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On Approximation by Matrix Means of the Multiple Fourier Series in the Hölder Metric

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Abstract. In this work, we shall give the degree of approximation for functions belonging to Hölder class by matrix summability method of multiple Fourier series in the Hölder metric.

1. Introduction and some Notations

Suppose that $f(x,y)$ is integrable in the sense of Lebesgue over the square $S^2 := S(-\pi,\pi;-\pi,\pi)$ and of period 2π in x and in y. If $f(x,y)$ is defined only on the square S^2, we extend it periodically onto the whole xy-plane. The double Fourier series of $f(x,y)$ can be written in the form

$$f(x,y) \sim \sum_{m,n\in\mathbb{N}} \lambda_{mn}[\eta_{mn}\cos mx\cos ny + \mu_{mn}\sin mx\cos ny + \rho_{mn}\cos mx\sin ny + \zeta_{mn}\sin mx\sin ny]$$

where

$$\lambda_{mn} = \begin{cases} 1/4, & m=n=0; \\ 1/2, & m>0, n=0 \vee m=0, n>0; \\ 1, & m>0, n>0. \end{cases}$$

and the coefficients η_{mn}, μ_{mn}, ρ_{mn} and ζ_{mn} are calculated by the formulas

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$$\eta_{mn} = \frac{1}{\pi^2}\iint\limits_{S^2} f(x,y)\cos mx \cos ny dxdy,$$

$$\mu_{mn} = \frac{1}{\pi^2}\iint\limits_{S^2} f(x,y)\sin mx \cos ny dxdy,$$

$$\rho_{mn} = \frac{1}{\pi^2}\iint\limits_{S^2} f(x,y)\cos mx \sin ny dxdy, \tag{1.1}$$

$$\zeta_{mn} = \frac{1}{\pi^2}\iint\limits_{S^2} f(x,y)\sin mx \sin ny dxdy,$$

for $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2 \ldots$. Now let

$$s_{mn}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n}[\eta_{ij}\cos ix \cos jy + \mu_{ij}\sin ix \cos jy$$

$$+\rho_{ij}\cos ix \sin jy + \zeta_{ij}\sin ix \sin jy].$$

The quantity $s_{mn}(x,y)$ $(m = 0, 1, 2, \ldots; n = 0, 1, 2 \ldots)$ are called the partial sums of double Fourier series. According to (1.1), we know that

$$s_{mn}(x,y) = \frac{1}{\pi^2}\iint\limits_{S^2} f(x+u, y+v)\frac{[\sin(m+1/2)u][\sin(n+1/2)v]}{4sin(u/2)sin(v/2)}dudv.$$

Moreover, let

$$\tau_{mn}(x,y) = \tau_{mn}(f; A, U; x, y) := \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mi}b_{nj}s_{ij}(x,y), \quad \forall m, n \geq 0$$

where $A \equiv (a_{m,i})$ and $U \equiv (b_{n,j})$ are lower triangular infinite matrices such that:

$$a_{m,i} = \begin{cases} \geq 0, & i \leq m; \\ 0, & i > m \end{cases} \quad (i, m = 0, 1, 2, \ldots) \quad \wedge \quad \sum_{i=0}^{m} a_{m,i} = 1 \tag{1.2}$$

and

$$b_{n,j} = \begin{cases} \geq 0, & j \leq n; \\ 0, & j > n \end{cases} \quad (j, n = 0, 1, 2, \ldots) \quad \wedge \quad \sum_{j=0}^{n} a_{n,j} = 1. \tag{1.3}$$

The double Fourier series of the function $f(x,y)$ is called to be (A,U)-summable to a finite number s, if $\tau_{mn}(x,y) \to s$ as $m, n \to \infty$. The condition of regularity for double matrix summability means are given by

$$
\begin{aligned}
&\sum_{i=0}^{m}\sum_{j=0}^{n} a_{mi}b_{nj} \to 1, \text{ as } \quad m,n\to\infty,\\
&\lim_{m,n}\sum_{j=0}^{n} |a_{mi}b_{nj}| = 0, \text{ for each } \quad i=1,2,\dots,\\
&\lim_{m,n}\sum_{i=0}^{m} |a_{mi}b_{nj}| = 0, \text{ for each } \quad j=1,2,\dots.
\end{aligned} \tag{1.4}
$$

Let

$$H_\alpha = \{f \in C_{2\pi} : |f(x)-f(y)| \le K|x-y|^\alpha\}$$

where K is a positive constant, not necessarily the same at each occurrence. It is known that H_α is a Banach space(see Prösdorff, [7]) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_C + \sup_{x\neq y} \Delta^\alpha f(x,y) \tag{1.5}$$

where

$$\Delta^\alpha f(x,y) = \frac{|f(x)-f(y)|}{|x-y|^\alpha} \ (x\neq y),$$

by convention $\Delta^0 f(x,y)=0$ and

$$\|f\|_C = \sup_{x\in[-\pi,\pi]} |f(x)|.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Prösdorff has been studied the degree of approximation in the Hölder metric and proved the following theorem:

Theorem A.([7]) *Let $f \in H_\alpha (0<\alpha\le 1)$ and $0\le\beta<\alpha\le 1$. Then*

$$\|\sigma_n(f)-f\|_\beta = O(1)\begin{cases} n^{\beta-\alpha} & ,0<\alpha<1;\\ n^{\beta-1}\ln n & ,\alpha=1\end{cases} \tag{1.6}$$

where $\sigma_n(f)$ is Fejér means of the Fourier series of f.

The case $\beta=0$ in Theorem A is owing to Alexits [1]. Chandra obtained a generalization of Theorem A in the Woronoi-Nörlund transform [2]. In [6], Mohapatra and Chandra considered the problem by matrix means of the Fourier series of $f\in H_\alpha$. In the one-dimensional case, these problems have been studied in detail. Naturally, similar problems are considered for the periodic functions with two variables. Stepanets investigated the problem of the approximation of functions $f(x,y)$, 2π-periodic with respect to each of the variables by the partial sums of their Fourier sums and under the some conditions in [9, 10]. In [5], Lal studied the approximation of functions belonging to Lipschitz class by matrix summability method for double Fourier series under the uniform norm.

The Hölder class for $f(x,y)$ continuous functions periodic in both variables with period 2π is defined as

$$H_{(\alpha,\beta)} = \{f : |f(x,y;z,w)| := |f(x,y) - f(z,w)| \leq C_1(|x-z|^\alpha + |y-w|^\beta)\}$$

for some $\alpha, \beta > 0$ and for all x, y, z, w where C_1 is a positive constant may depend on f, but not on x, y, z, w. This class of functions is also called Lipschitz class and denoted by $Lip(\alpha, \beta)$. It can be easily verified that $H_{\alpha,\beta}$ is a Banach space with the norm $\|\cdot\|_{\alpha,\beta}$ defined by

$$\|f\|_{\alpha,\beta} = \|f\|_C + \sup_{x\neq z\ ,\ y\neq w} \Delta^{\alpha,\beta} f(x,y;z,w) \tag{1.7}$$

where

$$\Delta^{\alpha,\beta} f(x,y;z,w) = \frac{|f(x,y) - f(z,w)|}{|x-z|^\alpha + |y-w|^\beta}\ (x \neq z\ ,\ y \neq w),$$

by convention $\Delta^{0,0} f(x,y;z,w) = 0$ and

$$\|f\|_C = \sup_{(x,y)\in S^2} |f(x,y)|.$$

Moreover, a function f in $Lip(\alpha, \beta)$ is said to belong to the little Lipschitz class $lip(\alpha, \beta)$ if

$$\lim_{z\to x,\, w\to y} (|x-z|^\alpha + |y-w|^\beta)^{-1} |f(x,y;z,w)| = 0$$

uniformly in (x,y). The aim of this paper is as follows. First, the approximation to functions $f(x,y)$ belonging to these Lipschitz classes is given by matrix summability method of double Fourier series in accordance with the norm in (1.7). Later the approximation is generalized to the N-multiple Fourier series.

Throughout this paper, we shall also use the following notations:

$$\Psi(u,v) := \Psi(x,y;u,v) := \frac{1}{4}\{f(x+u,y+v) + f(x+u,y-v)$$
$$+f(x-u,y+v) + f(x-u,y-v) - 4f(x,y)\}$$

and

$$F(u,v) = \Phi(u,v) - \Psi(u,v)$$

where $\Phi(u,v) := \Psi(z,w;u,v)$. Since $f(x,y) \in H_{(\alpha,\beta)}$, it is clear that

$$|F(u,v)| = O(|x-z|^\alpha + |y-w|^\beta). \tag{1.8}$$

2. In Case of Double Fourier Series

The approximation by matrix means for double Fourier series is as follows with respect to Hölder metric.

Theorem 2.1. *Assume $A \equiv (a_{m,i})$ and $U \equiv (b_{n,j})$ are lower triangular matrices where $(a_{m,i})$ and $(b_{n,j})$ are nondecreasing sequences with respect to $i \le m$ and $j \le n$ satisfying the conditions (1.2) and (1.3), respectively such that double matrix method (A, U) is regular. If $f(x,y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha,\beta)}$ for $0 < \alpha, \beta \le 1$, then*

$$\|\tau_{mn} - f\|_{\alpha,\beta} = O(1) \begin{cases} (m+1)^{-\alpha} + (n+1)^{-\beta} & , 0 < \alpha < 1,\ 0 < \beta < 1; \\ \dfrac{\log((m+1)\pi)}{(m+1)} + \dfrac{\log((n+1)\pi)}{(n+1)} & , \alpha = \beta = 1 \end{cases}$$

for $m, n = 0, 1, 2, \ldots$.

For small Lipschitz class, the analogy of the Theorem can be written if "O" is replaced by "o" as $m, n \to \infty$ independently one another, and $f \in Lip(\alpha, \beta)$ is replaced by $f \in lip(\alpha, \beta)$ for $0 < \alpha, \beta < 1$. We don't enter in details.

Furthermore, double matrix summability method gives us the following means for some important cases:

- $(C, 1, 1)$ means, when $a_{m,i} = \dfrac{1}{m+1}$ and $b_{n,j} = \dfrac{1}{n+1}$ for all i and j, respectively [3];
- (N, p_m, q_n) means, when $a_{m,i} = \dfrac{p_{m-i}}{P_m}$ and $b_{n,j} = \dfrac{q_{n-j}}{Q_n}$; where $P_m = \sum_{k=0}^{m} p_k \ne 0$ and $Q_n = \sum_{k=0}^{n} q_k \ne 0$ [4];
- $(H, 1, 1)$ means, when $a_{m,i} = \dfrac{1}{(m-i+1)\log m}$ and $b_{n,j} = \dfrac{1}{(n-j+1)\log n}$ [8].

Taking into account the first two case above, we write the following results.

Corollary 2.2. *If $f(x,y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha,\beta)}$ for $0 < \alpha, \beta \le 1$, then*

$$\|\sigma_{mn} - f\|_{\alpha,\beta} = O(1) \begin{cases} (m+1)^{-\alpha} + (n+1)^{-\beta} & , 0 < \alpha < 1,\ 0 < \beta < 1; \\ \dfrac{\log((m+1)\pi)}{(m+1)} + \dfrac{\log((n+1)\pi)}{(n+1)} & , \alpha = \beta = 1 \end{cases}$$

for $m, n = 0, 1, 2, \ldots$, where

$$\sigma_{mn}(x,y) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} s_{ij}(x,y), \quad \forall m, n \ge 0.$$

Corollary 2.3. *If $f(x,y)$ is a function of period 2π in x and y Lebesgue integrable in S^2 belonging to the class $H_{(\alpha,\beta)}$ for $0<\alpha,\beta<1$, then*

$$\|\mathbf{N}_{mn}-f\|_{\alpha,\beta}=O(1)\left\{(m+1)^{-\alpha}+(n+1)^{-\beta}\right\}$$

for $m,n=0,1,2,\ldots$, where

$$\mathbf{N}_{mn}(x,y)=\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m-i}q_{n-j}s_{ij}(x,y),\quad \forall m,n\geq 0.$$

Before giving the proof of Theorem 2.1, we need the following auxiliary results.

Lemma 2.4. *Let $(a_{m,i})$ and (b_{nj}) be real nonnegative and nondecreasing sequence with (1.2) and (1.3), respectively.*

(i) *For $0<u\leq 1/(m+1)$, we have $K_m(u)=O(m+1)$ where*

$$K_m(u):=\frac{1}{\pi}\sum_{i=0}^{m}a_{m,i}\frac{sin(i+\frac{1}{2})u}{sin(\frac{u}{2})}.$$

(ii) *For $0<v\leq 1/(n+1)$, we have $K_n(v)=O(n+1)$ where*

$$K_n(v):=\frac{1}{\pi}\sum_{j=0}^{n}b_{n,j}\frac{sin(j+\frac{1}{2})v}{sin(\frac{v}{2})}.$$

This is easily proved by an elementary calculation.

Lemma 2.5.([5]) *Assume that $(a_{m,i})$ and (b_{nj}) be real nonnegative and nondecreasing sequence with $i\leq m$ and $j\leq n$, respectively.*

(i) *For $1/(n+1)<v\leq\pi$ and any $n\in\mathbb{N}$, we have*

$$K_n(v)=O(\frac{B_{n,\sigma}}{v}),$$

where $B_{n,\sigma}=\sum\limits_{j=n-\sigma}^{n}b_{nj}$ and σ denote integer part of $\frac{1}{v}$.

(ii) *For $1/(m+1)<u\leq\pi$ and any $m\in\mathbb{N}$, we have*

$$K_m(u)=O(\frac{A_{m,\kappa}}{u}),$$

where $A_{m,\kappa}=\sum\limits_{i=m-\kappa}^{m}a_{mi}$ and κ denote integer part of $\frac{1}{u}$.

3. Proof of the Theorem 2.1

Proof. We know that

$$s_{ij}(x,y) - f(x,y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\Psi(u,v)\frac{[\sin(i+1/2)u][\sin(j+1/2)v]}{sin(u/2)sin(v/2)}dudv. \tag{3.1}$$

Taking into account (3.1) and $\tau_{mn}(x,y)$ that double matrix means of $s_{mn}(x,y)$, we write

$$\tau_{mn}(x,y) - f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mi}b_{nj}\left\{s_{ij}(x,y) - f(x,y)\right\}$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\Psi(u,v)\sum_{i=0}^{m}\sum_{j=0}^{n} a_{mi}b_{nj}\frac{[\sin(i+1/2)u][\sin(j+1/2)v]}{sin(u/2)sin(v/2)}dudv$$

$$= \int_0^{\pi}\int_0^{\pi}\Psi(u,v)K_m(u)K_n(v)dudv$$

Let us estimate that

$$\sup_{x\neq z,\, y\neq w}\frac{|\tau_{mn}(x,y) - f(x,y) - (\tau_{mn}(z,w) - f(z,w))|}{|x-z|^{\alpha} + |y-w|^{\beta}} = O(1). \tag{3.2}$$

$$|\tau_{mn}(x,y) - f(x,y) - (\tau_{mn}(z,w) - f(z,w))| = \left|\int_0^{\pi}\int_0^{\pi}F(u,v)K_m(u)K_n(v)dudv\right|$$

$$\leq \left(\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}} + \int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi} + \int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}} + \int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\right)|F(u,v)K_m(u)K_n(v)|dudv$$

$$=: J_1 + J_2 + J_3 + J_4. \tag{3.3}$$

Therefore, from (1.8) and Lemma 2.4, we obtain

$$\begin{aligned} J_1 &= \int_0^{1/(m+1)}\int_0^{1/(n+1)} |F(u,v)K_m(u)K_n(v)|dudv \\ &= (m+1)(n+1)\int_0^{1/(m+1)}\int_0^{1/(n+1)} |F(u,v)|dudv = O(|x-z|^{\alpha} + |y-w|^{\beta}) \end{aligned} \tag{3.4}$$

for $0<\alpha,\beta\le 1$. By using Lemma 2.4, Lemma 2.5 and again (1.8), then we have

$$\begin{aligned} J_2 &= \int_0^{1/(m+1)}\int_{1/(n+1)}^{\pi} |F(u,v)K_m(u)K_n(v)|dudv \\ &= (m+1)\int_0^{1/(m+1)}\int_{1/(n+1)}^{\pi} |F(u,v)|\frac{B_{n,\sigma}}{v}dudv = O(|x-z|^\alpha+|y-w|^\beta)\int_{1/(n+1)}^{\pi}\frac{B_{n,\sigma}}{v}dv \\ &\le O(|x-z|^\alpha+|y-w|^\beta)\int_{1/(n+1)}^{\pi}\frac{B_{n,1/v}}{v}dv \end{aligned}$$

$$(3.5) = O(|x-z|^\alpha+|y-w|^\beta)\int_{1/\pi}^{(n+1)}\frac{B_{n,t}}{t}dt = O(|x-z|^\alpha+|y-w|^\beta)$$

since $\frac{B_{n,t}}{t}$ is monotonic increasing. Similarly, we can prove that

$$(3.6) \quad J_3 = \int_{1/(m+1)}^{\pi}\int_0^{1/(n+1)} |F(u,v)K_m(u)K_n(v)|dudv = O(|x-z|^\alpha+|y-w|^\beta)$$

and

$$(3.7) \quad J_4 = \int_{1/(m+1)}^{\pi}\int_{1/(n+1)}^{\pi} |F(u,v)K_m(u)K_n(v)|dudv = O(|x-z|^\alpha+|y-w|^\beta).$$

By combining (3.3)-(3.7), we obtain (3.2). On the other hand, we know that from [5]

$$(3.8) \quad \|\tau_{mn}-f\|_C = O(1)\begin{cases} (m+1)^{-\alpha}+(n+1)^{-\beta} & ,0<\alpha,\beta<1; \\ \dfrac{\log((m+1)\pi e)}{(m+1)}+\dfrac{\log((n+1)\pi e)}{(n+1)} & ,\alpha=\beta=1 \end{cases}$$

for $m,n=0,1,2,\ldots$. Since $\log e<\log(m+1)\pi$ and $\log e<\log(n+1)\pi$, we omit the number "e" in the formula (3.8). Therefore, according to (3.2) and (3.8), the proof of Theorem 2.1 is completed. □

4. In Case of N-Multiple Fourier Series, $N\ge 3$.

Let $f(x_1,\ldots,x_N)$ is integrable over the N dimensional cube S^N and of period

2π in each variable. The N-multiple Fourier series of $f(x_1,\dots,x_N)$ can be written in the form

$$f(x_1,\dots,x_N)\sim\sum_{m_1\in\mathbb{Z}}\sum_{m_2\in\mathbb{Z}}\cdots\sum_{m_N\in\mathbb{Z}}c_{m_1,m_2,\dots,m_N}e^{i(m_1x_1+m_2x_2+\cdots+m_Nx_N)}.$$

where $c_{m_1,m_2,\dots,m_N}$ is the Fourier coefficients of f (see, [11, p. 300]). The series is denoted by $S[f]$ and the partial sums of it are given by

$$S_{m_1m_2\dots m_N}(x_1,\dots,x_N):=\pi^{-N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(x_1+t_1,\dots,x_N+t_N)\prod_{j=1}^{N}D_{m_j}(t_j)dt_1\dots dt_N$$

where $D_{m_j}(t_j)$ are the Dirichlet kernels for each j. Moreover, similar to the two-dimensional, we can write

$$\begin{aligned}\tau_{m_1m_2\cdots m_N}(x_1,\dots,x_N)&=:\tau_{m_1m_2\cdots m_N}(f;\{A_k\}_1^N;x_1,\dots,x_N)\\&:=\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\dots\sum_{i_N=0}^{m_N}a_{m_1i_1}\dots a_{m_Ni_N}S_{i_1i_2\cdots i_N}(x_1,\dots,x_N)\end{aligned}$$

for all $m_k\geq 0$. Here $\{A_k\}_{k=1}^N\equiv\{(a_{m_k,i_k})\}_{k=1}^N$ are lower triangular infinite matrices such that:

$$\text{(4.1)}\quad a_{m_k,i_k}=\begin{cases}\geq 0, & i_k\leq m_k;\\ 0, & i_k>m_k\end{cases}\qquad (m_k,i_k=0,1,2,\dots)\quad\wedge\quad\sum_{i_k=0}^{m_k}a_{m_k,i_k}=1$$

for each $k=1,2,\dots,N$. The N-multiple Fourier series of function $f(x_1,\dots,x_N)$ is called to be $(A_1,\dots,A_N)$-summable to a finite number ℓ, if $\tau_{m_1m_2\cdots m_N}(x_1,\dots,x_N)\to\ell$ as $m_1,m_2,\cdots,m_N\to\infty$. The condition of regularity for N-multiple matrix summability means are given by

$$\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\dots\sum_{i_N=0}^{m_N}(a_{m_1i_1}\dots a_{m_Ni_N})\to 1,\ \text{as}\quad m_1,m_2,\cdots,m_N\to\infty,$$

$$\lim_{m_i}\sum_{i_2=0}^{m_2}\sum_{i_3=0}^{m_3}\dots\sum_{i_N=0}^{m_N}(a_{m_1i_1}\dots a_{m_Ni_N})=0,\ \text{for each}\quad i_1=1,2,\dots,$$

$$\lim_{m_i}\sum_{i_1=0}^{m_1}\sum_{i_3=0}^{m_3}\dots\sum_{i_N=0}^{m_N}(a_{m_1i_1}\dots a_{m_Ni_N})=0,\ \text{for each}\quad i_2=1,2,\dots,$$

$$\vdots$$

$$\lim_{m_i}\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\dots\sum_{i_{N-1}=0}^{m_{N-1}}(a_{m_1i_1}\dots a_{m_Ni_N})=0,\ \text{for each}\quad i_N=1,2,\dots.$$

Next, we give the notion of Lipschitz classes of functions on S^N. Let $f(x_1,\ldots,x_N)$ be a continuous periodic function with period 2π in each variable. The function f belongs to the Lipschitz class $Lip(\alpha_1,\alpha_2,\ldots,\alpha_N)$(or $H_{(\alpha_1,\alpha_2,\ldots,\alpha_N)}$) for some $\alpha_1,\alpha_2,\ldots,\alpha_N\geq 0$ if there exists a constant K_1 such that

$$|f(x_1,\ldots,x_N;y_1,\ldots,y_N)|:=|f(x_1,\ldots,x_N)-f(y_1,\ldots,y_N)|\leq K_1\sum_{k=1}^{N}|x_k-y_k|^{\alpha_k}$$

for all x_k,y_k where $k=1,\ldots,N$. Furthermore, a function f in $Lip(\alpha_1,\alpha_2,\ldots,\alpha_N)$ is said to belong to little Lipschitz class $lip(\alpha_1,\alpha_2,\ldots,\alpha_N)$ if

$$\lim_{y_1\to x_1,\cdots,y_N\to x_N}\frac{|f(x_1,\ldots,x_N;y_1,\ldots,y_N)|}{\sum\limits_{k=1}^{N}|x_k-y_k|^{\alpha_k}}=0$$

uniformly in $(x_1,\ldots,x_N)$.

The function space $H_{(\alpha_1,\alpha_2,\ldots,\alpha_N)}$ is a Banach space with respect to the norm $\|\cdot\|_{\alpha_1,\alpha_2,\ldots,\alpha_N}$ defined by

$$\|f\|_{\alpha_1,\alpha_2,\ldots,\alpha_N}=\|f\|_C+\sup_{x_1\neq y_1,\ldots,x_N\neq y_N}\Delta^{\alpha_1,\alpha_2,\ldots,\alpha_N}f(x_1,\ldots,x_N;y_1,\ldots,y_N)$$

where

$$\Delta^{\alpha_1,\alpha_2,\ldots,\alpha_N}f(x_1,\ldots,x_N;y_1,\ldots,y_N)=\frac{|f(x_1,\ldots,x_N;y_1,\ldots,y_N)|}{\sum\limits_{k=1}^{N}|x_k-y_k|^{\alpha_k}},$$

for $x_1\neq y_1,\ldots,x_N\neq y_N$ by convention $\Delta^{0,\ldots,0}f(x_1,\ldots,x_N;y_1,\ldots,y_N)=0$ and

$$\|f\|_C=\sup_{(x_1,\ldots,x_N)\in S^N}|f(x_1,\ldots,x_N)|.$$

Now as an extension of Theorem 2.1, we write the following theorem.

Theorem 4.1. *Let $\{A_k\}_{k=1}^N\equiv\{(a_{m_k,i_k})\}_{k=1}^N$, $N\geq 3$, are lower triangular matrices where $\{(a_{m_k,i_k})\}_{k=1}^N$ are nondecreasing sequences with respect to $i_k\leq m_k$, $k=1,\ldots,N$, satisfying the conditions (4.1), respectively such that N-multiple matrix method $(A_1,A_2,\ldots,A_N)$ is regular. If $f(x_1,x_2,\ldots,x_N)$ is a function of period 2π in each variable Lebesgue integrable in S^N belonging to the class $H_{(\alpha_1,\alpha_2,\ldots,\alpha_N)}$ for $0<\alpha_1,\alpha_2,\ldots,\alpha_N\leq 1$, then*

$$\|\tau_{m_1m_2\cdots m_N}-f\|_{\alpha_1,\alpha_2,\ldots,\alpha_N}=O(1)\begin{cases}\sum\limits_{k=1}^{N}(m_k+1)^{-\alpha_k} & ,0<\alpha_1,\alpha_2,\ldots,\alpha_N<1;\\ \sum\limits_{k=1}^{N}\dfrac{\log((m_k+1)\pi)}{(m_k+1)} & ,\alpha_1=\alpha_2=\cdots=\alpha_N=1,\end{cases}$$

for $m_k = 0, 1, 2, \ldots,$ *where* $k = 1, 2, ..., N$ *and* $N \geq 3$ *is a fixed integer.*

Proof. One needs the extensions of Lemma 2.4 and Lemma 2.5 with respect to each variable from double to N-multiple. After this, the proof runs along the same lines as that of Theorem 2.1. □

Let $0 < \alpha_1, \alpha_2, \ldots, \alpha_N < 1$. The analogy of statement in the Theorem 2.1 can be written if "O" is replaced by "o" as $m_1, m_2, \cdots, m_N \to \infty$, and $f \in H_{(\alpha_1,\alpha_2,\ldots,\alpha_N)}$ is replaced by $f \in lip(\alpha_1, \alpha_2, \ldots, \alpha_N)$.

N- multiple matrix summability method gives us the $(C, 1, 1, \ldots, 1)$ means, when $a_{m_k,i_k} = \dfrac{1}{m_k + 1}$ for all i_k, $(k = 1, 2, \ldots, N)$ [11]. Then, it will be in the form

$$\sigma_{m_1 m_2 \cdots m_N}(x_1, \ldots, x_N) = \left(\prod_{k=1}^{N} \frac{1}{m_k + 1} \right) \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_N=0}^{m_N} S_{i_1 i_2 \cdots i_N}(x_1, \ldots, x_N)$$

Therefore, we observe the next result from the Theorem 4.1.

Corollary 4.2. *If* $f(x_1, x_2, \ldots, x_N)$ *is a function of period* 2π *in each variable Lebesgue integrable in* S^N *belonging to the class* $H_{(\alpha_1,\alpha_2,\ldots,\alpha_N)}$ *for* $0 < \alpha_1, \alpha_2, \ldots, \alpha_N \leq 1$*, then*

$$\|\sigma_{m_1 m_2 \cdots m_N} - f\|_{\alpha_1,\alpha_2,\ldots,\alpha_N} = O(1) \begin{cases} \sum\limits_{k=1}^{N} (m_k + 1)^{-\alpha_k} & , 0 < \alpha_1, \alpha_2, \ldots, \alpha_N < 1; \\ \sum\limits_{k=1}^{N} \dfrac{\log((m_k + 1)\pi)}{(m_k + 1)} & , \alpha_1 = \alpha_2 = \cdots = \alpha_N = 1 \end{cases}$$

for $m_k = 0, 1, 2, \ldots,$ *where* $k = 1, 2, ..., N$ *and* $N \geq 3$ *is a fixed integer.*

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