}
&(\Delta^+(D) + k)\gamma_{rk}(D) = \Delta^+(D)\gamma_{rk}(D) + k\gamma_{rk}(D) \\
&\geq k|V_0| + k(|V_1| + 2|V_2| + \ldots + k|V_k|) \\
&= k(|V_0| + |V_1| + \ldots + |V_k|) + k(|V_2| + 2|V_3| + \ldots + (k-1)|V_k|) \\
&= kn + k(|V_2| + 2|V_3| + \ldots + (k-1)|V_k|) \\
&\geq kn,
\end{aligned}$$

and this leads to the desired bound. □

The case $k = 1$ of Theorem 6 can be found in [13] as Theorem 15.57, and the case $k = 2$ of this bound was proved in [1].

3. Properties of the k-Rainbow Domatic Number

In this section we mainly present basic properties of $d_{rk}(D)$ and bounds on the k-rainbow domatic number of a graph.

Theorem 7. *If D is a digraph of order n, then*

$$\gamma_{rk}(D) \cdot d_{rk}(D) \leq kn.$$

Moreover, if $\gamma_{rk}(D) \cdot d_{rk}(D) = kn$, then for each kRD family $\{f_1, f_2, \dots, f_d\}$ on D with $d = d_{rk}(D)$, each function f_i is a $\gamma_{rk}(D)$-function and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a kRD family on D such that $d = d_{rk}(D)$. Then

$$\begin{aligned} d \cdot \gamma_{rk}(D) &= \sum_{i=1}^{d} \gamma_{rk}(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} |f_i(v)| \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V(D)} k = kn. \end{aligned}$$

If $\gamma_{rk}(D) \cdot d_{rk}(D) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the kRD family $\{f_1, f_2, \dots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} |f_i(v)| = \gamma_{rk}(D)$. Thus each function f_i is a $\gamma_{rk}(D)$-function, and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V(D)$. □

Corollary 8. *If k is a positive integer, and D is a digraph of order $n \geq k$, then*

$$d_{rk}(G) \leq n.$$

Proof. Since $n \geq k$, Proposition A leads to $\gamma_{rk}(D) \geq k$. Therefore it follows from Theorem 7 that

$$d_{rk}(D) \leq \frac{kn}{\gamma_{rk}(D)} \leq \frac{kn}{k} = n.$$

□

Corollary 9. *If k is a positive integer, and D is isomorphic to the complete digraph K_n^* of order $n \geq k$, then $d_{rk}(D) = n$.*

Proof. In view of Corollary 8, we have $d_{rk}(D) \leq n$. If $\{v_1, v_2, \dots, v_n\}$ is the vertex set of D then we define the function $f_i : V(D) \to \mathcal{P}(\{1, 2, \dots, k\})$ by $f_i(v_j) = \{1, 2, \dots, k\}$ for $i = j$ and $f_i(v_j) = \emptyset$ for $i \neq j$, where $i, j \in \{1, 2, \dots, n\}$. Then $\{f_1, f_2, \dots, f_n\}$ is a kRD family on D and thus $d_{rk}(D) = n$. □

Theorem 10. *If D is a digraph of order $n \geq k$, then*

$$\gamma_{rk}(D) + d_{rk}(D) \leq n + k.$$

Proof. Applying Theorem 7, we obtain

$$\gamma_{rk}(D) + d_{rk}(D) \leq \frac{kn}{d_{rk}(D)} + d_{rk}(D).$$

Note that $d_{rk}(D) \geq k$, by inequality (1.1), and that Corollary 8 implies that $d_{rk}(D) \leq n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n$, we obtain

$$\gamma_{rk}(D) + d_{rk}(D) \leq \max\left\{\frac{kn}{k} + k, \frac{kn}{n} + n\right\} = n + k,$$

and this is the desired bound. □

If D is isomorphic to the complete digraph K_n^* of order $n \geq k$, then $\gamma_{rk}(D) = k$ and $d_{rk}(D) = n$ by Corollary 9. Thus $\gamma_{rk}(K_n^*) \cdot d_{rk}(K_n^*) = nk$ and $\gamma_{rk}(K_n^*) + d_{rk}(K_n^*) = n + k$ when $n \geq k$. This example shows that Theorems 7 and 10 are sharp.

Corollary 11. *Let $k \geq 1$ be an integer, and let D be a digraph of order $n \geq k$. If $\gamma_{rk}(D) = n$, then $d_{rk}(D) = k$.*

Proof. Inequality (1.1) shows that $d_{rk}(D) \geq k$. Furthermore, it follows by Theorem 7 that

$$d_{rk}(D) \leq \frac{kn}{\gamma_{rk}(D)} = \frac{kn}{n} = k$$

and therefore $d_{rk}(D) = k$. □

Theorem 12. *For every digraph D,*

$$d_{rk}(D) \leq \delta^-(D) + k.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family on D such that $d = d_{rk}(D)$, and let v be a vertex of minimum indegree $\delta^-(D)$. Since $\sum_{u \in N^-[v]} |f_i(u)| \geq 1$ for all $i \in \{1, 2, \ldots, d\}$ and $\sum_{u \in N^-[v]} |f_i(u)| < k$ for at most k indices $i \in \{1, 2, \ldots, d\}$, we obtain

$$\begin{aligned} kd - k(k-1) &\leq \sum_{i=1}^{d} \sum_{u \in N^-[v]} |f_i(u)| \\ &= \sum_{u \in N^-[v]} \sum_{i=1}^{d} |f_i(u)| \\ &\leq \sum_{u \in N^-[v]} k = k(\delta^-(D) + 1), \end{aligned}$$

and this leads to the desired bound. □

The special case $k = 1$ of Theorem 12 can be found in [20].

To prove sharpness of Theorem 12, let $p \geq 2$ be an integer, and let D_i be a copy of the complete digraph K^*_{p+k+1} with vertex set $V(D_i) = \{v^i_1, v^i_2, \ldots, v^i_{p+k+1}\}$ for $1 \leq i \leq p$. Now let D be the digraph obtained from $\bigcup_{i=1}^{p} D_i$ by adding a new vertex v and joining v to each v^i_1 by the arcs vv^i_1 and $v^i_1 v$. Define the k-rainbow dominating functions $f_1, f_2, \ldots, f_{p+k}$ as follows: for $1 \leq i \leq p$ and $1 \leq s \leq k$

$$f_i(v^i_1) = \{1, \ldots, k\}, f_i(v^j_{i+1}) = \{1, \ldots, k\} \text{ if } j \in \{1, \ldots, p\}\backslash\{i\} \text{ and } f(x) = \emptyset \text{ otherwise},$$

$$f_{p+s}(v) = \{1\}, f_{p+s}(v^j_{p+s+1}) = \{1, \ldots, k\} \text{ if } j \in \{1, \ldots, p\} \text{ and } f(x) = \emptyset \text{ otherwise}.$$

It is easy to see that f_i is a k-rainbow dominating function on D for each i and $\{f_1, f_2, \ldots, f_{p+k}\}$ is a k-rainbow dominating family on D. Since $\delta^-(D) = p$, we have $d_{rk}(D) = \delta^-(G) + k$.

4. Nordhaus-Gaddum Type Results

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. A digraph D is *in-regular* when $\delta^-(D) = \Delta^-(D)$ and *r-regular* when $\delta^-(D) = \Delta^-(D) = \delta^+(D) = \Delta^+(D) = r$. As an application of (1.1) and Theorem 12, we will prove our first Nordhaus-Gaddum type inequality.

Theorem 13. *For every digraph D of order n,*

$$2k \leq d_{rk}(D) + d_{rk}(\overline{D}) \leq n + 2k - 1.$$

If $d_{rk}(D) + d_{rk}(\overline{D}) = n + 2k - 1$, then D is in-regular.

Proof. Using (1.1), the inequality $2k \leq d_{rk}(D) + d_{rk}(\overline{D})$ is immediate.

Since $\delta^-(\overline{D}) = n - 1 - \Delta^-(D)$, it follows from Theorem 12 that

$$\begin{aligned} d_{rk}(D) + d_{rk}(\overline{D}) &\leq (\delta^-(D) + k) + (\delta^-(\overline{D}) + k) \\ &= (\delta^-(D) + k) + (n - \Delta^-(D) - 1 + k) \\ &\leq n + 2k - 1 \end{aligned}$$

and this is the second bound. In addition, if D is not in-regular, then $\Delta^-(D) - \delta^-(D) \geq 1$, and the inequality chain above leads to the better bound $d_{rk}(D) + d_{rk}(\overline{D}) \leq n + 2k - 2$. This completes the proof. □

Corollary 9 implies that $d_{r1}(K^*_n) = n$ and hence $d_{rk}(K^*_n) + d_{rk}(\overline{K^*_n}) = n + 1$. Consequently, the upper bound in Theorem 13 is sharp for $k = 1$. The next result gives an upper bound for the k-rainbow domatic number of some special regular digraphs.

Theorem 14. *Let D be an r-regular digraph of order n. If D has a $\gamma_{rk}(D)$-function f such that $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$ or $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < r|V_1|$, where $V_i = \{v \in V(D) \colon |f(v)| = i\}$, then*

$$d_{rk}(D) \leq r + k - 1.$$

Proof. Let f be a $\gamma_{rk}(D)$-function, and let $V_i = \{v : |f(v)| = i\}$ for $i = 0, 1, \ldots, k$. Then $\gamma_{rk}(D) = |V_1| + 2|V_2| + \cdots + k|V_k|$ and $n = |V_0| + |V_1| + \cdots + |V_k|$. Following the proof of Theorem 6, we obtain

$$(4.1) \qquad (r+k)\gamma_{rk}(D) \geq kn + k(|V_2| + 2|V_3| + \cdots + (k-1)|V_k|) \geq kn.$$

Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family of D such that $d = d_{rk}(D)$. We observe that

$$(4.2) \qquad \sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(D)} |f_i(v)| = \sum_{v \in V(D)} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V(D)} k = kn.$$

Suppose to the contrary that $d \geq r + k$. If $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$, then (4.1) shows that $\gamma_{rk}(D) \geq (kn+k)/(r+k)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{rk}(D) \geq d \left\lceil \frac{kn+k}{r+k} \right\rceil \geq (r+k)\left(\frac{kn+k}{r+k}\right) = kn + k > kn,$$

a contradiction to (4.2). If $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < r|V_1|$, then $\gamma_{rk}(D) = |V_1|$ and $n = |V_0| + |V_1|$ and thus

$$(r+k)\gamma_{rk}(D) = k|V_1| + r|V_1| > k|V_1| + k|V_0| = kn.$$

This implies that $\gamma_{rk}(G) > kn/(r+k)$, and we obtain the following contradiction to (4.2)

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{rk}(D) > (r+k)\left(\frac{kn}{r+k}\right) = kn.$$

Therefore $d \leq r + k - 1$, and the proof is complete. □

Now we improve the upper bound given in Theorem 13 for regular digraphs and $k \geq 2$.

Theorem 15. *If $k \geq 2$ is an integer, and D is an r-regular digraph of order n, then*

$$d_{rk}(D) + d_{rk}(\overline{D}) \leq n + 2k - 2.$$

Proof. Since D is r-regular, we observe that $\overline{D}$ is $(n-r-1)$-regular. Assume that D has a $\gamma_{rk}(D)$-function f such that $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$ or $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < r|V_1|$, where $V_i = \{v \in V(D)\colon |f(v)| = i\}$. Then we deduce from Theorem 14 that $d_{rk}(D) \leq r + k - 1$. Using Theorem 12, we obtain the desired result as follows

$$d_{rk}(D) + d_{rk}(\overline{D}) \leq (r + k - 1) + (n - r - 1 + k) = n + 2k - 2.$$

It remains the case that every $\gamma_{rk}(D)$-function f of D fulfills $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| = r|V_1|$, where $V_i = \{v \in V(D)\colon |f(v)| = i\}$. Note that $n = |V_0| + |V_1|$. Furthermore, $|V_0| \geq 1$ and thus $|V_1| \geq k$. Since $\overline{D}$ is $(n-r-1)$-regular, it follows that $r \geq (n-1)/2$ or $n - r - 1 \geq (n-1)/2$. We assume, without loss of generality, that $r \geq (n-1)/2$.

If $|V_1| \geq 2k$, then $k|V_0| = r|V_1| \geq 2kr$ and thus $|V_0| \geq 2r$. This leads to the contradiction

$$n = |V_0| + |V_1| \geq 2r + 2k \geq n - 1 + 2k.$$

In the case $k + 1 \leq |V_1| \leq 2k - 1$, we define $V_1^i = \{v\colon f(v) = \{i\}\}$ for $i \in \{1, 2, \ldots, k\}$. Because of $|V_1| \leq 2k - 1$, we observe that $|V_1^i| = 1$ for at least one index $i \in \{1, 2, \ldots, k\}$. We assume, without loss of generality, that $|V_1^1| = 1$. Since each vertex of V_0 has an in-neighbor in V_1^1, we deduce that $|V_0| \leq r$. This implies that

$$k|V_0| \leq kr < r|V_1|,$$

a contradiction to the assumption $k|V_0| = r|V_1|$.

If $|V_1| = k$, then $|V_0| = r$ and so $n = r + k$. Hence $n - r - 1 = k - 1$. Since the k vertices of V_1 induce a complete digraph of order k in $\overline{D}$, we deduce from Corollary 9 that $d_{rk}(\overline{D}) \leq k$. Now Theorem 12 implies that

$$d_{rk}(D) + d_{rk}(\overline{D}) \leq (r + k) + k = n + k \leq n + 2k - 2.$$

Since we have discussed all possible cases, the proof is complete. □

The complete digraph K_n^* demonstrates that Theorem 15 does not hold for $k = 1$. However, we propose the following conjecture.

Conjecture. If $k \geq 2$ is an integer, and D is a digraph of order n, then

$$d_{rk}(D) + d_{rk}(\overline{D}) \leq n + 2k - 2.$$

Corollary 16. *If $k \geq 1$ is an integer, and D is a digraph of order n, then*

$$d_{rk}(D) \cdot d_{rk}(\overline{D}) \leq \frac{(n + 2k - 1)^2}{4}.$$

Proof. It follows from Theorem 13 that

$$\begin{aligned}(n+2k-1)^2 &\geq (d_{rk}(D)+d_{rk}(\overline{D}))^2\\ &= (d_{rk}(D)-d_{rk}(\overline{D}))^2+4d_{rk}(D)\cdot d_{rk}(\overline{D})\\ &\geq 4d_{rk}(D)\cdot d_{rk}(\overline{D}),\end{aligned}$$

and this leads to the desired bound. □

5. Cartesian Product and Strong Product of Directed Cycles

Let $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ be two digraphs which have disjoint vertex sets V_1 and V_2 and disjoint arc sets A_1 and A_2, respectively. The Cartesian product $D_1\Box D_2$ is the digraph with vertex set $V_1\times V_2$ and for any two vertices (x_1,x_2) and (y_1,y_2) of $D_1\Box D_2$, $(x_1,x_2)(y_1,y_2)\in A(D_1\Box D_2)$ if one of the following holds:

(a) $x_1=y_1$ and $x_2y_2\in A(D_2)$;

(b) $x_1y_1\in A(D_1)$ and $x_2=y_2$.

The strong product $D_1\otimes D_2$ is the digraph obtained from $D_1\Box D_2$ by adding the following arcs:

(c) $x_1y_1\in A(D_1)$ and $x_2y_2\in A(D_2)$.

The proof of the following results can be found in [1].

Proposition C. *If $m=2r$ and $n=2s$ for some positive integers r,s, then*

$$\gamma_{r2}(C_m\Box C_n)=\gamma_{r2}(C_m\otimes C_n)=\frac{mn}{2}.$$

Proposition D. *For $n\geq 2$, $\gamma_{r2}(C_3\Box C_n)=2n$.*

Proposition E. *If n is odd, then $\gamma_{r2}(C_2\Box C_n)=n+1$.*

Proposition F. *If $m=4r$ and $n=2s+1$ for some positive integers r,s, then $\gamma_{r2}(C_m\otimes C_n)=\frac{mn}{2}$.*

Proposition 17. *If m and n are even positive integers, then $d_{r2}(C_m\Box C_n)=4$.*

Proof. Let $m=2r$ and $n=2s$ for some positive integers r,s. It follows from Theorem 7 and Proposition C that $d_{r2}(C_m\Box C_n)\leq 4$ and $d_{r2}(C_m\otimes C_n)\leq 4$. Define $f_1,f_2,g_1,g_2:V(D)\to\mathcal{P}(\{1,2\})$ by:

$f_1((2i-1,2j-1))=\{1\}$, for each $1\leq i\leq r$ and $1\leq j\leq s, f_1((2i,2j))=\{2\}$ for each $1\leq i\leq r$ and $1\leq j\leq s$ and $f_1(x)=\emptyset$ otherwise,

$f_2((2i-1,2j-1))=\{2\}$, for each $1\leq i\leq r$ and $1\leq j\leq s, f_2((2i,2j))=\{1\}$ for each $1\leq i\leq r$ and $1\leq j\leq s$ and $f_2(x)=\emptyset$ otherwise,

$g_1((2i, 2j-1)) = \{1\}$, for each $1 \le i \le r$ and $1 \le j \le s, g_1((2i-1, 2j)) = \{2\}$ for each $1 \le i \le r$ and $1 \le j \le s$ and $g_1(x) = \emptyset$ otherwise,

$g_2((2i, 2j-1)) = \{2\}$, for each $1 \le i \le r$ and $1 \le j \le s, g_2((2i-1, 2j)) = \{1\}$ for each $1 \le i \le r$ and $1 \le j \le s$ and $g_2(x) = \emptyset$ otherwise.

It is easy to see that $\{f_1, f_2, g_1, g_2\}$ is a 2RD family of $C_m \Box C_n$ and $C_m \otimes C_n$, and so $d_{r2}(C_m \Box C_n) = d_{r2}(C_m \otimes C_n) = 4$. □

Proposition 18. *For* $n \ge 2$, $d_{r2}(C_3 \Box C_n) = 3$.

Proof. By Theorem 7 and Proposition D, we have $d_{r2}(C_3 \Box C_n) \le 3$.

If $n \equiv 0 \pmod 3$, then define $g_1, g_2, g_3 : V(C_3 \Box C_n) \to \mathcal{P}(\{1, 2\})$ as follows:

$g_1((1, 3i+1)) = g_1((2, 3i+2)) = g_1((3, 3i+3)) = \{1\}, g_1((1, 3i+3)) = g_2((2, 3i+1)) = g_1((3, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n}{3} - 1$ and $g_1(x) = \emptyset$ otherwise,

$g_2((1, 3i+2)) = g_2((2, 3i+3)) = g_2((3, 3i+1)) = \{1\}, g_2((1, 3i+1)) = g_2((2, 3i+2)) = g_2((3, 3i+3)) = \{2\}$ for $0 \le i \le \frac{n}{3} - 1$ and $g_2(x) = \emptyset$ otherwise,

$g_3((1, 3i+3)) = g_3((2, 3i+1)) = g_3((3, 3i+2)) = \{1\}, g_3((1, 3i+2)) = g_3((2, 3i+3)) = g_3((3, 3i+1)) = \{2\}$ for $0 \le i \le \frac{n}{3} - 1$ and $g_3(x) = \emptyset$ otherwise.

If $n \equiv 1 \pmod 3$, then define $g_1, g_2, g_3 : V(C_3 \Box C_n) \to \mathcal{P}(\{1, 2\})$ as follows:

$g_1((3, n)) = \{1\}, g_1((2, n)) = \{2\}, g_1((1, 3i+1)) = g_1((2, 3i+2)) = g_1((3, 3i+3)) = \{1\}, g_1((1, 3i+3)) = g_1((2, 3i+1)) = g_1((3, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-1}{3} - 1$ and $g_1(x) = \emptyset$ otherwise,

$g_2((1, n)) = \{1\}, g_2((3, n)) = \{2\}, g_2((2, 3i+1)) = g_2((3, 3i+2)) = g_2((1, 3i+3)) = \{1\}, g_2((2, 3i+3)) = g_2((3, 3i+1)) = g_2((1, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-1}{3} - 1$ and $g_2(x) = \emptyset$ otherwise,

$g_3((2, n)) = \{1\}, g_3((1, n)) = \{2\}, g_3((3, 3i+1)) = g_3((1, 3i+2)) = g_3((2, 3i+3)) = \{1\}, g_3((3, 3i+3)) = g_3((1, 3i+1)) = g_3((2, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-1}{3} - 1$ and $g_3(x) = \emptyset$ otherwise.

If $n \equiv 2 \pmod 3$, then define $g_1, g_2, g_3 : V(C_3 \Box C_n) \to \mathcal{P}(\{1, 2\})$ as follows:

$g_1((1, n)) = g_1((1, n-1)) = g_1((3, n)) = \{1\}, g_1((2, n-1)) = \{2\}, g_1((1, 3i+1)) = g_1((2, 3i+2)) = g_1((3, 3i+3)) = \{1\}, g_1((1, 3i+3)) = g_1((2, 3i+1)) = g_1((3, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-2}{3} - 1$ and $g_1(x) = \emptyset$ otherwise,

$g_2((2, n)) = g_2((2, n-1)) = g_2((1, n)) = \{1\}, g_2((3, n-1)) = \{2\}, g_2((2, 3i+1)) = g_2((3, 3i+2)) = g_2((1, 3i+3)) = \{1\}, g_2((2, 3i+3)) = g_2((3, 3i+1)) = g_2((1, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-2}{3} - 1$ and $g_2(x) = \emptyset$ otherwise,

$g_3((3, n)) = g_3((3, n-1)) = g_3((2, n)) = \{1\}, g_3((1, n-1)) = \{2\}, g_3((3, 3i+1)) = g_3((1, 3i+2)) = g_3((2, 3i+3)) = \{1\}, g_3((3, 3i+3)) = g_3((1, 3i+1)) = g_3((2, 3i+2)) = \{2\}$ for $0 \le i \le \frac{n-2}{3} - 1$ and $g_3(x) = \emptyset$ otherwise.

It is easy to see that $\{g_1, g_2, g_3\}$ is a 2RDF family of $C_3 \Box C_n$ and so $d_{r2}(C_3 \Box C_n) \ge 3$. Thus $d_{r2}(C_3 \Box C_n) = 3$. □

Proposition 19. *If n is odd, then* $2 \leq d_{r2}(C_2 \Box C_n) \leq 3$.

Proof. By Theorem 7 and Proposition E, we have $d_{r2}(C_2 \Box C_n) \leq 3$. To prove lower bound, define $g_1, g_2 : V(C_2 \Box C_n) \to \mathcal{P}(\{1,2\})$ by

$g_1((1,1)) = \{1\}, g_1((1,2i)) = \{1\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $g_1((2,2i-1)) = \{2\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $g_1(x) = \emptyset$ otherwise, and

$g_2((1,1)) = \{2\}, g_2((1,2i)) = \{2\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $g_2((2,2i-1)) = \{1\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $g_2(x) = \emptyset$ otherwise.

Clearly $\{g_1, g_2\}$ is a 2RDF family of $C_2 \Box C_n$ and so $d_{r2}(C_2 \Box C_n) \geq 2$. □

Proposition 20. *If $m = 4r$ and $n = 2s+1$ for some positive integers r, s, then* $d_{r2}(C_m \otimes C_n) = 4$.

Proof. By Theorem 7, we have $d_{r2}(C_m \otimes C_n) \leq 4$. Define $g_1, g_2, g_3, g_4 : V(C_m \otimes C_n) \to \mathcal{P}(\{1,2\})$ as follows:

$g_1((4i+1,1)) = \{1\}, g_1((4i+3,1)) = \{2\}$ for $0 \leq i \leq r-1,$ $g_1((4i+2,2j)) = g_1((4i+4,2j+1)) = \{1\}, g_1((4i+4,2j)) = g_1((4i+2,2j+1)) = \{2\}$ for $0 \leq i \leq r-1$ and $1 \leq j \leq s,$ and $g_1(x) = \emptyset$ otherwise,

$g_2((4i+1,1)) = \{2\}, g_2((4i+3,1)) = \{1\}$ for $0 \leq i \leq r-1,$ $g_2((4i+2,2j)) = g_2((4i+4,2j+1)) = \{2\}, g_2((4i+4,2j)) = g_2((4i+2,2j+1)) = \{1\}$ for $0 \leq i \leq r-1$ and $1 \leq j \leq s,$ and $g_2(x) = \emptyset$ otherwise,

$g_3((4i+2,1)) = \{1\}, g_3((4i+4,1)) = \{2\}$ for $0 \leq i \leq r-1,$ $g_3((4i+3,2j)) = g_3((4i+1,2j+1)) = \{1\}, g_3((4i+1,2j)) = g_3((4i+3,2j+1)) = \{2\}$ for $0 \leq i \leq r-1$ and $1 \leq j \leq s,$ and $g_3(x) = \emptyset$ otherwise,

$g_4((4i+2,1)) = \{2\}, g_4((4i+4,1)) = \{1\}$ for $0 \leq i \leq r-1,$ $g_4((4i+3,2j)) = g_4((4i+1,2j+1)) = \{2\}, g_4((4i+1,2j)) = g_4((4i+3,2j+1)) = \{1\}$ for $0 \leq i \leq r-1$ and $1 \leq j \leq s,$ and $g_4(x) = \emptyset$ otherwise.

It is easy to see that $\{g_1, g_2, g_3, g_4\}$ is a 2RDF family of $C_m \otimes C_n$ and so $d_{r2}(C_m \otimes C_n) \geq 4$. Thus $d_{r2}(C_m \otimes C_n) = 4$. □

We conclude this paper with a problem.

Problem. Determine the exact value of $d_{r2}(C_m \Box C_n)$ and $d_{r2}(C_m \otimes C_n)$ for all m and n.

References

[1] J. Amjadi, A. Bahremandpour, S. M. Sheikholeslami and L. Volkmann, *The rainbow domination number of a digraph*, Kragujevac J. Math., **37**(2013), 257–268.

[2] J. Amjadi, N. Mohammadi, S. M. Sheikholeslami and L. Volkmann, *The k-rainbow bondage number of a digraph*, Discuss. Math. Graph Theory, **35**(2015), 261–270.

[3] B. Brešar, M. A. Henning and D. F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math., **12**(2008), 213–225.

[4] B. Brešar and T. K. Šumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math., **155**(2007), 2394–2400.

[5] G. J. Chang, J. Wu and X. Zhu, *Rainbow domination on trees*, Discrete Appl. Math., **158**(2010), 8–12.

[6] G. Chartrand, F. Harary and B. Q. Yue, *On the out-domination and in-domination numbers of a digraph*, Discrete Math., **197,198**(1999), 179–183.

[7] T. Chunling, L. Xiaohui, Y. Yuansheng and L. Meiqin, *2-rainbow domination of generalized Petersen graphs* $P(n, 2)$, Discrete Appl. Math., **157**(2009), 1932–1937.

[8] N. Dehgardi, S. M. Sheikholeslami and L. Volkmann, *The rainbow domination subdivision numbers of graphs*, Mat. Vesnik Mat. Vesnik, **67**(2015), 102–114.

[9] M. Falahat, S. M. Sheikholeslami and L. Volkmann, *New bounds on the rainbow domination subdivision number*, Filomat, **28**(2014), 615–622.

[10] S. Fujita, M. Furuya and C. Magnant, *k-Rainbow domatic numbers*, Discrete. Appl. Math., **160**(2012), 1104–1113.

[11] J. Ghoshal, R. Laskar and D. Pillone, *Topics on domination in directed graphs*, in [13], 401–437.

[12] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[13] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in graphs, Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[14] C. Lee, *On the domination number of a digraph*, PhD thesis, Department of Mathematics, Michigan State University, 1994.

[15] C. Lee, *Domination in digraphs*, J. Korean Math. Soc., **35**(1998), 843–853.

[16] D. Meierling, S. M. Sheikholeslami and L. Volkmann, *Nordhaus-Gaddum bounds on the k-rainbow domatic number of a graph*, Applied Math. Letters, **24**(2011), 1758–1761.

[17] S. M. Sheikholeslami and L. Volkmann, *The k-rainbow domatic number of a graph*, Discuss. Math. Graph Theory, **32**(2012), 129–140.

[18] G. Xu, *2-rainbow domination of generalized Petersen graphs* $P(n, 3)$, Discrete Appl. Math., **157**(2009), 2570–2573.

[19] X. D. Zhang, J. Liu, X. Chen and J. Meng, *On domination number of Cartesian product of directed cycles*, Inform. Process. Lett., **111**(2010), 36–39.

[20] B. Zelinka, *Semidomatic numbers of directed graphs*, Math. Slovaca, **34**(1984), 371–374.