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Forced Oscillation Criteria for Nonlinear Hyperbolic Equations via Riccati Method

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Abstract. In this paper, we consider the nonlinear hyperbolic equations with forcing term. Some sufficient conditions for the oscillation are derived by using integral averaging method and a generalized Riccati technique.

1. Introduction

We shall provide oscillation results of solution of the hyperbolic equation

$$\text{(E)} \qquad \frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right) + p(t)\frac{\partial}{\partial t}u(x,t) \\ -a(t)\Delta u(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) \\ +\sum_{i=1}^{m} q_i(x,t)\varphi_i(u(x,\sigma_i(t))) = f(x,t),\ (x,t)\in\Omega\equiv G\times(0,\infty),$$

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary ∂G. Recently, the oscillation of solution of hyperbolic equation via Riccati method has been investigated by many authors, see for example [2], [6], [7], In particular, Shoukaku [6] established the oscillation results of solution of the equation (E). In the work of [6], restriction is imposed on forcing term $f(x,t)$ to be oscillatory function.

Gaef and Spikes [3], Wong and Agarwal [8], Li [4] and Agawal, et al [1] obtained several oscillation results for second order nonlinear differential equations. Their results used the different assumption of forcing term from the work of [6].

Motivated by the work of [1], in this paper we will obtain the oscillation results

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of the hyperbolic equation (E), and remove the assumption of the forcing term such as the work [6].

We assume throughout this paper that:

(H1) $r(t) \in C^1([0,\infty);(0,\infty))$, $p(t) \in C([0,\infty);\mathbb{R})$,
$a(t),\ b_i(t) \in C([0,\infty);[0,\infty))$ $(i=1,2,\dots,k)$,
$q_i(x,t) \in C(\overline{\Omega};[0,\infty))$ $(i=1,2,\dots,m)$, $f(x,t) \in C(\overline{\Omega};\mathbb{R})$;

(H2) $\tau_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \tau_i(t) = \infty$ $(i=1,2,\dots,k)$,
$\sigma_i(t) \in C^1([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i=1,2,\dots,m)$;

(H3) $\varphi_i(s) \in C^1(\mathbb{R};\mathbb{R})$ $(i=1,2,\dots,m)$ are convex on $[0,\infty)$, and $\varphi_i(s) \geq 0$ and $\varphi_i(-s) = -\varphi_i(s)$ for $s \geq 0$.

We consider the following Dirichlet and Robin boundary boundary conditions

(B1) $$u = \psi \quad \text{on} \quad \partial G \times [0,\infty),$$

(B2) $$\frac{\partial u}{\partial \nu} + \mu u = \tilde{\psi} \quad \text{on} \quad \partial G \times [0,\infty),$$

where ν denotes the unit exterior normal vector to ∂G and $\psi, \tilde{\psi} \in C(\partial G \times (0,\infty);\mathbb{R})$, $\mu \in C(\partial G \times (0,\infty);[0,\infty))$.

Definition 1. By a *solution* of Eq. (E) we mean a function $u \in C^2(\overline{G} \times [t_{-1},\infty)) \cap C(\overline{G} \times [\tilde{t}_{-1},\infty))$ which satisfies (E), where

$$t_{-1} = \min\left\{0, \min_{1\leq i\leq k}\left\{\inf_{t\geq 0} \tau_i(t)\right\}\right\}, \quad \tilde{t}_{-1} = \min\left\{0, \min_{1\leq i\leq m}\left\{\inf_{t\geq 0} \sigma_i(t)\right\}\right\}.$$

Definition 2. A solution u of Eq. (E) is said to be *oscillatory* in Ω if u has a zero in $G \times (t,\infty)$ for any $t > 0$. That is, there exists a point $t_1 > t$ such that $u(x,t_1) = 0$.

Definition 3. We say that functions H_1, H_2 belong to a function class $\mathbb{H}$, denoted by $H_1, H_2 \in \mathbb{H}$, if $H_1, H_2 \in C(D;[0,\infty))$ satisfy

$$H_i(t,t) = 0,\ H_i(t,s) > 0\ (i=1,2) \quad \text{for } t > s,$$

where $D = \{(t,s) : 0 < s \leq t < \infty\}$. Moreover, the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on D such that

$$\frac{\partial H_1}{\partial t}(s,t) = h_1(s,t)H_1(s,t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s)H_2(t,s),$$

where $h_1,\ h_2 \in C_{loc}(D;\mathbb{R})$.

2. Reduction to One-Dimensional Problems

In this section we reduce the multi-dimensional oscillation problems for (E) to one-dimensional oscillation problems. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G. Now we define

$$q_i(t) = \min_{x \in \overline{G}} q_i(x,t).$$

The following notation will be used:

$$\begin{aligned} U(t) &= K_\Phi \int_G u(x,t)\Phi(x)dx, \quad \tilde{U}(t) = \frac{1}{|G|}\int_{\partial G} u(x,t)dx, \\ F(t) &= K_\Phi \int_G f(x,t)\Phi(x)dx, \quad \tilde{F}(t) = \frac{1}{|G|}\int_{\partial G} f(x,t)dx, \\ \Psi(t) &= K_\Phi \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS, \quad \tilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G} \tilde{\psi} dS, \end{aligned}$$

where $K_\Phi = (\int_G \Phi(x)dx)^{-1}$ and $|G| = \int_G dx$.

Theorem 1. *If every eventually positive solution $y(t)$ of the functional differential inequalities*

(1)
$$(r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \leq \pm G(t)$$

satisfies $\liminf_{t\to\infty} y(t) = 0$, then every solution $u(x,t)$ of the problem (E), (B1) *is oscillatory in Ω or sastifies*

(2)
$$\liminf_{t\to\infty} |U(t)| = 0,$$

where

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(\tau_i(t))\Psi(\tau_i(t)).$$

Proof. Suppose to the contrary that there is a nonoscillatory solution u of the problem (E), (B1) which does not satisfy (2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$ because the case where $u(x,t) < 0$ can be treated similarly. Since (H2) holds, we see that $u(x, \tau_i(t)) > 0$ $(i = 1, 2, \ldots, k)$ and $u(x, \sigma_i(t)) > 0$ $(i = 1, 2, \ldots, m)$ in $G \times [t_1, \infty)$ for some

$t_1 \geq t_0$. Multiplying (E) by $K_\Phi \Phi(x)$ and integrating over G, we obtain

$$\begin{aligned}(3)\qquad & (r(t)U'(t))' + p(t)U'(t) \\ & -a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx - \sum_{i=1}^{k} b_i(t)K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx \\ & + \sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx = F(t),\ t \geq t_1.\end{aligned}$$

From Green's formula it follows that

$$(4)\qquad K_\Phi \int_G \Delta u(x,t)\Phi(x)dx \leq -\Psi(t),\ t \geq t_1,$$

$$(5)\qquad K_\Phi \int_G \Delta u(x,\tau_i(t))\Phi(x)dx \leq -\Psi(\tau_i(t)),\ t \geq t_1.$$

An application of Jensen's inequality shows that

$$(6)\qquad \sum_{i=1}^{m} K_\Phi \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))),\ t \geq t_1.$$

Combining (3)–(6) yields

$$(r(t)U'(t))' + p(t)U'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))) \leq G(t),\ t \geq t_1.$$

Therefore $U(t)$ is a positive solution of (1) which does not satisfy (2). This contradicts the hypothesis and completes the proof. □

Theorem 2. *If every eventually positive solution $y(t)$ of the functional differential inequalities*

$$(7)\qquad (r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \leq \pm\tilde{G}(t)$$

satisfies $\liminf_{t\to\infty} y(t) = 0$, then every solution $u(x,t)$ of the problem (E), (B2) *is oscillatory in Ω or satisfies*

$$(8)\qquad \liminf_{t\to\infty} |\tilde{U}(t)| = 0,$$

where

$$\tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(\tau_i(t))\tilde{\Psi}(\tau_i(t)).$$

Proof. Suppose to the contrary that there is a nonoscillatory solution u of problem (E), (B2) which does not satisfy (8). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (H2) holds, we see that $u(x, \tau_i(t)) > 0$ $(i = 1, 2, \ldots, k)$ and $u(x, \sigma_i(t)) > 0$ $(i = 1, 2, \ldots, m)$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Dividing (E) by $|G|$ and integrating over G, we obtain

$$\begin{aligned}(9)\qquad & (r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) \\ & -\frac{a(t)}{|G|}\int_G \Delta u(x,t)dx - \sum_{i=1}^{k}\frac{b_i(t)}{|G|}\int_G \Delta u(x,\tau_i(t))dx \\ & +\frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx = \tilde{F}(t),\ t \geq t_1.\end{aligned}$$

It follows from Green's formula that

$$(10)\qquad \frac{1}{|G|}\int_G \Delta u(x,t)dx \leq \tilde{\Psi}(t),\ t \geq t_1,$$

$$(11)\qquad \frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \leq \tilde{\Psi}(\tau_i(t)),\ t \geq t_1.$$

Applying Jensen's inequality, we observe that

$$(12)\qquad \frac{1}{|G|}\sum_{i=1}^{m}\int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))),\ t \geq t_1.$$

Combining (9)–(12) yields

$$(r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(\tilde{U}(\sigma_i(t))) \leq \tilde{G}(t),\ t \geq t_1.$$

Hence, $\tilde{U}(t)$ is a positive solution of (7) which does not satisfy (8). This contradicts the hypothesis and completes the proof. □

3. Second Order Functional Differential Inequality

We obatin the sufficient conditions for every positive solution $y(t)$ of the functional differential inequality

$$(13)\qquad (r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \leq f(t)$$

to satisfy $\liminf_{t\to\infty} y(t) = 0$, where $f(t) \in C([0,\infty);\mathbb{R})$. We assume the following hypotheses:

(H4) $\varphi_j'(t) > 0$, $\varphi_j'(t)$ is nondecreasing for $t > 0$ and some $j \in \{1, 2, \ldots, m\}$;

(H5) there exists a positive constant σ such that

$$\sigma_j'(t) \geq \sigma \quad \text{and} \quad \sigma_j(t) \leq t;$$

(H6) there exists a positive constant K such that

$$q_j(t) \geq K|f(t)|.$$

Theorem 3. *If the Riccati inequalities for $i = 1, 2$*

$$x'(t) + \frac{1}{2}\frac{1}{p_i(t)}x^2(t) \leq -q(t) \tag{14}$$

have no solution on $[T, \infty)$ for all large T, then eventually positive solution of (13) *satisfies $\liminf_{t\to\infty} y(t) = 0$, where*

$$p_1(t) = \tilde{K}e^{R(t)},\ p_2(t) = p_1(\sigma_j(t)),\ R(t) = \log r(t) + \int_{t_0}^{t} \frac{p(s)}{r(s)}ds,$$
$$q(t) = \frac{e^{R(t)}}{r(t)}\left\{q_j(t) - K|f(t)|\right\}$$

for every positive constant $\tilde{K}$.

Proof. Suppose that $y(t)$ is an eventually positive solution of (13) on $[t_0, \infty)$ for some $t_0 > 0$, and $\liminf_{t\to\infty} y(t) > 0$. Hence, there exists $k_1 > 0$ such that $y(t) \geq k_1$, $t \geq t_1$ for some $t_1 \geq t_0$. It follows from (13) that

$$\left(e^{R(t)}y'(t)\right)' + q_j(t)\frac{e^{R(t)}}{r(t)}\varphi_j(y(\sigma_j(t))) \leq \frac{e^{R(t)}}{r(t)}f(t),\ t \geq t_1. \tag{15}$$

Since $\varphi_j(y(\sigma_j(t))) > \varphi_j(k_1) \equiv K_1$, $t \geq t_2$ for some $t_2 \geq t_1$, we can see from (H6) that

$$\left(e^{R(t)}y'(t)\right)' \leq -\frac{e^{R(t)}}{r(t)}\left\{K_1 q_j(t) - |f(t)|\right\} \leq 0,\ t \geq t_2. \tag{16}$$

Then we consider $y'(t) < 0$ or $y'(t) \geq 0$ for $t \geq t_2$.

Case 1. $y'(t) < 0$ for $t \geq t_2$. Setting

$$z(t) = \frac{e^{R(t)}y'(t)}{\varphi_j(y(t))},$$

then

$$
\begin{aligned}
(17)\quad z'(t) &= \frac{(e^{R(t)}y'(t))'}{\varphi_j(y(t))} - e^{R(t)}y'(t)\frac{y'(t)\varphi_j'(y(t))}{\varphi_j^2(y(t))} \\
&\le -q_j(t)\frac{e^{R(t)}}{r(t)}\frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(y(t))} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(y(t))} - e^{-R(t)}\varphi_j'(y(t))z^2(t) \\
&\le -q_j(t)\frac{e^{R(t)}}{r(t)} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(k_1)} - e^{-R(t)}\varphi_j'(k_1)z^2(t) \\
&\le -\frac{e^{R(t)}}{r(t)}\left\{q_j(t) - \frac{|f(t)|}{K_1}\right\} - e^{-R(t)}\varphi_j'(k_1)z^2(t)
\end{aligned}
$$

which contradicts the fact that $z(t)$ is negative solution of (14).
Case 2. $y'(t) \ge 0$ for $t \ge t_2$. Since $y(t) > 0$, $y'(t) \ge 0$ eventually, we see that $y(\sigma_j(t)) \ge k_1$ for some $k_1 > 0$. Let

$$
w(t) = \frac{e^{R(t)}y'(t)}{\varphi_j(y(\sigma_j(t)))}.
$$

By using $e^{R(t)}y'(t)$ is nonincreasing, we have

$$
\begin{aligned}
(18)\qquad w'(t) &= \frac{(e^{R(t)}y'(t))'}{\varphi_j(y(\sigma_j(t)))} - e^{R(t)}y'(t)\frac{\sigma_j'(t)y'(\sigma_j(t))\varphi_j'(y(\sigma_j(t)))}{\varphi_j^2(y(\sigma_j(t)))} \\
&\le -q_j(t)\frac{e^{R(t)}}{r(t)} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(y(\sigma_j(t)))} \\
&\qquad -e^{-R(\sigma_j(t))}\varphi_j'(y(\sigma_j(t)))\sigma_j'(t)w^2(t) \\
&\le -\frac{e^{R(t)}}{r(t)}\left\{q_j(t) - \frac{|f(t)|}{\varphi_j(k_1)}\right\} \\
&\qquad -e^{-R(\sigma_j(t))}\varphi_j'(k_1)\sigma w^2(t),\ t \ge t_2.
\end{aligned}
$$

Therefore $w(t)$ is a positive solution of (14). This contradicts the hypothesis and completes the proof. □

Theorem 4. *If for some $T \ge 0$ and for $i = 1, 2$, there exist $H_1, H_2 \in \mathbb{H}$ and some $c \in (a, b)$ such that $T \le a < b$ and*

$$
\begin{aligned}
(19)\qquad & \frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\left\{q(s) - \frac{1}{2}\lambda_1^2(s,a)p_i(s)\right\}\phi(s)ds \\
&+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\left\{q(s) - \frac{1}{2}\lambda_2^2(b,s)p_i(s)\right\}\phi(s)ds > 0,
\end{aligned}
$$

then eventually positive solution of (13) *satisfies* $\liminf_{t\to\infty} y(t) = 0$*, where* $\phi(t) \in C^1((T,\infty);(0,\infty))$ *and*

$$
\lambda_1(s,t) = \frac{\phi'(s)}{\phi(s)} + h_1(s,t), \quad \lambda_2(t,s) = \frac{\phi'(s)}{\phi(s)} - h_2(t,s).
$$

Proof. Suppose that $y(t)$ is a positive solution of (13) on $[t_0, \infty)$ for some $t_0 > 0$, and $\liminf_{t\to\infty} y(t) > 0$. At first, we assume that $y(t) > 0$ on (a, b) for $a, b \geq t_0$. Proceeding as the same proof of Theorem 3, we have the inequality (14). Multiplying (14) by $H_2(t, s)$ and $\phi(s)$, integrating over $[c, t]$ for $t \in [c, b)$ and letting $t \to b^-$, we see easily that

$$\frac{1}{H_2(b,c)} \int_c^b H_2(b,s) \left\{ q(s) - \frac{1}{2}\lambda_2^2(b,s)p_i(s) \right\} \phi(s)ds \leq x(c)\phi(c). \tag{20}$$

Similarly, multiplying (14) by $H_1(s, t)$and $\phi(s)$, integrating over $[t, c]$ for $t \in (a, b]$ and letting $t \to a^+$, we have

$$\frac{1}{H_1(c,a)} \int_a^c H_1(s,a) \left\{ q(s) - \frac{1}{2}\lambda_1^2(s,a)p_i(s) \right\} \phi(s)ds \leq -x(c)\phi(c). \tag{21}$$

Adding (20) and (21), we can lead to the contradiction. Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By assumptions, for each $i \in \mathbb{N}$, there exists $a_i,\ b_i,\ c_i \in [0, \infty)$ such that $T_i \leq a_i < c_i < b_i$, and (19) holds with $a,\ b,\ c$ replaced by $a_i,\ b_i,\ c_i$, respectively. From that, every nontrivial solution $y(t)$ of (13) has no zero $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \geq T_i,\ i \in \mathbb{N}$, we see that $y(t)$ is a eventually positive solution of (13). This contradiction proves that Theorem 4 holds. □

Theorem 5. *For some functions $H_1, H_2 \in \mathbb{H}$, each $T \geq 0$ and for $i = 1, 2$, if*

$$\limsup_{t\to\infty} \int_T^t H_1(s,T) \left\{ q(s) - \frac{1}{2}\lambda_1^2(s,T)p_i(s) \right\} \phi(s)ds > 0 \tag{22}$$

and

$$\limsup_{t\to\infty} \int_T^t H_2(t,s) \left\{ q(s) - \frac{1}{2}\lambda_2^2(t,s)p_i(s) \right\} \phi(s)ds > 0, \tag{23}$$

then eventually positive solution of (13) *satisfies* $\liminf_{t\to\infty} y(t) = 0$*, where* $\phi(t) \in C^1((T_0, \infty); (0, \infty))$ *for some* $T_0 > 0$.

Proof. For any $T \geq t_0$, let $a = T$. In (22) we choose $T = a$. Then there exists $c > a$ such that for $t \in (a, c]$

$$\int_a^c H_1(s,a) \left\{ q(s) - \frac{1}{2}\lambda_1^2(s,a)p_i(s) \right\} \phi(s)ds > 0 \tag{24}$$

(cf. [9,Theorem 8.8.5]). In (23) we choose $T = c$. Then there exists $b > c$ such that for $t \in [c, b)$

$$\int_c^b H_2(b,s) \left\{ q(s) - \frac{1}{2}\lambda_2^2(b,s)p_i(s) \right\} \phi(s)ds > 0. \tag{25}$$

Combining (22) and (23) we obtain (19). The conclusion come from Theorem 4, and the proof is completed. □

4. Oscillation Criteria for Eq. (E)

4.1. Oscillation results by Riccati inequality

We are going to use the following lemma which is due to Usami [5].

Lemma. *If there exists a function $\phi(t) \in C^1([T_0,\infty);(0,\infty))$ such that*

$$\int_{T_1}^{\infty}\left(\frac{\bar{p}(t)|\phi'(t)|^{\beta}}{\phi(t)}\right)^{\frac{1}{\beta-1}}dt<\infty,\ \int_{T_1}^{\infty}\frac{1}{\bar{p}(t)(\phi(t))^{\beta-1}}dt=\infty,$$
$$\int_{T_1}^{\infty}\phi(t)\bar{q}(t)dt=\infty$$

for some $T_1 \geq T_0$, then the Riccati inequality

$$x'(t)+\frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^{\beta}\leq-\bar{q}(t),$$

where $\beta>1$, $\bar{p}(t)\in C([T_0,\infty);(0,\infty))$ and $\bar{q}(t)\in C([T_0,\infty);\mathbb{R})$, has no solution on $[T,\infty)$ for all large T.

Combinig Theorems 1-3, we obtain following theorems.

Theorem 6. *Assume that* (H1)–(H5) *hold, and that*

(H7) *there exists a positive constant K such that*

$$q_j(t)\geq K|G(t)|.$$

If for $i=1,2$,

$$\int_{T_1}^{\infty}\left(\frac{p_i(t)\phi'(t)^2}{\phi(t)}\right)dt<\infty,\quad\int_{T_1}^{\infty}\frac{1}{p_i(t)\phi(t)}dt=\infty,$$
$$\int_{T_1}^{\infty}\phi(t)Q(t)dt=\infty,$$

then every solution $u(x,t)$ of (E), (B1) *is oscillatory in Ω or satisfies* (2)*, where*

$$Q(t)=\frac{e^{R(t)}}{r(t)}\Big\{q_j(t)-K|G(t)|\Big\}.$$

Theorem 7. *Assume that* (H1)–(H5) *hold, and that*

(H8) *there exists a positive constant K such that*

$$q_j(t)\geq K|\tilde{G}(t)|.$$

If for $i = 1, 2$,

$$\int_{T_1}^{\infty} \left(\frac{p_i(t)\phi'(t)^2}{\phi(t)} \right) dt < \infty, \qquad \int_{T_1}^{\infty} \frac{1}{p_i(t)\phi(t)} dt = \infty,$$
$$\int_{T_1}^{\infty} \phi(t)\tilde{Q}(t)dt = \infty,$$

then every solution $u(x,t)$ *of* (E), (B2) *is oscillatory in* Ω *or satisfies* (8), *where*

$$\tilde{Q}(t) = \frac{e^{R(t)}}{r(t)} \Big\{ q_j(t) - K|\tilde{G}(t)| \Big\}.$$

Example 1. We consider the problem

$$\frac{\partial}{\partial t}\left(e^t \frac{\partial}{\partial t}u(x,t)\right) + e^t \frac{\partial}{\partial t}u(x,t) - \left(e^t + e^{\frac{t}{2}}\right)\Delta u(x,t) \tag{26}$$
$$+2e^t u\left(x, t - \frac{\pi}{2}\right) = e^{\frac{t}{2}} \sin x \sin t, \ (x,t) \in (0,\pi) \times (0,\infty),$$
$$u(0,t) = u(\pi,t) = 0, \ t > 0. \tag{27}$$

Here $n = k = m = 1$ $r(t) = e^t$, $p_1(t) = e^{2t}$, $p_2(t) = e^{2t-\pi/2}$, $q_1(x,t) = 2e^t$, $\sigma_1(t) = t - \pi/2$ and $f(x,t) = e^t \sin x \sin t$. It is easily verified that $\Phi(x) = \sin x$ and

$$q_1(t) \equiv 2e^t \geq \frac{\pi}{4}|e^{\frac{t}{2}} \sin t| \equiv |G(t)|.$$

By choosing $\phi(t) = e^{-3t}$, the conditions of Theorem 6 are satisfied. Therefore, we conclude that every solution u of the problem (26), (27) is oscillatory in $(0,\pi) \times (0,\infty)$ or satisfies (2). For example, $u = \sin x \sin t$ is such a solution.

Example 2. Consider the problem

$$\frac{\partial}{\partial t}\left(e^{-t} \frac{\partial}{\partial t}u(x,t)\right) + 2e^{-t} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) \tag{28}$$
$$+e^{\frac{t}{2}} u\left(x, \frac{t}{2}\right) = \left(e^{-t} + 1\right)\cos x, \ (x,t) \in \left(0, \frac{\pi}{2}\right) \times (0,\infty),$$
$$-u_x(0,t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = -e^{-t}, \ t > 0. \tag{29}$$

Here $n = k = m = 1$ $r(t) = e^{-t}$, $p_1(t) = e^t$, $p_2(t) = e^{t/2}$, $q_1(x,t) = 2e^{-t}$, $a(t) = 1$, $\sigma_1(t) = t/2$ and $f(x,t) = (e^{-t} + 1)\cos x$. A simple calculation yields $\tilde{G}(t) = 2/\pi$ and

$$q_1(t) \equiv e^{\frac{t}{2}} \geq \frac{2}{\pi} \equiv |\tilde{G}|.$$

By choosing $\phi(t) = e^{-\frac{3}{2}t}$ we note that the conditions of Theorem 7 holds. Therefore, every solution u of the problem (28), (29) is oscillatory in $(0,\pi) \times (0,\infty)$ or satisfies

(8). For example, $u = e^{-t} \cos x$ is such a solution.

4.2. Interval oscillation results

Combining Theorems 1–2 and 4, we have following theorems.

Theorem 8. *Assume that* (H1)–(H5) *and that* (H7) *hold. If for some* $T \geq 0$ *and for* $i = 1, 2$, *there exist* $H_1, H_2 \in \mathbb{H}$ *and some* $c \in (a, b)$ *such that* $T \leq a < b$, (19) *with* $q(s)$ *replaced by* $Q(s)$, *then every solution* $u(x, t)$ *of* (E), (B1) *is oscillatory in* Ω *or satisfies* (2).

Theorem 9. *Assume that* (H1)–(H5) *and* (H8) *hold. If for some* $T \geq 0$ *and for* $i = 1, 2$, *there exist* $H_1, H_2 \in \mathbb{H}$ *and some* $c \in (a, b)$ *such that* $T \leq a < b$, (19) *with* $q(s)$ *replaced by* $\tilde{Q}(s)$, *then every solution* $u(x, t)$ *of* (E), (B2) *is oscillatory in* Ω *or satisfies* (8).

Combining Theorems 1–2 and 5, we obtain two theorems.

Theorem 10. *Assume that* (H1)–(H5) *and* (H7) *hold. For some functions* $H_1, H_2 \in \mathbb{H}$, *some* $T \geq 0$ *and for* $i = 1, 2$, *if* (22) *and* (23) *with* $q(s)$ *replaced by* $Q(s)$ *hold, then every solution* $u(x, t)$ *of* (E), (B1) *is oscillatory in* Ω *or satisfies* (2).

Theorem 11. *Assume that* (H1)–(H5) *and* (H8) *hold. For some functions* $H_1, H_2 \in \mathbb{H}$, *some* $T \geq 0$ *and for* $i = 1, 2$, *if* (22) *and* (23) *with* $q(s)$ *replaced by* $\tilde{Q}(s)$ *hold, then every solution* $u(x, t)$ *of* (E), (B2) *is oscillatory in* Ω *or satisfies* (8).

Remark. Our results in this paper hold without the hypotheses (H5) and (H6), if condition $\sigma_j(t) = t$ satisfied.

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