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Multiperiod Mean Absolute Deviation Uncertain Portfolio Selection

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ABSTRACT
Multiperiod portfolio selection problem attracts more and more attentions because it is in accordance with the practical investment decision-making problem. However, the existing literature on this field is almost undertaken by regarding security returns as random variables in the framework of probability theory. Different from these works, we assume that security returns are uncertain variables which may be given by the experts, and take absolute deviation as a risk measure in the framework of uncertainty theory. In this paper, a new multiperiod mean absolute deviation uncertain portfolio selection models is presented by taking transaction costs, borrowing constraints and threshold constraints into account, which an optimal investment policy can be generated to help investors not only achieve an optimal return, but also have a good risk control. Threshold constraints limit the amount of capital to be invested in each stock and prevent very small investments in any stock. Based on uncertain theories, the model is converted to a dynamic optimization problem. Because of the transaction costs, the model is a dynamic optimization problem with path dependence. To solve the new model in general cases, the forward dynamic programming method is presented. In addition, a numerical example is also presented to illustrate the modeling idea and the effectiveness of the designed algorithm.

Keywords: Uncertain Variable, Multiperiod Uncertain Portfolio Selection, Uncertain Measure, Mean Absolute Deviation, The Forward Dynamic Programming Method

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1. INTRODUCTION

Portfolio optimization problem concerns with an individual who is trying to allocate one's capital to a selected number of securities in order to achieve the investment goal. In a seminal paper, Markowitz (1952) presented the idea of an optimal portfolio selection by taking into account the trade-off between the portfolio expected return and its risk which is measured by the variance of the portfolio. Since then, variance has been widely used as a risk measure, and a large number of models have been investigated. Konno and Yamazaki (1991) provided a linear model for portfolio optimization in which the absolute deviation was used to measure the risk of the portfolio. In particular, when the returns of securities are multivariate-normally distributed, the model is equivalent to Markowitz's mean-variance model. Based on absolute deviation, numerous models were developed. Ie., Simaan (1997) provided a thorough comparison of the mean variance model and the mean absolute deviation model; Speranza (1993) used the semi-absolute deviation to measure the risk and formulated a portfolio selection model. However, the previous papers mainly consider single period portfolio selection problem. In fact, typically portfolio strategies are periodically rebalanced in a planning horizon, since the investor will adjust his/her portfolio to purse the better strategy from time to time. Therefore, multiperiod portfolio selection models are in accordance with the practical situation.

Up to now, multiperiod portfolio selection problem attracted more and more attentions both in practice and in theory. The first formulation of the multiperiod portfolio selection problem has already been given in the book of Markowitz (1959) followed by the papers of Mossin (1968), Samuelson (1969) and Merton and Samuelson (1974). Although it is heavily discussed in recent literature (see e.g., Li and Ng, 2000; Zhu *et al*., 2004; Güpınar and Rustem, 2007; Çelikyurt and Özekici, 2007; Calafiore, 2008; Yan *et al*., 2009, 2012; Yu *et al*. 2010, 2012; Wu and Li, 2012; Li and Li, 2012; Zhang *et al*., 2012, 2014; Liu *et al*., 2012, 2013; Zhang and Zhang, 2014; Bodnar *et al*., 2015), to the best of our knowledge, a closed-form solution is not available in the general case up to now. Closed-form solutions are presented only under the assumption of independence, ie. Li and Ng (2000) used dynamic programming approach to deal with the multiperiod mean variance portfolio selection problem by using the idea of embedding the problem in a tractable auxiliary problem. Then, they obtained breakthrough result, that is, the optimal mean-variance port-folio policy and the efficient frontier; Zhu *et al*. (2004) incorporated a control of the probability of bankruptcy in the generalized mean variance formulation for multiperiod portfolio optimization; Yu *et al*. (2010, 2012) discussed a dynamic portfolio optimization problem with risk control for the absolute deviation model; Wu and Li (2012) investigate a non-self-financing portfolio optimization problem under the framework of multiperiod mean-variance with Markov regime switching and a stochastic cash flow; Li and Li (2012) represented a multiperiod portfolio optimization problem for asset-liability management of an investor who intends to control the probability of bankruptcy before reaching the end of an investment horizon. For more general model, the solution is frequently determined by a numerical procedure ie. van Binsbergen and Brandt (2007) compared the numerical performance of value function iterations with portfolio weight iterations in the context of the simulation-based dynamic programming approach; Mansini *et al*. (2007) presented multiperiod mean CVaR portfolio selection model; Güpınar and Rustem (2007) extend the multiperiod mean-variance optimization framework to worst-case design with multiple rival return and risk scenarios; Yan *et al*. (2009, 2012) proposed a hybrid genetic algorithm with particle swarm optimizer to solve a class of multiperiod semi-variance portfolio selection with a four-factor futures price model and a multiperiod semi-variance portfolio selection; Zhang *et al*. (2012, 2014), and Liu *et al*. (2012, 2013) respectively proposed genetic algorithm, hybrid intelligent algorithm and differential evolution algorithm to solve several kinds of multiperiod fuzzy portfolio selection models; Zhang and Zhang (2014) proposed the discrete approximate iteration method to solve the multiperiod fuzzy portfolio selection model with cardinality constraints; Köksalan and Şakar (2014) consider expected return, conditional value at risk, and liquidity criteria in a multiperiod portfolio optimization setting modeled by stochastic programming.

In real life, we frequently do not have enough data to estimate the probability distribution of security returns, which implies that random portfolio selection models are difficult to be employed. In this situation, a better way is to estimate security returns by experienced experts such as fund managers, which implies that security returns are fuzzy variables. Several researchers (Wang and Zhu, 2002; Terol *et al*., 2006; Fang *et al*., 2006; Vercher *et al*., 2007;, Zhang *et al*., 2007, 2009; Huang, 2008; Li *et al*., 2010; Liu and Liu, 2002; Huang, 2008; Li *et al*., 2010; Zhang and Liu, 2014) have utilized fuzzy set theory to investigate portfolio selection problem by regarding security returns as fuzzy variables instead of random variables. Different from random variables and fuzzy variables, Liu (2007) proposed the concept of uncertain variable and established uncertainty theory to study the behavior of uncertain phenomena. As an application, Qin *et al*. (2009) introduced the singleperiod mean-variance model for portfolio selection under uncertain environment. Similarly, Li and Qin (2014) proposed a mean- semi absolute deviation model for uncertain portfolio selection.

The contribution of this work is as follows. We originally represent uncertain absolute deviation to measure portfolio risk, and propose a new multiperiod mean absolute deviation uncertain portfolio selection model with borrowing constraints, transaction costs and threshold constraints. We design a novel forward dynamic programming method for solution. Finally, we give an example to illustrate the idea of the model and demonstrate the effectiveness of the designed algorithm.

This paper is organized as follows. In Section 2, several concepts, properties of uncertain measure, the definitions of the uncertain mean and the uncertain absolute deviation are introduced, respectively. In Section 3, the borrowing constraints, transaction costs and threshold constraints are formulated into the multiperiod portfolio, and a new multiperiod uncertain portfolio selection model is proposed. The forward dynamic programming method is proposed to solve it in Section 4. In Section 5, a numerical example is also presented to illustrate the modeling idea and the effectiveness of the designed algorithm. Finally, some conclusions are given in Section 6.

2. PRELIMINARIES

Let Γ be a nonempty set, and let A be a σ-algebra over Γ. Each element of A is called an event. A set function is called an uncertain measure (Liu, 2007) if and only if it satisfies

Axiom 1. (Normality) $\mathrm{M}\{\Gamma\} = 1$;

Axiom 2. (Monotonicity) $\mathrm{M}\{A\} \le \mathrm{M}\{B\}$ whenever $A \subseteq B$;

Axiom 3. (Self-duality) $\mathrm{M}\{A\}+\mathrm{M}\{A^c\} = 1$ for any event A;

Axiom 4. (Subadditivity) $M(\bigcup_i A_i) \le \sum_{i=1}^{\infty} M(A_i)$ for any countable sequence of events $\{A_i\}$.

Definition 1. (Liu, 2007) Let Γ be a nonempty set, and let A be a σ-algebra over it. If M is an uncertain measure, then the triplet (Γ, A, M) is called an uncertainty space.

Definition 2. (Liu, 2007) Uncertain variable ξ is defined as a measurable function from an uncertainty space (Γ, A, M) to the set of real numbers $\Re$. That is, for any Borel set B, we have

$$\{\gamma \in \Gamma, \xi(\gamma) \in B\} \in A \tag{1}$$

Definition 3. (Liu, 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}dx - \int_{-\infty}^{0} M\{\xi \le x\}dx \tag{2}$$

provided that at least one of the two integrals is finite.

Based on **Definition 3**, Liu (2009) deduced the following two theorems.

Theorem 1. (Liu (2009)) Let ξ be an uncertain variable with finite expected value. Then, for any real numbers a and b , it holds that

$$E[a\,\xi + b] = a\,E[\xi] + b \tag{3}$$

Theorem 2. (Linearity of Expected Value Operator, Liu (2009) Let ξ and η be independent uncertain variables with finite expected values. Then, for any real numbers a and b, it holds that

$$E(a\xi + b\eta) = aE(\xi) + bE(\eta) \tag{4}$$

Definition 4. (Liu, 2007) An uncertain variable ξ can be characterized by an uncertainty distribution which is a function $\Phi : \Re \rightarrow [0, 1]$ is defined as

$$\Phi(t) = M\{\xi \le t\} \tag{5}$$

Definition 5. Let ξ be an uncertain variable with finite expected value e. Then the absolute deviation of ξ is defined by

$$AD(\xi) = E[|\xi - e|] \tag{6}$$

If ξ is an uncertain variable with expected value e, then its absolute deviation is used to measure the spread of its distribution about e.

Theorem 3. Let ξ be an uncertain variable with finite expected value e. Then its uncertain absolute deviation is defined as

$$AD(\xi) = \int_e^{+\infty} (1 - \Phi(r))dr + \int_{-\infty}^{e} \Phi(r)dr \tag{7}$$

Proof. From the **Definition 5** and **Definition 3,** it follows that

$$\begin{aligned}
AD(\xi) &= E[|\xi - \mathrm{e}|] \\
&= \int_0^{+\infty} M\{|\xi - \mathrm{e}| \ge x\}dx - \int_{-\infty}^{0} M\{|\xi - \mathrm{e}| \le x\}dx \\
&= \int_0^{+\infty} M\{|\xi - \mathrm{e}| \ge x\}dx \\
&= \int_0^{+\infty} M\{\xi - \mathrm{e} \ge x\}dx + \int_0^{+\infty} M\{\xi - \mathrm{e} \le -x\}dx \\
&= \int_e^{+\infty} M\{\xi \ge r\}dr + \int_{-\infty}^{e} M\{\xi \le r\}dr \\
&= \int_e^{+\infty} (1 - M\{\xi \le r\})dr + \int_{-\infty}^{e} M\{\xi \le r\}dr \\
&= \int_e^{+\infty} (1 - \Phi(r))dr + \int_{-\infty}^{e} \Phi(r)dr
\end{aligned}$$

Thus, the proof of the theorem is ended. □

Theorem 4. Let ξ be an uncertain variable with finite expected value e. Then for any nonnegative real numbers λ, it holds

$$AD(\lambda\xi) = \lambda AD(\xi) \tag{8}$$

Proof. From the **Definition 5**, it follows that

$$AD(\lambda\xi) = E[|\lambda\xi - \lambda e|] = \lambda E[|\xi - e|] = \lambda AD(\xi)$$

Thus, the proof of the theorem is ended. □

Theorem 5. Let ξ be an uncertain variable with finite expected value e. Then for any nonnegative real numbers λ and for any real numbers η, it holds

$$AD(\lambda\xi + \eta) = \lambda AD(\xi) \tag{9}$$

Proof. From the **Definition 5,** it follows that

$$\begin{aligned}
AD(\lambda\xi + \eta) &= E[|\lambda\xi + \eta - (\lambda e + \eta)|] \\
&= E[\lambda|\xi - e|] = \lambda E[|\xi - e|] \\
&= \lambda AD(\xi)
\end{aligned}$$

Thus, the proof of the theorem is ended.□

If $r = (a, b, c)$ be a triangle uncertain variable, then uncertainty distribution Φ(r) can be described as:

$$\Phi(r) = \begin{cases} 0, & \text{if } r \le a, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b, \\ \dfrac{r+c-2b}{2(c-b)}, & \text{if } b \le r \le c, \\ 1, & \text{if } r \ge c. \end{cases} \tag{10}$$

The triangle uncertain variable is denoted by $r(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

Theorem 6. If ξ (a, b, c) be a triangle uncertain variable, the expected value of ξ can be given by:

$$E(\xi)=\frac{a+2b+c}{4} \tag{11}$$

Proof. From the **Definition 3** and **Theorem 6**, it follows that

$$\begin{aligned}E(\xi)&=\int_0^{+\infty}M\{\xi\geq r\}dr-\int_{-\infty}^0 M\{\xi\leq r\}dr\\&=\int_0^{+\infty}(1-M\{\xi\leq r\})dr-\int_{-\infty}^0 M\{\xi\leq r\}dr\\&=\int_0^{+\infty}(1-\Phi(r))dr-\int_{-\infty}^0\Phi(r)dr\end{aligned} \tag{12}$$

According to Eq. (10), the right-hand side of Eq. (12) is

$$\begin{aligned}\int_0^{+\infty}(1-\Phi(r))dr-\int_{-\infty}^0\Phi(r)dr&=\int_0^a(1-0)dr+\int_a^b(1-\frac{r-a}{2(b-a)})dr\\&+\int_b^c(1-\frac{r+c-2b}{2(c-b)})dr+\int_c^{+\infty}(1-1)dr\\&=a+\frac{3b-3a}{4}+\frac{c-b}{4}=\frac{a+2b+c}{4}\end{aligned} \tag{13}$$

According to Eq. (12) and Eq. (13), we can get

$$\begin{aligned}&\int_0^{+\infty}(1-\Phi(r))dr-\int_{-\infty}^0\Phi(r)dr=\int_0^a(1-0)dr\\&+\int_a^b(1-\frac{r-a}{2(b-a)})dr+\int_b^c(1-\frac{r+c-2b}{2(c-b)})dr\\&+\int_c^{+\infty}(1-1)dr=a+\frac{3b-3a}{4}+\frac{c-b}{4}=\frac{a+2b+c}{4}\end{aligned}$$

Thus, the proof of the theorem is ended. □

Theorem 7. Let ξ (a, b, c) be a triangle uncertain variable, which $E(\xi)=\frac{a+2b+c}{4}$. Then, the uncertain absolute deviation of ξ can be given by:

$$AD(\xi)=\begin{cases}\dfrac{4b^2-12ab+4bc-6ac+9a^2+c^2}{32(b-a)}, & \text{if } c-b\leq b-a\\[2ex]\dfrac{(3c-a-2b)^2}{32(c-b)}, & \text{if } c-b\geq b-a\end{cases} \tag{14}$$

Proof. From the **Theorem 3**, it follows that

$$\begin{aligned}AD(\xi)&=\int_e^{+\infty}(1-\Phi(r))dr+\int_{-\infty}^e\Phi(r)dr\\&=\int_{\frac{a+2b+c}{4}}^{+\infty}(1-\Phi(r))dr+\int_{-\infty}^{\frac{a+2b+c}{4}}\Phi(r)dr\end{aligned} \tag{15}$$

If $c-b\leq b-a$, the right-hand side of Eq. (15) is

$$\begin{aligned}&\int_{\frac{a+2b+c}{4}}^{+\infty}(1-\Phi(r))dr+\int_{-\infty}^{\frac{a+2b+c}{4}}\Phi(r)dr=\int_{\frac{a+2b+c}{4}}^{b}(1-\frac{r-a}{2(b-a)})dr\\&+\int_b^c(1-\frac{r+c-2b}{2(c-b)})dr+\int_c^{+\infty}(1-1)dr+\int_0^a 0dr+\int_a^{\frac{a+2b+c}{4}}\frac{r-a}{2(b-a)}dr\\&=\frac{20b^2-28ab-12bc+10ac+9a^2+c^2}{64(b-a)}+\frac{c-b}{4}+\frac{(2b-3a+c)^2}{64(b-a)}\\&=\frac{24b^2-40ab-8bc+4ac+18a^2+2c^2}{64(b-a)}+\frac{16(c-b)(b-a)}{64}\\&=\frac{8b^2-24ab+8bc-12ac+18a^2+2c^2}{64(b-a)}\\&=\frac{4b^2-12ab+4bc-6ac+9a^2+c^2}{32(b-a)}\end{aligned} \tag{16}$$

If $c-b\geq b-a$, the right-hand side of Eq. (15) is

$$\begin{aligned}&\int_{\frac{a+2b+c}{4}}^{+\infty}(1-\Phi(r))dr+\int_{-\infty}^{\frac{a+2b+c}{4}}\Phi(r)dr=\int_{\frac{a+2b+c}{4}}^{c}(1-\frac{r+c-2b}{2(c-b)})dr\\&+\int_c^{+\infty}(1-1)dr+\int_0^a 0dr+\int_a^b\frac{r-a}{2(b-a)}dr+\int_b^{\frac{a+2b+c}{4}}\frac{r+c-2b}{2(c-b)}dr\\&=\frac{(3c-a-2b)^2}{64(c-b)}+\frac{(3c-a-2b)^2}{64(c-b)}\\&=\frac{(3c-a-2b)^2}{32(c-b)}\end{aligned} \tag{17}$$

According to Eq. (16) and Eq. (17), we can get

$$AD(\xi)=\begin{cases}\dfrac{4b^2-12ab+4bc-6ac+9a^2+c^2}{32(b-a)}, & \text{if } c-b\leq b-a\\[2ex]\dfrac{(3c-a-2b)^2}{32(c-b)}, & \text{if } c-b\geq b-a\end{cases}$$

Thus, the proof of the theorem is ended. □

3. THE MULTIPERIOD PORTFOLIO SELECTION MODEL

Assume that there are n risky assets and one risk-free asset in financial market for trading. An investor wants to allocate his/her initial wealth W_1 among n+1 assets at the beginning of period 1, and obtains the final wealth at the end of period T. He/She can reallocate his/her wealth among the n risky assets at the beginning of each of the following T consecutive investment periods. Suppose that the return rates of the n risky assets at each period are denoted as triangular uncertain variables, and the returns of portfolios among different periods are independent of each other. For the sake of description, let us first introduce the following notations:

- x_{i0} the initial investment proportion of risky asset i at period 0;
- x_t the portfolio at period t, where $x_t=(x_{1t}, x_{2t}, \cdots, x_{nt})$;
- x_{ft} the investment proportion of risk-free asset at period t, where $x_{ft}=1-\sum_{i=1}^{n}x_{it}$;

x^b_{ft} the lower bound of the investment proportion of risk-free asset at period t, where $x_{ft} \geq x^b_{ft}$;
R_{it} the return of risky asset i at period t;
r_{pt} the return rate of the portfolio x_t at period t;
r_{bt} the borrowing rate of the risk-free asset at period t;
r_{lt} the lending rate of the risk-free asset at period t;
l_{it} the lower bound constraints of x_{it};
u_{it} the upper bound constraints of x_{it};
r_{Nt} the net return rate of the portfolio x_t at period t;
W_t the crisp form of the holding wealth at the beginning of period t;
c_{it} the unit transaction cost of risky asset i at period t.

3.1 Return, Risk and Transaction Costs

In this section, we employ the uncertain mean value of the net return on the portfolio at each period to measure the return of portfolio. The risk on the return rate of portfolio at each period is quantified by the uncertain absolute deviation. The return rate of security i at period t, $R_{it} = (a_{it}, b_{it}, c_{it})$, is triangular uncertain variable for all $i = 1, \cdots, n$ and $t = 1, \cdots, T$.

The uncertain mean value of the portfolio $x_t = (x_{1t}, x_{2t}, \cdots, x_{nt})'$ at period t can be expressed as

$$r_{pt} = \sum_{i=1}^{n} E(R_{it})x_{it} = \sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right), \quad t=1,\cdots,T \tag{18}$$

where $r_{ft} = \begin{cases} r_{lt}, & 1-\sum_{i=1}^{n}x_{it} \geq 0 \\ r_{bt}, & 1-\sum_{i=1}^{n}x_{it} \leq 0 \end{cases}$, $r_{bt} \geq r_{lt}$. When $1-\sum_{i=1}^{n}x_{it} \geq 0$, it denotes that lending is allowed on the risk-free asset; When $1-\sum_{i=1}^{n}x_{it} \leq 0$, it represents that borrowing is allowed on the risk-free asset.

We assume in the sequel that the transaction costs at period t is a V shape function of difference between the tth period portfolio $x_t = (x_{1t}, x_{2t}, \cdots, x_{nt})$ and the $t-1$th period portfolio $x_{(t-1)} = (x_{1(t-1)}, x_{2(t-1)}, \cdots, x_{n(t-1)})$. That is to say, the transaction cost for asset i at period t can be expressed by

$$C_{it} = c_{it}\left|x_{it} - x_{i(t-1)}\right| \tag{19}$$

Hence, the total transaction costs of the portfolio $x_t = (x_{1t}, x_{2t}, \cdots, x_{nt})$ at period t can be represented as

$$C_t = \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|, \; t=1,\cdots,T \tag{20}$$

Thus, the net return rate of the portfolio x_t at period t can be denoted as

$$r_{Nt} = \sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right) - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|. \; t=1,\cdots,T \tag{21}$$

Then, the crisp form of the holding wealth at the beginning of the period t can be written as

$$\begin{aligned} W_{t+1} &= W_t(1+r_{Nt}) \\ &= W_t\left(1+\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it} + r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right) - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|\right), \; t=1,\cdots,T \end{aligned} \tag{22}$$

The absolute deviation of the portfolio x_t can be expressed as

$$AD_t(x_t) = AD_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \cdots + r_{nt}x_{nt}) \tag{23}$$

The main characteristic of this model is that the risk of a portfolio is measured by the absolute deviation of the return rate of assets instead of the variance.

Threshold constraints limit the amount of capital to be invested in each stock and prevent very small investments in any stock. The threshold constraints of multiperiod portfolio selection can be expressed as

$$0 \leq x_{it} \leq u_{it} \tag{24}$$

where u_{it} are respectively the upper bounds constraints of x_{it}.

For a rational investor, he/she wishes not only to maximize expected return but also to minimize the risk which is measured by the variance of the rate of return on a portfolio. So he/she must make a tradeoff between the two objectives. Let $(1-\theta)$ and θ be the weights associated with criteria r_{pt} and $AD_t(x_t)$ respectively. Then the investor attempts to maximize

$$\begin{aligned} F_t(r_{Nt}, AD_t(x_t)) = &\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it} \\ &+ r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right) - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right| \\ &- \theta AD_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \cdots + r_{nt}x_{nt}) \end{aligned} \tag{25}$$

Here the parameter θ can be interpreted as the risk aversion factor of the investor. The greater the factor θ is, the more risk aversion the investor has. In this paper, we assume that the investor is of risk aversion, i.e., $\theta \geq 0$.

3.2 The Basic Multiperiod Portfolio Optimization Models

When the investors can give a tolerable level of risk at period t, and want to maximize the terminal wealth at the given level of risk, we have the multiperiod uncertain mean absolute deviation model as follows:

$$\max \sum_{t=1}^{T}\left[\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)\right.$$
$$\left.-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right| -\theta AD_t(r_{1t}x_{1t}+r_{2t}x_{2t}+\cdots+r_{nt}x_{nt})\right]$$

$$s.t\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)\right.\right. & (a)\\ \left.\left.-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right)\right)W_t & \\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b} & (b)\\ 0\leq x_{it}\leq u_{it},\ i=1,\cdots,n,\ t=1,\cdots,T & (c)\end{cases} \quad (26)$$

where v_t denotes the maximum risk level the investors can tolerate. Constraint (a) denotes the wealth accumulation constraint; constraint (b) indicates the investment proportion of risk-free asset at period t must exceed the given lower bound x^b_{ft}; constraint (c) represents threshold constraints of x_{it}.

According to Qin *et al.* (2011), if $r_{1t}, r_{2t}, \cdots, r_{nt}$ are independent triangular uncertain variables, and $x_{it} \geq 0$, $i = 1, \cdots, n$,

$$AD_t(r_{1t}x_{1t}+r_{2t}x_{2t}+\cdots+r_{nt}x_{nt})=\sum_{i=1}^{n}x_{it}AD_t(r_{it}). \quad (27)$$

where

$$AD(r_{it})=\begin{cases}\dfrac{4b_{it}^2-12a_{it}b_{it}+4b_{it}c_{it}-6a_{it}c_{it}+9a_{it}^2+c_{it}^2}{32(b-a)}, & \text{if } c_{it}-b_{it}\leq b_{it}-a_{it}\\ \dfrac{(3c_{it}-a_{it}-2b_{it})^2}{32(c_{it}-b_{it})}, & \text{if } c_{it}-b_{it}\geq b_{it}-a_{it}\end{cases} \quad (28)$$

According to Eq. (27), the Model (26) can be turned into as follows:

$$\max \sum_{t=1}^{T}\left[\left(\sum_{i=1}^{n}\left[\frac{a_{it}+2b_{it}+c_{it}}{4}\right]x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})\right.\right.$$
$$\left.\left.-\sum_{i=1}^{n}c_i\left|x_{it}-x_{i(t-1)}\right|\right) -\theta\left(\sum_{i=1}^{n}AD_t(r_{it})x_{it}\right)\right]$$

$$s.t\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)\right.\right.\\ \left.\left.-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right)\right)W_t\\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ 0\leq x_{it}\leq u_{it},\ i=1,\cdots,n,\ t=1,\cdots,T\end{cases} \quad (29)$$

Let $y_{it}=\left|x_{it}-x_{i(t-1)}\right|$. Then the Model (29) can be turned into as follows.

$$\max \sum_{t=1}^{T}\left[\left(\sum_{i=1}^{n}\left[\frac{a_{it}+2b_{it}+c_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)-\sum_{i=1}^{n}c_{it}y_{it}\right)-\theta\left(\sum_{i=1}^{n}AD_t(r_{it})x_{it}\right)\right]$$

$$s.t\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{n}\left(\frac{a_{it}+2b_{it}+c_{it}}{4}\right)x_{it}+r_{ft}\left(1-\sum_{i=1}^{n}x_{it}\right)\right.\right.\\ \left.\left.-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{i(t-1)}\right|\right)\right)W_t\\ y_{it}\geq x_{it}-x_{i(t-1)}\\ y_{it}\geq -(x_{it}-x_{i(t-1)})\\ 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ 0\leq x_{it}\leq u_{it},\ i=1,\cdots,n,\ t=1,\cdots,T\end{cases} \quad (30)$$

4. SOLUTION ALGORITHM

In this section, the forward dynamic programming method is proposed to solve the Model (30).

4.1 The Forward Dynamic Programming Method

The sub-problem of period t of the Model (30) can be transformed into

$$\max\left(\sum_{i=1}^{n}[\frac{a_{it}+2b_{it}+c_{it}}{4}]x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}y_{it}\right)$$
$$-\theta\left(\sum_{i=1}^{n}\left[AD_t(r_{it})x_{it}\right]\right)$$

$$s.t\begin{cases} 1-\sum_{i=1}^{n}x_{it}\geq x_{ft}^{b}\\ y_{it}\geq x_{it}-x_{i(t-1)}\\ y_{it}\geq -(x_{it}-x_{i(t-1)})\\ 0\leq x_{it}\leq u_{it},\ i=1,\cdots,n\end{cases} \quad (31)$$

In the following section, we provide the detailed procedure of the forward dynamic programming method for finding optimal solutions to the Model (30). The procedure of the algorithm can be showed as follows:

Algorithm The forward dynamic programming method:

Step1. When $t = 1$, W_1 and $x_0 = (x_{10}, \cdots, x_{n0})'$ have been given, the sub-problem of period 1 of the Model (30) can be transformed into

$$\max\left(\sum_{i=1}^{n}[\frac{a_{i1}+2b_{i1}+c_{i1}}{4}]x_{i1}+r_{f1}(1-\sum_{i=1}^{n}x_{i1})-\sum_{i=1}^{n}c_{i1}y_{i1}\right)$$
$$-\theta\left(\sum_{i=1}^{n}[AD_1(r_{i1})x_{i1}]\right)$$

$$s.t\begin{cases} W_2=\left(1+\sum_{i=1}^{n}(\frac{a_{i1}+2b_{i1}+c_{i1}}{4})x_{i1}\right. \\ \left.\quad +r_{f1}(1-\sum_{i=1}^{n}x_{i1})-\sum_{i=1}^{n}c_{i1}y_{i1}\right)W_1 \\ 1-\sum_{i=1}^{n}x_{i1}\geq x_{f1}^{b} \\ \mathrm{y}_{i1}\geq x_{i1}-x_{i0},\mathrm{y}_{i1}\geq-(x_{i1}-x_{i0}) \\ 0\leq x_{i1}\leq u_{i1},\ i=1,\cdots,n \end{cases} \quad (32)$$

The optimal solution of period t = 1, $x_1^*=(x_{11}^*,\cdots,x_{n1}^*)'$ can be obtained solving the Model (32) by the interior-point algorithms (Fang and Puthenpura, 1993). At the same time,

$$\max\left(\sum_{i=1}^{n}\left[\frac{a_{i1}+2b_{i1}+c_{i1}}{4}\right]x_{i1}+r_{f1}\left(1-\sum_{i=1}^{n}x_{i1}\right)-\sum_{i=1}^{n}c_{i1}y_{i1}\right)$$
$$-\theta\left(\sum_{i=1}^{n}\left[AD_1(r_{i1})x_{i1}\right]\right)$$

and W_2^* can be obtained, respectively.

Step 2. When $t = m$ ($m \geq 1$ and $m \in Z^+$), W_{m+1}^* and $x_m^* = (x_{1m}^*,\cdots,x_{nm}^*)'$ have been obtained, the sub-problem of period m of the Model (30) can be transformed into

$$\max\left(1+\sum_{i=1}^{n}(\frac{a_{i(m+1)}+2b_{i(m+1)}+c_{i(m+1)}}{4})x_{i(m+1)}\right.$$
$$\left.+r_{f1}(1-\sum_{i=1}^{n}x_{i(m+1)})-\sum_{i=1}^{n}c_{i(m+1)}y_{i(m+1)}\right)$$
$$-\theta\left(\sum_{i=1}^{n}AD_{(m+1)}(r_{i(m+1)})x_{i(m+1)}\right)$$

$$s.t\begin{cases} W_{(m+2)}=\left(1+\sum_{i=1}^{n}\left(\frac{a_{i(m+1)}+2b_{i(m+1)}+c_{i(m+1)}}{4}\right)x_{i(m+1)}\right. \\ \quad +r_{f(m+1)}(1-\sum_{i=1}^{n}x_{i(m+1)}) \\ -\sum_{i=1}^{n}c_{i(m+1)}y_{i(m+1)})W_{(m+1)}^{*} \\ 1-\sum_{i=1}^{n}x_{i(m+1)}\geq x_{f(m+1)}^{b} \\ \mathrm{y}_{i(m+1)}\geq x_{i(m+1)}-x_{im}^{*} \\ \mathrm{y}_{i(m+1)}\geq-(x_{i(m+1)}-x_{im}^{*}) \\ 0\leq x_{i(m+1)}\leq u_{i(m+1)},\ i=1,\cdots,n \end{cases} \quad (33)$$

The optimal solution of period t = 1, $x_{(m+1)}^*=(x_{1(m+1)}^*,\cdots,x_{n(m+1)}^*)'$ can be obtained solving the Model (33) by the interior-point algorithms (Fang and Puthenpura, 1993). At the same time,

$$\left(1+\sum_{i=1}^{n}\left(\frac{a_{i(m+1)}+2b_{i(m+1)}+c_{i(m+1)}}{4}\right)x_{i(m+1)}^{*}\right.$$
$$\left.+r_{f1}\left(1-\sum_{i=1}^{n}x_{i(m+1)}^{*}\right)-\sum_{i=1}^{n}c_{i(m+1)}y_{i(m+1)}^{*}\right)$$
$$-\theta\left(\sum_{i=1}^{n}AD_{(m+1)}(r_{i(m+1)})x_{i(m+1)}^{*}\right)$$

and W_{m+2}^* can be obtained, respectively.

Step 3. If $t = T$, then the maximization of the terminal utility

$$\left(1+\sum_{i=1}^{n}(\frac{a_{iT}+2b_{iT}+c_{iT}}{4})x_{iT}^{*}+r_{fT}(1-\sum_{i=1}^{n}x_{iT}^{*})-\sum_{i=1}^{n}c_{iT}y_{iT}^{*}\right)$$
$$-\theta\left(\sum_{i=1}^{n}AD_T(r_{iT})x_{iT}^{*}\right)$$

and W_{T+1}^* can be obtained, respectively. Otherwise $t = m+1$, then turn Step 2.

The global optimal solutions of the sub-problem of period t of the Model (30), ie. the Model (32) and Model (33) can be obtained by the interior-point algorithms (Fang and Puthenpura, 1993). So, the global optimal solution of the Model (30) can also be obtained by the forward dynamic programming method, ie., the global optimal solution of Model (26) can also be obtained.

5. NUMERICAL EXAMPLE

In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from Shanghai Stock Exchange for his investment. The stocks codes are respectively S_1, ⋯, S_{30}. He/She intends to make five periods of investment with initial wealth W_1 = 1 and his wealth can be adjusted at the beginning of each period. He/she assumes that the returns, risk and turnover rates of the thirty stocks at each period are represented as trapezoidal fuzzy numbers. We collect historical data of them from April 2006 to March 2015 and set every three months as a period to handle the historical data. By using the simple estimation method in Vercher *et al.* (2007) to handle their historical data, the triangular possibility distributions of the return rates of assets at each period can be obtained as shown in Appendix A. According to Eq. (14) and Appendix A, $AD_t(r_{it})$ (i = 1, ⋯, 30; t = 1, ⋯, 5) can be obtained as shown in Appendix B.

Suppose that the transaction costs of assets of the two periods investment take the same value c_{it} = 0.003 (i = 1, ⋯, 30; t = 1, ⋯, 5), the lower bound of the investment proportion of risk-free asset x^b_{ft} = -0.5, the borrowing rate of the risk-free asset r_{bt} = 0.017, the lending rate of the risk-free asset r_{lt} = 0.009, t = 1, ⋯, 5 ,the lower l_{it} = 0 and upper bound constraints u_{it} = 0.2 (i = 1, ⋯, 30; t = 1, ⋯, 5).

In case when the preference coefficients θ = 0, 0.25, 0.5, 1, ⋯, 5.75, the multiperiod uncertain mean absolute deviation portfolio selection model maximizing the terminal utility is set as follows:

$$\max\sum_{t=1}^{5}\left[\left(\sum_{i=1}^{30}\left[\frac{a_{it}+2b_{it}+c_{it}}{4}\right]x_{it}+r_{ft}\left(1-\sum_{i=1}^{30}x_{it}\right)-\sum_{i=1}^{30}c_{it}y_{it}\right)-\theta\left(\sum_{i=1}^{30}[AD_t(r_{it})x_{it}]\right)\right]$$

$$s.t\begin{cases} W_{t+1}=\left(1+\left(\sum_{i=1}^{30}\left(a_{it}+\frac{\beta_{it}-\alpha_{it}}{4}\right)x_{it}-\sum_{i=1}^{30}c_{it}y_{it}\right)\right)W_t \\ y_{it}\ge x_{it}-x_{i(t-1)} \\ y_{it}\ge -(x_{it}-x_{i(t-1)}) \\ 1-\sum_{i=1}^{30}x_{it}\ge x_{ft}^{b} \\ l_{it}\le x_{it}\le u_{it},\ i=1,\cdots,30,\ t=1,\cdots,5 \end{cases} \tag{34}$$

If $\theta = 1$, the optimal solution of Model (34) will be obtained as the Table 1 using the forward dynamic programming method.

When $\theta = 1$, the optimal investment strategy at period 1 is $x_{11} = 0.2$, $x_{31} = 0.1$, $x_{41} = 0.2$, $x_{81} = 0.2$, $x_{131} = 0.2$, $x_{171} = 0.2$, $x_{261} = 0.2$, $x_{281} = 0.2$, $x_{f1} = -0.5$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 1, asset 3, asset 4, asset 8, asset 13, asset 17, asset 26, asset 28, risk-free asset and otherwise asset by the proportions of 20%,10%, 20%, 20%, 20%, 20%, 20% , 20%, -50% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 1, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 2.1327.

If $\theta = 3.5$, the optimal solution of Model (34) will be obtained as the Table 2 using the forward dynamic programming method.

When $\theta = 3.5$, the available terminal wealth is 1.6163.

To display the influence of θ on the optimal solution of multiperiod, its value is set as 1 and 3.5, respectively, and the Model (34) for portfolio decision-making will be used afterwards. After using the forward dynamic programming method, the corresponding optimal investment strategies can be obtained as shown in Table 1 and Table 2. From Table 1 and Table 2, it can be seen that some of risk assets of the optimal solutions of $\theta = 6$ and $\theta = 7$ are same. There are two assets in period 1, i.e. asset 3, asset 17. There are two assets in period 2, i.e. asset 15, asset 17. There are one asset in period 3, i.e. asset 15. There are four assets in period 5, i.e. asset 13, asset 15, asset 17, asset 20.

When $\theta = 0, 0.25, 0.5, \cdots, 5.75$, of Model (34) will be obtained as the Table 3 using the forward dynamic

Table 1. The optimal solution when $\theta = 1$

Asset i / t	The optimal investment proportions								
1	Asset 1 0.2	Asset 3 0.1	Asset 4 0.2	Asset 8 0.2	Asset 13 0.2	Asset 17 0.2	Asset 26 0.2	Asset 28 0.2	x_{f1} –0.5
2	Asset 1 0.2	Asset 8 0.2	Asset13 0.2	Asset 15 0.2	Asset 17 0.2	Asset 20 0.2	Asset 22 0.1	Asset 28 0.2	x_{f2} –0.5
3	Asset 1 0.2	Asset 4 0.2	Asset 8 0.1	Asset 12 0.2	Asset 13 0.2	Asset 15 0.2	Asset 17 0.2	Asset 28 0.2	x_{f3} –0.5
4	Asset 1 0.2	Asset 8 0.2	Asset12 0.2	Asset 13 0.2	Asset 15 0.2	Asset 17 0.2	Asset 26 0.1	Asset 28 0.2	x_{f4} –0.5
5	Asset 1 0.2	Asset 8 0.2	Asset12 0.2	Asset 13 0.2	Asset 15 0.2	Asset 17 0.2	Asset 20 0.2	Asset 28 0.1	x_{f5} –0.5

Table 2. The optimal solution when $\theta = 3.5$

Asset i / t	The optimal investment proportions								
1	Asset 3 0.2	Asset 17 0.2	Asset 22 0.2	Asset 25 0.2	x_{f1} 0.2	otherwise asset 0			
2	Asset 15 0.2	Asset 17 0.2	Asset 24 0.2	Asset 30 0.2	x_{f2} 0.2	otherwise asset 0			
3	Asset 3 0.2	Asset 15 0.2	Asset 24 0.2	x_{f3} 0.4	otherwise asset 0	otherwise asset 0			
4	Asset 6 0.2	Asset 8 0.2	Asset 15 0.2	Asset 20 0.2	Asset 25 0.2	x_{f4} 0	otherwise asset 0		
5	Asset 8 0.1	Asset 13 0.2	Asset15 0.2	Asset 17 0.2	Asset 20 0.2	Asset 22 0.2	Asset 25 0.2	Asset 30 0.2	x_{f5} –0.5

programming method.

Where W_6 is denoted the terminal wealth of the portfolio.

From Table 3, the Figure 1 which reflect the relationship between the preference coefficients θ and the terminal wealth of the Model (34) can be obtained as follows.

In the used data sets, the experiments in this paper correspond to the values of θ in the interval [0, 5.75]. It can be seen that, as will be seen in Fig. 1, the terminal wealth becomes smaller, when preference coefficient θ which $0 \leq \theta \leq 5.5$, become larger, the terminal wealth is same, when $5.5 \leq \theta \leq 6$; which reflects the influence of preference coefficient θ on portfolio selection.

6. CONCLUSIONS

In this paper, we consider the multi-period portfolio selection problem in uncertain environment. We use the uncertain mean value and the absolute deviation to measure the return and the risk of the multiperiod portfolio, respectively. A new multi-period portfolio optimization models with transaction cost, borrowing constraints and threshold constraints are proposed. Based on the uncertain theories, the proposed model is transformed into a dynamic optimization problem. Because of the transaction cost, the multiperiod portfolio selection model is a dynamic optimization problem with path dependence. The forward dynamic programming method is designed to obtain the optimal portfolio strategy. Finally, an examples is given to illustrate the behavior of the proposed model and the designed algorithm using real data from the Shanghai Stock Exchange.

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Table 3. The optimal terminal wealth of the portfolio when θ = 0, 0.25, 0.5, ⋯, 5.75

θ	0	0.25	0.5	0.75	1	1.25	1.50	1.75	2.00	2.25
W_6	2.1640	2.1637	2.1589	2.1493	2.1327	2.1138	2.0823	2.0705	2.0463	2.0320
θ	2.5	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75
W_6	2.0116	1.9985	1.9865	1.9505	1.6163	1.3496	1.2499	1.2312	1.1586	1.1283
θ	5	5.25	5.5	5.75						
W_6	1.1268	1.1085	1.085	1.085						

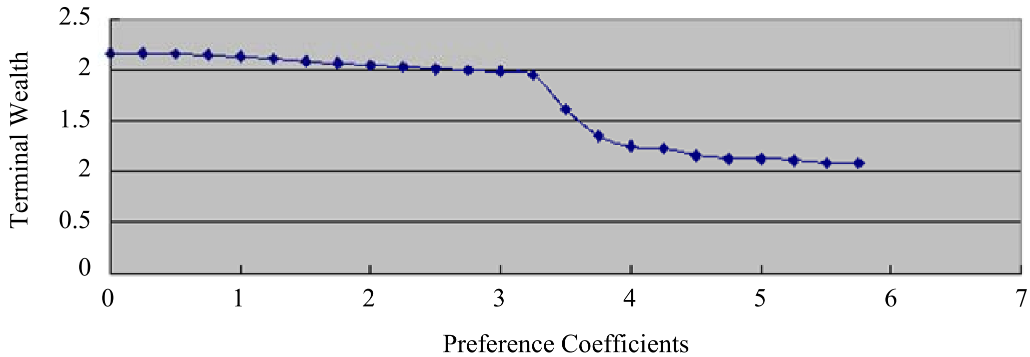

Figure 1. The relationship between the θ and the terminal wealth of the Model (34).

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<Appendix A>

The codes of thirty stocks are respectively S_1 (600000), S_2 (600005), S_3 (600015), S_4 (600016), S_5 (600019), S_6 (600028), S_7 (600030), S_8 (600036), S_9 (600048), S_{10} (600050), S_{11} (600104), S_{12} (600362), S_{13} (600519), S_{14} (600900), S_{15} (601088), S_{16} (601111), S_{17} (601166), S_{18} (601168), S_{19} (601318), S_{20} (601328), S_{21} (601390), S_{22} (601398), S_{23} (601600), S_{24} (601601), S_{25} (601628), S_{26} (601857), S_{27} (601919), S_{28} (601939), S_{29} (601988), S_{30} (601998). The triangle uncertain distributions, $\xi_{it} = (a_{it}, b_{it}, c_{it})$,of the return rates of assets at each period can be obtained as shown in Table 4.1 to Table 4.10.

Table 4.1 The fuzzy return rates on assets of five periods investment

Asset *t* / *i*		Asset 1			Asset 2			Asset 3	
1	0.0381	0.1430	0.2586	0.0093	0.0750	0.2414	0.0251	0.1083	0.1683
2	0.0568	0.1449	0.2585	0.0105	0.0813	0.2413	0.0404	0.1085	0.1688
3	0.0577	0.1458	0.2585	0.0191	0.0857	0.2413	0.0414	0.1139	0.1687
4	0.0896	0.1516	0.2586	0.0351	0.0930	0.2413	0.0592	0.1152	0.1692
5	0.0923	0.1532	0.2586	0.0391	0.1053	0.2412	0.0602	0.1172	0.1688

Table 4.2 The fuzzy return rates on assets of five periods investment

Asset *t* / *i*		Asset 4			Asset 5			Asset 6	
1	0.0441	0.1172	0.1985	0.001	0.0801	0.1417	0.0429	0.1064	0.1680
2	0.0460	0.1203	0.1985	0.0075	0.0847	0.1418	0.0439	0.1073	0.1681
3	0.0506	0.1255	0.1985	0.0397	0.0900	0.1417	0.0390	0.1083	0.1681
4	0.0541	0.1274	0.1984	0.0399	0.0906	0.1418	0.0328	0.1091	0.1681
5	0.0656	0.1289	0.1989	0.0431	0.0926	0.1418	0.0402	0.1129	0.1680

Table 4.3 The fuzzy return rates on assets of five periods investment

Asset *t* / *i*		Asset 7			Asset 8			Asset 9	
1	0.0236	0.0798	0.2492	0.0423	0.1238	0.2261	0.0117	0.0639	0.1590
2	0.0264	0.0907	0.2657	0.0499	0.1259	0.2262	0.0117	0.0790	0.1656
3	0.0437	0.0992	0.2492	0.0512	0.1277	0.2262	0.0212	0.0818	0.1657
4	0.0478	0.1029	0.2491	0.0845	0.1383	0.2261	0.0216	0.0861	0.1661
5	0.0535	0.1069	0.2492	0.0845	0.1457	0.2262	0.0234	0.0884	0.1657

Table 4.4 The fuzzy return rates on assets of five periods investment

Asset *t* / *i*		Asset 10			Asset 11			Asset 12	
1	0.0052	0.0377	0.0791	0.0172	0.0575	0.1857	0.0073	0.1243	0.3083
2	0.0118	0.0410	0.0789	0.0189	0.0592	0.1856	0.0296	0.1303	0.3084
3	0.0151	0.0469	0.0790	0.0295	0.0669	0.1857	0.0553	0.1380	0.3084
4	0.0166	0.0480	0.0789	0.0324	0.0724	0.1857	0.0648	0.1491	0.3084
5	0.0174	0.0492	0.0790	0.0288	0.0741	0.1857	0.0788	0.1540	0.3084

Table 4.5 The fuzzy return rates on assets of five periods investment

Asset *t* / *i*		Asset 13			Asset 14			Asset 15	
1	0.0805	0.2049	0.3380	0.0154	0.0254	0.0977	0.0103	0.0893	0.2356
2	0.0920	0.2102	0.3379	0.0063	0.0667	0.1110	0.0503	0.1518	0.2377
3	0.0958	0.2194	0.3380	0.0137	0.0700	0.1111	0.0505	0.1538	0.2378
4	0.0977	0.2225	0.3379	0.0216	0.0716	0.1111	0.1031	0.1565	0.2377
5	0.1209	0.2238	0.3380	0.0322	0.0731	0.1110	0.1047	0.1600	0.2378

Table 4.6 The fuzzy return rates on assets of five periods investment

Asset t / i	Asset 16			Asset 17			Asset 18		
1	0.0093	0.0615	0.3434	0.0477	0.1665	0.2040	0.0018	0.1075	0.2661
2	0.0030	0.0625	0.2931	0.0566	0.1550	0.2322	0.0046	0.1183	0.2415
3	0.0142	0.0656	0.2932	0.0660	0.1553	0.2322	0.0184	0.1349	0.2615
4	0.0287	0.0747	0.2932	0.0731	0.1575	0.2322	0.0242	0.1467	0.3414
5	0.0329	0.0835	0.2931	0.1044	0.1579	0.2323	0.0542	0.1664	0.3414

Table 4.7 The fuzzy return rates on assets of five periods investment

Asset t / i	Asset 19			Asset 20			Asset 21		
1	0.0264	0.1000	0.1896	0.0266	0.0825	0.1678	0.0019	0.0536	0.0939
2	0.0200	0.0916	0.1550	0.0584	0.1217	0.1836	0.016	0.0704	0.1924
3	0.0220	0.0928	0.1550	0.0598	0.1218	0.1836	0.0172	0.0838	0.1925
4	0.0440	0.0940	0.1550	0.0727	0.1243	0.1836	0.0199	0.0880	0.1925
5	0.0448	0.0952	0.1552	0.1002	0.1269	0.1837	0.0214	0.0917	0.1924

Table 4.8 The fuzzy return rates on assets of five periods investment

Asset t / i	Asset 22			Asset 23			Asset 24		
1	0.0113	0.1083	0.1767	0.0284	0.0413	0.1162	0.0572	0.1030	0.1171
2	0.0552	0.1170	0.1909	0.0317	0.0454	0.1494	0.0310	0.0947	0.1340
3	0.0577	0.1200	0.1908	0.0322	0.0531	0.1494	0.0356	0.0984	0.1339
4	0.0579	0.1235	0.1909	0.0144	0.0574	0.1495	0.0457	0.1005	0.1339
5	0.0766	0.1254	0.1908	0.0002	0.0718	0.1495	0.0528	0.1021	0.1340

Table 4.9 The fuzzy return rates on assets of five periods investment

Asset t / i	Asset 25			Asset 26			Asset 27		
1	0.0557	0.0989	0.1513	0.0571	0.1283	0.2379	0.0143	0.0690	0.2324
2	0.0369	0.1021	0.1612	0.0370	0.1276	0.2539	0.0015	0.0638	0.2466
3	0.0470	0.1037	0.1611	0.0380	0.1329	0.2539	0.0080	0.0606	0.2366
4	0.0730	0.1044	0.1611	0.0620	0.1432	0.2537	0.0130	0.0662	0.2356
5	0.0847	0.1090	0.1611	0.0668	0.1445	0.2538	0.0108	0.0619	0.2266

Table 4.10 The fuzzy return rates on assets of five periods investment

Asset t / i	Asset 28			Asset 29			Asset 30		
1	0.0823	0.1551	0.2549	0.0361	0.0994	0.1471	0.0419	0.1074	0.1928
2	0.0593	0.1382	0.2351	0.0482	0.1123	0.1621	0.0401	0.1037	0.1475
3	0.0716	0.1395	0.2351	0.0486	0.1134	0.1622	0.0403	0.1048	0.1474
4	0.0647	0.1426	0.2350	0.0541	0.1157	0.1621	0.0486	0.1060	0.1474
5	0.0706	0.1470	0.2350	0.0563	0.1175	0.1621	0.0711	0.1061	0.1474

<Appendix B>

According Table 4.1 to Table 4.10, and Eq. (25), $AD_t(R_{it})$ (i = 1, ⋯, 30; t = 1, ⋯, 5) can be obtained as shown in Table 5.1 to Table 5.2.

Table 5.1 The uncertain absolute deviation of assets of five periods investment

Asset t / i	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8
1	0.0551	0.0599	0.0232	0.0386	0.0179	0.0280	0.0588	0.0461
2	0.0506	0.0593	0.0268	0.0381	0.0183	0.0282	0.0620	0.0443
3	0.0504	0.0571	0.0261	0.0331	0.0255	0.0263	0.0532	0.0439
4	0.0428	0.0533	0.0333	0.0340	0.0255	0.0245	0.0521	0.0358
5	0.0422	0.0516	0.0330	0.0395	0.0259	0.0258	0.0507	0.0356

Table 5.2 The credibilistic absolute deviation of assets of five periods investment

Asset t / i	Asset 9	Asset 10	Asset 11	Asset 12	Asset 13	Asset 14	Asset 15	Asset 16
1	0.0374	0.0185	0.0440	0.0760	0.0563	0.0228	0.0573	0.0894
2	0.0386	0.0168	0.0435	0.0708	0.0615	0.0143	0.0360	0.0765
3	0.0363	0.0160	0.0408	0.0647	0.0589	0.0150	0.0357	0.0740
4	0.0362	0.0156	0.0398	0.0620	0.0588	0.0162	0.0399	0.0704
5	0.0356	0.0123	0.0405	0.0587	0.0543	0.0195	0.0335	0.0688

Table 5.3 The credibilistic absolute deviation of assets of five periods investment

Asset t / i	Asset 17	Asset 18	Asset 19	Asset 20	Asset21	Asset 22	Asset 23	Asset 24
1	0.0285	0.0666	0.0409	0.0356	0.0119	0.0229	0.0240	0.0230
2	0.0361	0.0592	0.0338	0.0342	0.0453	0.0339	0.0319	0.0197
3	0.0393	0.0608	0.0217	0.0349	0.0443	0.0333	0.0311	0.0203
4	0.0417	0.0801	0.0278	0.0278	0.0435	0.0333	0.0346	0.0229
5	0.0322	0.0735	0.0269	0.0214	0.0430	0.0287	0.0373	0.0255

Table 5.4 The credibilistic absolute deviation of assets of five periods investment

Asset t / i	Asset 25	Asset 26	Asset 27	Asset 28	Asset 29	Asset 30
1	0.0240	0.0456	0.0568	0.0434	0.0279	0.0379
2	0.0254	0.0545	0.0638	0.0441	0.0273	0.0233
3	0.0285	0.0542	0.0599	0.0411	0.0272	0.0271
4	0.0224	0.0482	0.0581	0.0426	0.0288	0.0257
5	0.0196	0.0470	0.0564	0.0411	0.0292	0.0191