Dynamical Behaviors of a Discrete Predator-Prey System with Beddington-DeAngelis Functional Response

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ABSTRACT. In this paper, we consider a discrete predator-prey system obtained from a continuous Beddington-DeAngelis type predator-prey system by using the method in [9]. In order to investigate dynamical behaviors of this discrete system, we find out all equilibrium points of the system and study their stability by using eigenvalues of a Jacobian matrix for each equilibrium points. In addition, we illustrate some numerical examples in order to substantiate theoretical results.

1. Introduction

Classical two-species continuous time systems such as a Lotka-Volteface system have been used to investigate the interaction between ecological populations(see [2, 6, 7, 9, 11, 14, 15, 17]). However, sometimes it is necessary to consider discrete-time systems described by difference equations, discrete dynamical systems or iterative maps([4, 5, 12, 18]). Such population systems can be written in terms of a sequence $\{x_n\}$, for example, the well-known logistic difference equation is modeled as

$$(1.1) \qquad x_{n+1} = rx_n(1 - x_n),$$

where x_n denotes the population of a single species in the n-th generation and r is

the intrinsic growth rate.

In order to describe the relationship between two species, the functional responses are important(cf, [6]). One of the well know functional responses with predator interference is the Beddington DeAngelis functional response, which was introduced by Beddington [3] and DeAngelis et al. [8]. In fact, there are significant evidences to suggest that functional responses with predator interference occur quite frequently in laboratory and natural systems [16]. Thus, based on the above discussion, in the paper, we consider the following predator-prey system with Beddington-DeAngelis functional response ([9, 10, 19]).

$$(1.2) \quad \begin{cases} \dfrac{dx_1}{dt_1} = rx_1(1 - \dfrac{x_1}{K}) - \dfrac{a_1 x_1 y_1}{b_1 y_1 + x_1 + c_1}, \\ \dfrac{dy_1}{dt_1} = -d_1 y_1 + \dfrac{e a_1 x_1 y_1}{b_1 y_1 + x_1 + c_1}, \end{cases}$$

where $x_1(t), y_1(t)$ represent the population density of the prey and the predator at time t, respectively. Usually, K is called the carrying capacity of the prey. The constant r is called the intrinsic growth rate of the prey. The constants e, a_1 are the conversion rate and d_1 is the death rate of the predator, respectively. The term $b_1 y$ measures the mutual interference between predators. The parameter c_1 means the handling time of predator to catch prey.

To simplify system (1.2) with scaling parameters, let

$$(1.3) \qquad rt_1 = t, x_1 = Kx, y_1 = \dfrac{rK}{a_1}y, \dfrac{d_1}{r} = D, \dfrac{b_1 r}{a_1} = b, \dfrac{c_1}{K} = c, \dfrac{e a_1}{r} = a.$$

Then the following dimensionless system can be obtained

$$(1.4) \quad \begin{cases} \dfrac{dx}{dt} = x(1 - x) - \dfrac{xy}{by + x + c}, \\ \dfrac{dy}{dt} = -Dy + \dfrac{axy}{by + x + c}. \end{cases}$$

From biological point of view, we must assume that

$$(1.5) \qquad\qquad\qquad\qquad\qquad x < 1.$$

Now, we will adopt the method used in [9] to obtain the following discrete time analogue of system (1.4).

$$(1.6) \qquad T : \begin{cases} x(n+1) = x(n) \exp\left\{1 - x(n) - \dfrac{y(n)}{by(n) + x(n) + c}\right\}, \\ y(n+1) = y(n) \exp\left\{-D + \dfrac{ax(n)}{by(n) + x(n) + c}\right\}. \end{cases}$$

With an initial condition (x_0, y_0), the iteration of system (1.6) uniquely determines a trajectory of the states of population output in the following form

(1.7) $$(x_n, y_n) = T^n(x_0, y_0),$$

where $n = 0, 1, 2, \cdots$.

The main purpose of this paper is to investigate dynamical properties of system (1.6) by taking into account the stability of the equilibrium points of the system and to substantiate theoretical results by displaying some numerical examples.

2. Stability of the Equilibrium Points of System (1.6)

We first discuss the existence of the equilibria of system (1.6). It is obvious that there are at least two equilibrium points, $E_0(0,0)$, $E_1(1,0)$ of system (1.6). In order to find out positive equilibrium points of system (1.6) we need to consider the simultaneous equation satisfying

(2.1) $$\begin{cases} 1 - x - \dfrac{y}{by + x + c} = 0, \\ -D + \dfrac{ax}{by + x + c} = 0. \end{cases}$$

From elementary calculation, we have two solutions with respect to x as follows;

(2.2) $$x = \frac{ab - a + D \pm \sqrt{(ab - a + D)^2 + 4abcD}}{2ab}.$$

Since $ab - a + D - \sqrt{(ab - a + D)^2 + 4abcD} < 0$ we take into account the equilibrium point $E_2(x_*, y_*)$ of system (1.6), where

(2.3) $$x_* = \frac{ab - a + D + \sqrt{(ab - a + D)^2 + 4abcD}}{2ab}, \quad y_* = \frac{1}{b}\left(\left(\frac{a}{D} - 1\right)x_* - c\right).$$

For the positiveness of the equilibrium point E_2, the condition $\dfrac{(a - D)x_* - cD}{bD} > 0$ must be satisfied. In addition, it follow from (1.5) that $x_* < 1$, which gives the condition $a + cD > D$. Thus from now on we will assume that the following conditions hold;

(2.4) $$a > D \quad \text{and} \quad \frac{cD}{a - D} < x_*.$$

The Jacobian matrix of system (1.6) at a point (x, y) is

(2.5)
$$J(x, y) = \begin{pmatrix} \left(1 - x + \dfrac{xy}{(by + x + c)^2}\right)e_1(x, y) & \dfrac{-x(x + c)}{(by + x + c)^2}e_1(x, y) \\ \dfrac{ay(by + c)}{(by + x + c)^2}e_2(x, y) & \left(1 - \dfrac{abxy}{(by + x + c)^2}\right)e_2(x, y) \end{pmatrix},$$

where $e_1(x, y) = \exp(1 - x - \frac{y}{by+x+c})$ and $e_2(x, y) = \exp(-D + \frac{ax}{by+x+c})$. The corresponding characteristic equation to the Jacobian matrix $J(x, y)$ can be obtained as

$$(2.6) \qquad \lambda^2 - \mathrm{tr}(J(x, y))\lambda + \det(J(x, y)) = 0,$$

where $\mathrm{tr}(J(x, y))$ is the trace and $\det(J(x, y))$ is the determinant of the Jacobian matrix $J(x, y)$.

Let λ_1 and λ_2 be the two roots of equation (2.6), which are called eigenvalues of the point (x, y). We have the following definitions.

(1) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then (x, y) is called a *sink* and it is locally asymptotically stable;

(2) If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then (x, y) is called a *source* and it is locally unstable;

(3) If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), then (x, y) is called a *saddle*;

(4) If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then (x, y) is called *non-hyperbolic*.

Now, we will investigate the stability of the equilibrium points of system (1.6).

Theorem 2.1. *The equilibrium point E_0 is a saddle.*

Proof. It is easy to calculate the Jacobian matrix $J(E_0)$ at E_0. In fact, the matrix $J(E_0)$ is given by

$$(2.7) \qquad J(E_0) = \begin{pmatrix} \exp(1) & 0 \\ 0 & \exp(-D) \end{pmatrix}.$$

Then the eigenvalues of the matrix $J(E_0)$ are $\exp(1)$ and $\exp(-D)$. Thus we can show that the equilibrium point E_0 is a saddle.   □

Theorem 2.2. *For the equilibrium point E_1, we have the following topological types:*

(i) *E_1 is a sink if $a < (1 + c)D$;*
(ii) *E_1 is non-hyperbolic if $a = (1 + c)D$;*
(iii) *E_1 is a saddle if $a > (1 + c)D$.*

Proof. Since the Jacobian matrix $J(E_1)$ at E_1 is

$$(2.8) \qquad J(E_1) = \begin{pmatrix} 0 & -\frac{1}{c+1} \\ 0 & \exp(\frac{a}{c+1} - D) \end{pmatrix},$$

we can know that the eigenvalues of the matrix $J(E_1)$ are 0 and $\exp(\frac{a}{c+1} - D)$. Therefore we have the results (i), (ii) and (iii).   □

In order to investigate the stability of the positive equilibrium point $E_2(x_*, y_*)$ of system (1.6), we give the following lemma, which can be easily proved by the relations between roots and coefficients of the characteristic equation (2.6) (see [1],[13]).

Lemma 2.3.([1],[13]) *Let B and C be the trace and the determinant of the Jacobian matrix in (2.5), respectively and let $F(\lambda) = \lambda^2 - B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. Then*
    (i) *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;*
    (ii) *$|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;*
    (iii) *$|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;*
    (iv) *$\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;*
    (v) *λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.*

Theorem 2.4. *Assume that the condition (2.4) is satisfied. Then there exists the positive equilibrium E_2. Moreover, we have the following topological types for the point E_2:*
    (i) *E_2 is a sink if $\alpha x_* > \gamma$ and $\beta x_* < \delta$;*
    (ii) *E_2 is a source if $\alpha x_* > \gamma$ and $\beta x_* > \delta$;*
    (iii) *E_2 is a saddle if $\alpha x_* < \gamma$;*
    (iv) *E_2 is non-hyperbolic if $\alpha x_* = \gamma$, $x_* \neq \dfrac{2a + D - abD}{a + D - abD}$ and $x_* \neq \dfrac{D(1 - ab)}{a + D - abD}$,*
    where $\alpha = -2a - 2D + aD + 3abD - D^2$, $\beta = -a - D + aD + 2abD - D^2$, $\gamma = -4a - 2D - aD + 3abD + D^2 + 2cD^2$ and $\delta = D(-1 - a + 2ab + D + 2cD)$.

Proof. Using equation (2.1), the Jacobian matrix $J(E_2)$ at E_2 can be obtained as follows;

$$(2.9) \qquad J(E_2) = \begin{pmatrix} (\frac{D}{a} + 1)(1 - x_*) & \frac{D}{a}(b(1 - x_*) - 1) \\ (a - D)(1 - x_*) & 1 - bD(1 - x_*) \end{pmatrix},$$

In fact, the value B in Lemma 2.3, the trace of the matrix $J(E_2)$, can be obtained by elementary calculation as

$$(2.10) \qquad B = (1 + \frac{D}{a} - bD)(1 - x_*) + 1$$

and the value C in Lemma 2.3, the determinant of the matrix $J(E_2)$, can be also obtained as

$$(2.11) \qquad C = -2bD(1 - x_*)^2 + (1 + \frac{D}{a} + \frac{(a - D)D}{a})(1 - x_*).$$

Now, consider the function $F(\lambda) = \lambda^2 - B\lambda + C$. From elementary calculations, we can know that $F(1) = 1 - B + C > 0$ since $\sqrt{(ab - a + D)^2 + 4abcD} > 0$. Therefore, we can use Lemma 2.3 to prove the results. Now think about the value $F(-1) = -2bD(1 - x_*)^2 + \dfrac{(D(2 - D) + a(2 + D - bD))(1 - x_*)}{a} + 2$. Since x_* satisfies $abx_*^2 - (ab - a + D)x_* - cD = 0$, $F(-1)$ can be written as

(2.12)

$$F(-1) = \frac{1}{a}((-2a - 2D + aD + 3abD - D^2)x_* + 4a + 2D + aD - 3abD - D^2 - 2cD^2).$$

By applying the above similar way to the value C, we can obtain

$$(2.13) \quad C = \frac{1}{a}((-a - D + aD + 2abD - D^2)x_* + a + D + aD - 2abD - D^2 - 2cD^2).$$

Note that $F(-1) = 0$ and $C = 1$ hold if and only if $\alpha x_* = \gamma$ and $\beta x_* = \delta$ are satisfied, respectively, where $\alpha = -2a - 2D + aD + 3abD - D^2$, $\beta = -a - D + aD + 2abD - D^2$, $\gamma = -4a - 2D - aD + 3abD + D^2 + 2cD^2$ and $\delta = D(-1 - a + 2ab + D + 2cD)$. Therefore, it follows from Lemma 2.3 that the results of this theorem hold. $\square$
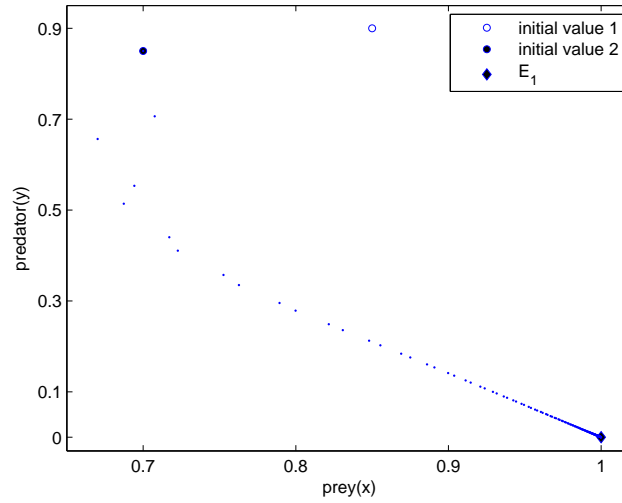

Figure 1: Phase portraits of system (1.6) with $a = 0.5, b = 1.5, c = 0.5$ and $D = 0.4$ when $(x_0, y_0) = (0.85, 0.9)$ or $(0.7, 0.85)$.

3. Numerical Simulations

In this section, we illustrate some phase portraits via numerical simulations in order to substantiate our theoretical results.

First, let us take parameters in system (1.6) as follows;

$$(3.1) \qquad\qquad a = 0.5, b = 1.5, c = 0.5 \text{ and } D = 0.4.$$

Then it follows from Theorem 2.2 that the equilibrium E_1 is a sink since $a < (1+c)D$. The Figure 1 is shown this phenomenon when we set two initial conditions as $(x_0, y_0) = (0.85, 0.9)$ and $(0.7, 0.85)$.

Next, for the parameters $a = 0.7, b = 0.3, c = 0.6, D = 0.2$ we can know that the equilibrium point E_2 is a sink since the parameters satisfy the condition of (i)

of Theorem 2.4. In this case, we can have the point $E_2(0.3333, 0.7778)$ and a phase portrait of system (1.6) is displayed in Figure 2.
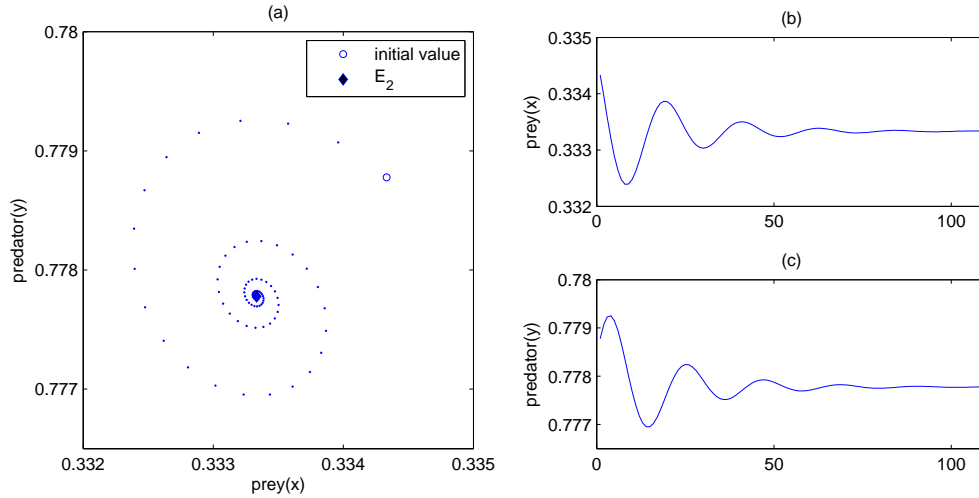

Figure 2: (a) A phase portrait of system (1.6) with $a = 0.7, b = 0.3, c = 0.6, D = 0.2$ and $E_2(0.3333, 0.7778)$. (b) Time series of the prey. (c) Time series of the predator.

For illustrating source phenomenon of the equilibrium E_2, let the parameters be as follows;

$$(3.2) \qquad a = 0.7, b = 0.3, c = 0.4 \text{ and } D = 0.1.$$

It is easy to see from Theorem 2.4 that the point $E_2 = (0.0975, 0.6157)$ is a source. In fact, if one takes initial conditions contained in a suitable neighborhood of the point E_2, the trajectories starting with this initial conditions are away from the point E_2 as shown in Figure 3. In addition, this figure demonstrates that system (1.6) could have a limit cycle. However, it is not easy to show the existence of it theoretically. Thus this problem is left for future work.
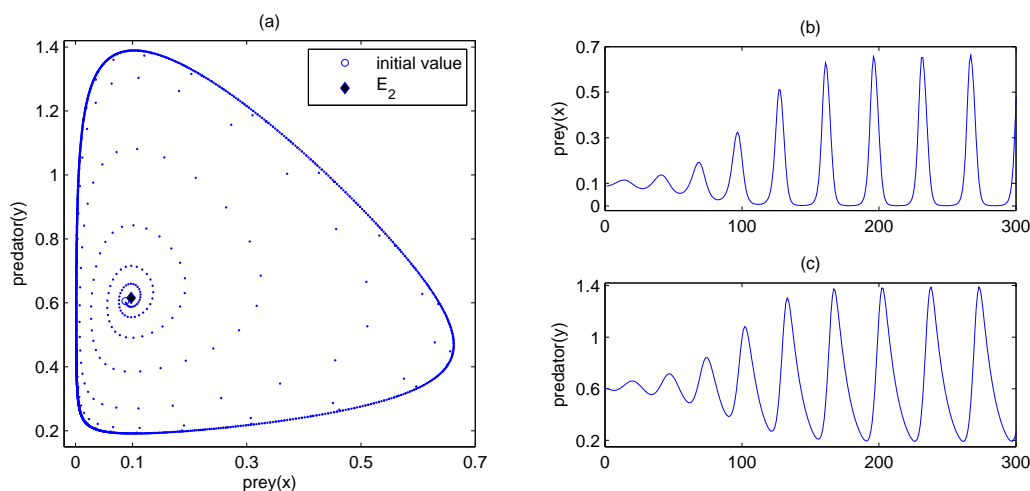
Figure 3: (a) A phase portrait of system (1.6) with $a = 0.7, b = 0.3, c = 0.4, D = 0.1$ and $E_2(0.0975, 0.6157)$. (b) Time series of the prey. (c) Time series of the predator.

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