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On Prime Cordial Labeling of Graphs

Abdullah Aljouiee
Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University, P. O. Box 90189, Riyadh 11613, Saudi Arabia
e-mail : joa111@gmail.com

Abstract. A graph G of order n has prime cordial labeling if its vertices can be assigned the distinct labels $1, 2 \cdots, n$ such that if each edge xy in G is assigned the label 1 in case the labels of x and y are relatively prime and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In this paper, we give a complete characterization of complete graphs which are prime cordial and we give a prime cordial labeling of the closed helm $\bar{H}_n$, and present a new way of prime cordial labeling of P_n^2. Finally we make a correction of the proof of Theorem 2.5 in [12].

1. Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notation and terminology of graph theory as in [2].

The notion of prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy and Howalla [11]. A graph G of order n with vertex set $V(G)$ is said to have prime labeling if its vertices are labeled with distinct integers $1, 2, \cdots, n$ such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have prime labeling. So far, there has been a little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with one vertex of degree at least 3 and with every other vertex has degree at most 2), olive trees (i.e., rooted trees consisting of k branches such that the i^{th} branch is a path of length i) and all trees of order up to 50. The notion of cordial labeling of graphs was introduced by Cahit [1] in 1987. Sundaram, Ponraj and Somasundaram [10] have introduced the notion of prime cordial labelings motivated by the prime and cordial labelings. A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection f from $V(G)$ to

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$\{1, 2, \cdots n\}$ where $n = |V(G)|$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ at most by 1.

For $i = 0, 1$, let $q_i(G)$ denote the number of edge labeled i under a prime cordial function f. In [10], Sundaram and others proved that the following graphs are prime cordial: C_n if and only if $n \geq 6$, P_n if and only if $n \neq 3$ or 5 ; $K_{1,n}$ (n odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \neq 3$. They also proved that if G is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of $K_{1,n}$ with the vertex of G labeled with 2 is prime cordial, and if G is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of $K_{1,2n}$ with the vertex of G labeled with 2 is prime cordial. They further proved that K_n is not prime cordial for $4 \leq n \leq 181$ and $K_{m,n}$ is not prime cordial for a number of special cases of m and n. Vaidya and Shah [13] proved that W_n is prime cordial if and only if $n \geq 8$. See ([3]-[13]) for related results. The reference [4] surveys the current state of knowledge for all variations of graph labelings appearing in this paper.

A graph G of order n is prime if and only if G is isomorphic to a spanning subgraph of the graph R_n of order n with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$ and whose edge set is defined as $E(R_n) = \{v_i v_j : \gcd(i, j) = 1\}$. We call R_n the maximal prime graph of order n and $\rho(n) = |E(R_n)|$, is the maximum number of edges in a prime graph of order n. Seoud and Youssef [9] proved that $\rho(n) = |E(R_n)| = \sum_{i=1}^{n} \phi(i) - 1$, where ϕ is the Euler's phi function. It follows that $\rho(n)$ is the maximum number of edges labeled 1 in a prime cordial graph of order n.

2. Main Results

In [10] Sundaram and others conjectured that $\sum_{i=2}^{n} \phi(i) \geq \frac{1}{2}\binom{n}{2} + 1$. The following is a proof of this conjecture.

Theorem 2.1. *With the setting above, the following inequality holds*

$$\sum_{i=2}^{n} \phi(i) \geq \frac{1}{2}\binom{n}{2} + 1, \text{ for all n} \geq 3.$$

Proof. The case $n = 3, 4$ can be checked manually. So we assume that $n \geq 5$.
We see that $\Big|\{(i,j) : 1 \leq i < j \leq n;\ \gcd(i,j) = 1\}\Big| = \sum_{i=2}^{n} \phi(i)$. Therefore, $\Big|\{(i,j) : 1 \leq i < j \leq n,\ \gcd(i,j) = 1\}\Big| = 1 + 2\sum_{i=2}^{n} \phi(i)$. But this is equivalent to showing that

$$\Big|\{(i,j) : 1 \leq i < j \leq n,\ \gcd(i,j) = 1\}\Big| \geq \frac{n(n-1)}{2} + 3.$$

Let $f(n)$ denote the LHS quantity. Observe that $f(n)$ counts the number of pairs $(x, y) \in \{1, 2, \ldots, n\}^2$ such that there is no prime p such that p dividing both x and y. Using the principal of inclusion-exclusion, we find that

$$f(n) = n^2 - \sum_p \left\lfloor \frac{n}{p} \right\rfloor^2 + \sum_{p<q} \left\lfloor \frac{n}{pq} \right\rfloor^2 - \sum_{p<q<r} \left\lfloor \frac{n}{pqr} \right\rfloor^2 + \cdots,$$

where the indices $p, q, r, \ldots$ are prime numbers. It follows that

$$\begin{aligned} f(n) &\geq n^2 - \sum_p \left(\frac{n}{p}\right)^2 + \sum_{p<q} \left\lfloor \frac{n}{pq} \right\rfloor^2 - \sum_{p<q<r} \left(\frac{n}{pqr}\right)^2 + \cdots \\ &> n^2 \left(1 - \sum_p \frac{1}{p^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} - \cdots\right), \end{aligned}$$

(where only sums with the odd number of primes appear). The RHS can be computed exactly, as we shall explain below. We know that

$$\prod_p \left(1 - \frac{1}{p^2}\right) = 1 - \sum_p \frac{1}{p^2} + \sum_{p<q} \frac{1}{p^2q^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} + \cdots,$$

but also $\prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \sum_{n\geq 1} \frac{1}{n^2} = \xi(2) = \frac{\pi^2}{6}$. So,

$$1 - \sum_p \frac{1}{p^2} + \sum_{p<q} \frac{1}{p^2q^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} + \cdots = \frac{6}{\pi^2}.$$

On the other hand, we have

$$\begin{aligned} 1 + \sum_p \frac{1}{p^2} + \sum_{p<q} \frac{1}{p^2q^2} + \sum_{p<q<r} \frac{1}{p^2q^2r^2} \cdots &= \prod_p \left(1 - \frac{1}{p^2}\right) = \prod_p \frac{1 - \frac{1}{p^4}}{1 - \frac{1}{p^2}} \\ &= \frac{\xi(2)}{\xi(4)} = \frac{\pi^2}{6} \Big/ \frac{\pi^4}{90} = \frac{15}{\pi^2}. \end{aligned}$$

Subtracting the above two results from each other, we find that

$$1 - \sum_p \frac{1}{p^2} - \sum_{p<q<r} \frac{1}{p^2q^2r^2} - \cdots = 1 - \frac{1}{2}\left(\frac{15}{\pi^2} - \frac{6}{\pi^2}\right) = 1 - \frac{9}{2\pi^2}.$$

Therefore, $f(n) \geq \left(1 - \frac{9}{2\pi^2}\right) n^2 \geq 0.544n^2$, which is greater than $\frac{n(n-1)}{2} + 3$ for all $n \geq 5$. This completes the proof. □

We denote $\alpha(n)$, the maximum number of edges in a prime cordial graph of order n. The following corollary gives an exact formula for $\alpha(n)$.

Corollary 2.2. $\alpha(n) = n(n-1) - 2\rho(n) + 1$.

Proof. Let $\lambda(n)$ be the maximum number of edges labeled 0 in a prime cordial graph of order n. Hence, $\lambda(n) = \dfrac{n(n-1)}{2} - \rho(n)$. From Theorem 2.1 $2\lambda(n) \leq \alpha(n) \leq 2\lambda(n) + 1$ for every $n \geq 1$, then $\alpha(n) = n(n-1) - 2\rho(n) + 1$. □

The following corollary gives a necessary condition for a graph of order n and size q to be prime cordial.

Corollary 2.3. *If G is a prime cordial graph of order n and size q, then $q \leq \alpha(n)$.*

The following table shows the values of $\rho(n)$ and $\alpha(n)$ for all $n \leq 12$

n	1	2	3	4	5	6	7	8	9	10	11	12
$\rho(n)$	0	1	3	5	9	11	17	21	27	31	41	45
$\alpha(n)$	0	1	1	3	3	9	9	15	19	29	29	43

Seoud and Salim [8] proved that K_n does not have prime cordial labeling for $2 < n < 500$ and conjectured that K_n is not prime cordial for all $n > 2$. Since the number of edges labeled 1 in K_n is equal to $\sum_{i=2}^{n} \phi(i)$ which is always odd for every $n \geq 2$, then K_n, $n \equiv 0$ or 1(mod 8) is not prime cordial, because in this case, the graph is of size $0 \equiv$ (mod 4). This contradicts that the number of edges labeled 1 is odd. However, by Corollary 2.3, we will obtain the following theorem

Theorem 2.4. *K_n is not prime cordial for all $n \geq 3$.*

Proof. From Theorem 2.1, $\rho(n) \geq \dfrac{n(n-1)}{4} + 1$, we have

$$\begin{aligned}
\rho(n) > \frac{n(n-1)}{4} + \frac{1}{2} &\Rightarrow 2\rho(n) > \frac{n(n-1)}{2} + 1 \\
&\Rightarrow \frac{n(n-1)}{2} + 2\rho(n) > n(n-1) + 1 \\
&\Rightarrow \frac{n(n-1)}{2} > n(n-1) - 2\rho(n) + 1 = \alpha(n).
\end{aligned}$$

That is $|E(K_n)| > \alpha(n)$ and the graph is not prime cordial from Corollary 2.3. □

The helm H_n $(n \geq 3)$ is the graph obtained from a wheel W_n by attaching a pendant edge at each vertex of the n-cycle, while the closed helm $\bar{H}_n$ is the graph obtained from a helm by joining each pendant vertex to form a cycle. We show that a closed helm $\bar{H}_n$ have prime cordial labeling for all $n \geq 6$.

Theorem 2.5. *$\bar{H}_n$ is prime cordial for all $n \geq 6$.*

Proof. Necessity, a direct computation shows that if $\bar{H}_3$, $\bar{H}_4$ or $\bar{H}_5$ has a prime cordial labeling, then $q_0(\bar{H}_3) \leq 4$, $q_0(\bar{H}_4) \leq 7$ or $q_0(\bar{H}_5) \leq 9$. For sufficiency, let

$V(\bar{H}_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and
$E(\bar{H}_n) = \{v_i v_j, u_i u_j : i - j \equiv \pm 1(\text{mod n})\} \cup \{v_i u_i : 1 \le i \le n\} \cup \{v_0 v_i : 1 \le i \le n\}$ and let $f : V(\bar{H}_n) \to \{1, 2, \ldots, 2n+1\}$. If $n = 6$, we give a prime cordial labeling f as follows: $(f(v_0), f(v_1), f(v_2), \ldots, f(v_n)) = (6, 2, 4, 8, 10, 12, 3)$ and $(f(u_1), f(u_2), \ldots, f(u_n) = (1, 5, 7, 11, 13, 9)$. If $n \ge 7$, we define f as follows: $f(v_0) = 2, f(v_1) = 3, f(v_i) = 2(i+1), 2 \le i \le n-1, f(v_n) = 4$ $f(u_1) = 9, f(u_2) = 5, f(u_3) = 7, f(u_4) = 11, f(u_5) = 15, f(u_6) = 13, f(u_j) = 2j+3, 7 \le j \le n-1, f(u_n) = 1$. The number of edges labeled 0 obtained from the even vertex labels on the inner cycle is equal to $n-2$ beside one more from the vertex labels 3 and 6. The number of edges labeled 0 obtained from the label of the apex vertex and the rim vertices of the inner cycle is equal to $n-1$ and finally the vertex labels 3;9 and 12;15 give two edges labeled 0. Hence $q_0(\bar{H}_n) = 2n = q_1(\bar{H}_n)$ and this completes the proof. □

Vaidya and Shah [12] proved that P_n^2 is a prime cordial if $n = 6$ and $n \ge 8$. Here we introduce a simple proof for this result

Theorem 2.6. $P_n^2, n \ge 3$ *is prime cordial if and only if* $n \ge 6, n \ne 7$.

Proof. Necessity follows from Corollary 2.3. For sufficiency, first we give a prime cordial labeling f of $P_n^2, n = 6, 8, 9$ and 10 in the following pattern: $(f(v_1), f(v_2), \ldots, f(v_n))$ where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$:
$n = 6 : (2, 4, 6, 3, 1, 5), n = 8 : (2, 4, 8, 6, 3, 1, 5, 7), n = 9 : (2, 4, 8, 6, 3, 9, 1, 5, 7), n = 10 : (2, 4, 8, 10, 6, 3, 1, 9, 5, 7)$.

Now, let n be even and $n \ge 12$. We describe the prime cordial labeling as the above pattern:$(2, 4, 8, 10, \ldots, n, 6, 3, 1, 9, 5, 7, 11, 13, \ldots, n-1)$. Then $q_0(P_n^2) = n-1$ and $q_1(P_n^2) = n-2$, and P_n^2 is prime cordial in this case. If n is odd and $n \ge 11$, we label P_{n-1}^2 as in the former case and then we label the remaining vertex by the label n. Then $q_0(P_n^2) = n-2$ and $q_1(P_n^2) = n-1$, and again P_n^2 is prime cordial.□

Vaidya and Shah [12] proved that C_n^2 is a prime cordial for $n \ge 10$. The proof is incorrect, in fact, the labeling function does not work in some cases. For example, C_{21}^2 is not prime cordial under this labeling since $\left| q_0(C_{21}^2) - q_1(C_{21}^2) \right| = 2$ and more generally the case $n \equiv 21 \pmod{30}$ does not work. In the following theorem we correct this result.

Theorem 2.7. $C_n^2, n \ge 4$ *is prime cordial if and only if* $n \ge 10$.

Proof. If $4 \le n \le 8$, then C_n^2 is not prime cordial by Corollary 2.3. If $n = 9$, then $q_0(C_9^2) \le 8$ for any prime cordial labeling function and hence C_9^2 is not prime cordial. Conversely, first we give a prime cordial labeling f of $C_n^2, n = 10$ in the following pattern: $(f(v_1), f(v_2), \ldots, f(v_n))$ where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$: $n = 10 : (4, 8, 10, 2, 6, 3, 9, 1, 5, 7)$. If n is even and $n \ge 12$, we describe the prime cordial labeling as the above pattern: $(4, 8, 10, \ldots, n, 2, 6, 3, 9, 1, 5, 7, 11, \ldots, n-1)$. The consecutive vertex labels of even labels give $\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 2\right)$ edges of label 0 and the vertex labels 6, 3 and 9 give 3 edges labeled 0. Hence $q_0(C_n^2) = n$ and $q_1(C_n^2) =$

n. If $n = 11, 13$, we describe the prime cordial labeling as the above pattern: $n = 11 : (10, 2, 4, 8, 6, 3, 9, 1, 7, 11, 5), n = 13 : (4, 8, 10, 2, 12, 6, 3, 9, 1, 5, 7, 11, 13)$. If n is odd and $n \geq 15$, we give the vertex prime cordial labeling in the following pattern: $(4, 8, 10, 14, 16, \ldots, n-1, 2, 12, 6, 3, 9, 1, 5, 7, 11, 13, \ldots, n)$. The consecutive vertex labels of even labels give $\left(\frac{n-3}{2}\right) + \left(\frac{n-5}{2}\right)$ edges of label 0 and the vertex labels 6, 3 and 9 give 3 edges labeled 0. Finally the vertex labels 3 and 12 give a an edge labeled 0. Hence $q_0(C_n^2) = n$ and $q_1(C_n^2) = n$. □

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