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EXISTENCE AND CONTROLLABILITY OF FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY IN BANACH SPACES

SUBRAMANIAN KAILASAVALLI[1], SELVARAJ SUGANYA[2], AND MANI MALLIKA ARJUNAN[2†]

[1]DEPARTMENT OF MATHEMATICS, PSNA COLLEGE OF ENGINEERING AND TECHNOLOGY, DINDIGUL-624 622, TAMIL NADU, INDIA
E-mail address: skvalli2k5@gmail.com

[2]DEPARTMENT OF MATHEMATICS, C. B. M. COLLEGE, KOVAIPUDUR, COIMBATORE - 641 042, TAMIL NADU, INDIA
E-mail address: selvarajsuganya2014@yahoo.in,arjunphd2007@yahoo.co.in

ABSTRACT. In view of ideas for semigroups, fractional calculus, resolvent operator and Banach contraction principle, this manuscript is generally included with existence and controllability (EaC) results for fractional neutral integro-differential systems (FNIDS) with state-dependent delay (SDD) in Banach spaces. Finally, an examples are also provided to illustrate the theoretical results.

1. INTRODUCTION

The notion of fractional differential equation (FDE) is increasing as a basic scope of research because of the reality it is better in problems in connection with relating hypothesis of conventional differential equations. Fractional order models are often observed to be more adequate than integer order models in some real world problems as fractional derivatives supply an extraordinary application for the depiction of memory and hereditary properties of various materials and procedures. These days, it has been shown that the differential models including derivatives of fractional order occur in lots of engineering and scientific professions as the mathematical modeling of systems and strategies in numerous domains, for case in point, basic sciences, navigation, feed back amplifiers, and neuron modelling so on. For crucial confirmations about fractional frameworks, one can make reference to the treatises [1, 2], and the papers [3–9], and the references cited therein. FDE with delay characteristics happen in a number of domains such as medical and physical with SDD or non-constant delay. Presently, existence

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† Corresponding author.

and controllability results of mild solutions for such problems turned quite interesting and numerous researchers working on it. Just lately, number of papers have been released on the fractional order problems with SDD, see for instance [10–19].

On the other side, the thought of controllability has accepted a central part all through the historical backdrop of cutting edge control theory. This is the qualitative property of control frameworks and is of specific significance in control theory. A lot of dynamical systems are such that the control does not affect the complete state of the dynamical system yet just a part of it. Then again, frequently in real industrial procedures it is conceivable to notice just a specific piece of the complete state of the dynamical framework. Along these lines, it is essential to figure out if or not control of the complete state of the dynamical framework is conceivable. In this way, here the thought of complete controllability and approximate controllability exists. Generally discussing, controllability usually indicates that it is conceivable to steer dynamical framework from a arbitrary beginning state to the coveted last state utilizing the set of acceptable controls.

The existence, controllability and other qualitative and quantitative attributes of FDE are the most advancing area of pursuit, in particular, see [20–29]. These days, Santos et al. [16, 23, 24] reviewed the existence of solutions for FIDE with unbounded or SDD delay in Banach spaces. Shu et al. [25] examined the existence outcomes for FDE with nonlocal conditions of order $\alpha \in (1,2)$. In [26, 27], the writers present sufficient circumstances for the existence and approximate controllability of fractional order neutral differential and stochastic differential system with infinite delay. Kexue et al. [28] analyzed the controllability of nonlocal FDE of order $\alpha \in (1,2]$. Sakthivel et al. [29] acknowledged the approximate controllability of fractional dynamical system by making use of appropriate fixed point theorem. Lately, in [17–19], the authors outlined the approximate controllability results for FNIDS with SDD by applying the acceptable fixed point theorem. However, EaC results for FNIDS with SDD in $\mathscr{B}_h$ phase space adages have not yet been completely examined.

Inspired by the effort of the previously stated papers [14–16], the principle inspiration driving this manuscript is to research the EaC of mild solutions for FNIDS with SDD of the models

$$D_t^\alpha\left[x(t)+\mathscr{G}\left(t,x_{\varrho(t,x_t)},\int_0^t e_1(t,s,x_{\varrho(s,x_s)})ds\right)\right]=\mathscr{A}x(t)+\int_0^t\mathscr{B}(t-s)x(s)ds$$
$$+\mathscr{F}\left(t,x_{\varrho(t,x_t)},\int_0^t e_2(t,s,x_{\varrho(s,x_s)})ds\right)$$
$$+\mathscr{H}\left(t,x_{\varrho(t,x_t)},\int_0^t e_3(t,s,x_{\varrho(s,x_s)})ds\right),\quad t\in\mathscr{I}=[0,T], \tag{1.1}$$
$$x_0=\varsigma(t)\in\mathscr{B}_h,\quad x'(0)=0,\ t\in(-\infty,0], \tag{1.2}$$

and the corresponding controllability structure

$$D_t^\alpha\left[x(t)+\mathscr{G}\left(t,x_{\varrho(t,x_t)},\int_0^t e_1(t,s,x_{\varrho(s,x_s)})ds\right)\right]=\mathscr{A}x(t)+\int_0^t\mathscr{B}(t-s)x(s)ds$$

$$+ \mathscr{F}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_2(t,s,x_{\varrho(s,x_s)})ds\right) + \mathscr{H}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_3(t,s,x_{\varrho(s,x_s)})ds\right)$$

$$+ (\mathcal{C}u)(t), \quad t \in \mathscr{I} = [0,T], \tag{1.3}$$

$$x_0 = \varsigma(t) \in \mathscr{B}_h, \quad x'(0) = 0,\ t \in (-\infty, 0], \tag{1.4}$$

where the unknown $x(\cdot)$ needs values in the Banach space $\mathbb{X}$ having norm $\|\cdot\|$, $\mathscr{I} = [0,T]$ is an operational interval, D_t^α is the Caputo fractional derivative of order $\alpha \in (1,2)$, $\mathscr{A}$, $(\mathscr{B}(t))_{t\geq 0}$ are closed linear operators described on a regular domain which is dense in $(\mathbb{X}, \|\cdot\|)$ and $D_t^\alpha \sigma(t)$ symbolize the Caputo derivative of $\alpha > 0$ characterized by

$$D_t^\alpha \sigma(t) := \int_0^t \widetilde{\mu}_{n-\alpha}(t-s)\frac{d^n}{ds^n}\sigma(s)ds,$$

where $n \geq \alpha$ and $\widetilde{\mu}_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. Further, $\mathscr{G}, \mathscr{F}, \mathscr{H} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}, e_i : \mathscr{D} \times \mathscr{B}_h \to \mathbb{X}$, $i = 1,2,3$; $\mathscr{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I} : 0 \leq s \leq t \leq T\}$, $\varrho : \mathscr{I} \times \mathscr{B}_h \to (-\infty, T]$ are apposite functions, $\mathcal{C}$ is a bounded linear operator from a Banach space U into $\mathbb{X}$; the control function $u(\cdot) \in L^2(\mathscr{I}, U)$, a Banach space of admissible control functions, and $\mathscr{B}_h$ is a phase space characterized in Preliminaries.

For almost any continuous function x characterized on $(-\infty, T]$ and any $t \geq 0$, we designate by x_t the part of $\mathscr{B}_h$ characterized by $x_t(\theta) = x(t+\theta)$ for $\theta \leq 0$. Now $x_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ likely the current time t.

We push ahead as takes after. Section 2 is focused on call to mind of some crucial perspectives that will be utilized in this work to accomplish our primary results. In Section 3 and 4, we declare and present the EaC results about by proposes of Banach fixed point theorem. In Section 5, as a last point, an appropriate cases are equipped to replicate the efficiency of the conceptual idea.

To the best of our insight, there is no work gave an account of the EaC results for FNIDS with SDD, which is communicated in the structures (1.1)–(1.2) and (1.3)–(1.4). To pack this gap, in this manuscript, we contemplate this fascinating model.

2. Preliminaries

In this section, we present some primary components which are required to confirm the principal outcomes. Let $\mathscr{L}(\mathbb{X})$ symbolizes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed with the uniform operator topology, having its norm recognized as $\|\cdot\|_{\mathscr{L}(\mathbb{X})}$. Let $C(\mathscr{I}, \mathbb{X})$ symbolize the space of all continuous functions from $\mathscr{I}$ into $\mathbb{X}$, having its norm recognized as $\|\cdot\|_{C(\mathscr{I},\mathbb{X})}$. Moreover, $B_r(x, \mathbb{X})$ symbolizes the closed ball in $\mathbb{X}$ with the middle at x and the distance r. It needs to be outlined that, once the delay is infinite, then we should talk about the theoretical phase space $\mathscr{B}_h$ in a beneficial way. In this manuscript, we deliberate phase spaces $\mathscr{B}_h$ which are same as described in [30]. So, we bypass the details.

We expect that the phase space $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Hale and Kato (see case in point in [31, 32]).

If $x : (-\infty, T] \to \mathbb{X}, T > 0$, is continuous on $\mathscr{I}$ and $x_0 \in \mathscr{B}_h$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(P_1) x_t is in $\mathscr{B}_h$;

(P_2) $\|x(t)\|_{\mathbb{X}} \leq H\|x_t\|_{\mathscr{B}_h}$;

(P_3) $\|x_t\|_{\mathscr{B}_h} \leq \mathscr{D}_1(t)\sup\{\|x(s)\|_{\mathbb{X}} : 0 \leq s \leq t\} + \mathscr{D}_2(t)\|x_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(\cdot)$.

(P_4) The function $t \to \varsigma_t$ is well described and continuous from the set

$$\mathcal{R}(\varrho^-) = \{\varrho(s, \varsigma) : (s, \varsigma) \in [0, T] \times \mathscr{B}_h\},$$

into $\mathscr{B}_h$ and there is a continuous and bounded function $J^\varsigma : \mathcal{R}(\varrho^-) \to (0, \infty)$ to ensure that $\|\varsigma_t\|_{\mathscr{B}_h} \leq J^\varsigma(t)\|\varsigma\|_{\mathscr{B}_h}$ for every $t \in \mathcal{R}(\varrho^-)$.

Recognize the space

$$\mathscr{B}_T = \{x : (-\infty, T] \to \mathbb{X} : x|_{\mathscr{I}} \text{ is continuous and } x_0 \in \mathscr{B}_h\},$$

where $x|_{\mathscr{I}}$ is the constraint of x to the real compact interval on $\mathscr{I}$. The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|x\|_{\mathscr{B}_T} = \|\varsigma\|_{\mathscr{B}_h} + \sup\{\|x(s)\|_{\mathbb{X}} : s \in [0, T]\}, \ x \in \mathscr{B}_T.$$

Lemma 2.1. *[33, Lemma 2.1] Let $x : (-\infty, T] \to \mathbb{X}$ be a function in a way that $x_0 = \varsigma$, and if $(P4)$ hold, then*

$$\|x_s\|_{\mathscr{B}_h} \leq (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h} + \mathscr{D}_1^* \sup\{\|x(\theta)\|_{\mathbb{X}} : \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $J^\varsigma = \sup_{t \in \mathcal{R}(\varrho^-)} J^\varsigma(t), \quad \mathscr{D}_1^* = \sup_{s \in [0,T]} \mathscr{D}_1(s), \quad \mathscr{D}_2^* = \sup_{s \in [0,T]} \mathscr{D}_2(s).$

To be able to acquire our outcomes, we believe that the subsequent FIDS

$$D_t^\alpha x(t) = \mathscr{A}x(t) + \int_0^t \mathscr{B}(t-s)x(s)ds, \tag{2.1}$$

$$x(0) = \varsigma \in \mathbb{X}, \quad x'(0) = 0, \tag{2.2}$$

has an associated α-resolvent operator of bounded linear operators $(\mathcal{R}_\alpha(t))_{t \geq 0}$ on $\mathbb{X}$.

Definition 2.1. *[23, Definition 2.1] A one parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t \geq 0}$ on $\mathbb{X}$ is called a α-resolvent operator of (2.1)-(2.2) if the subsequent conditions are fulfilled.*

(a) *The function $\mathcal{R}_\alpha(\cdot) : [0, \infty) \to \mathscr{L}(\mathbb{X})$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in \mathbb{X}$ and $\alpha \in (1, 2)$.*

(b) *For $x \in D(\mathscr{A})$, $\mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(\mathscr{A})]) \bigcap C^1((0, \infty), \mathbb{X})$, and*

$$D_t^\alpha \mathcal{R}_\alpha(t)x = \mathscr{A}\mathcal{R}_\alpha(t)x + \int_0^t \mathscr{B}(t-s)\mathcal{R}_\alpha(s)xds, \tag{2.3}$$

$$D_t^{\alpha}\mathcal{R}_{\alpha}(t)x = \mathcal{R}_{\alpha}(t)\mathscr{A}x + \int_0^t \mathcal{R}_{\alpha}(t-s)\mathscr{B}(s)xds, \tag{2.4}$$

for every $t \geq 0$.

The existence of a α-resolvent operator for the model (2.1)-(2.2) was analyzed in [20]. To be able to research our model, we need to consider the conditions $(P1)-(P3)$ which are same as stated in [23], consequently we preclude it. In view of the conditions $(P1)-(P3)$, in the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \vartheta)$,

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\lambda| > r, \, |arg(\lambda)| < \theta\},$$

for $\Gamma_{r,\theta}, \Gamma_{r,\theta}^i$, $i = 1,2,3$, are the paths $\Gamma_{r,\theta}^1 = \{te^{i\theta} : t \geq r\}, \Gamma_{r,\theta}^2 = \{re^{i\xi} : -\theta \leq \xi \leq \theta\}, \Gamma_{r,\theta}^3 = \{te^{-i\theta} : t \geq r\}$, and $\Gamma_{r,\theta} = \bigcup_{i=1}^{3} \Gamma_{r,\theta}^i$ oriented counterclockwise. Furthermore, $\rho_{\alpha}(G_{\alpha})$ are the sets

$$\rho_{\alpha}(G_{\alpha}) = \{\lambda \in \mathbb{C} : G_{\alpha}(\lambda) := \lambda^{\alpha-1}(\lambda^{\alpha} I - \mathscr{A} - \widehat{\mathscr{B}}(\lambda))^{-1} \in \mathscr{L}(\mathbb{X})\}.$$

Presently, we determine the operator family $(\mathcal{R}_{\alpha}(t))_{t\geq 0}$ by

$$\mathcal{R}_{\alpha}(t) = \begin{cases} \frac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} G_{\alpha}(\lambda) d\lambda, & t > 0, \\ I, & t = 0. \end{cases} \tag{2.5}$$

Now, we are in a position to present some conventional outcomes from current works.

Theorem 2.1 ([20, Theorem 2.1]). *Assume that conditions (P1)–(P3) are fulfilled. Then there exists a unique α-resolvent operator for problem (2.1)-(2.2).*

Theorem 2.2 ([20, Lemma 2.5]). *The function $\mathcal{R}_{\alpha} : [0,\infty) \to \mathscr{L}(\mathbb{X})$ is strongly continuous and $\mathcal{R}_{\alpha} : (0,\infty) \to \mathscr{L}(\mathbb{X})$ is uniformly continuous.*

Hereafter, we expect that the conditions $(P1)-(P3)$ are fulfilled. Further, we need to talk about the mild solution for the model (1.1)-(1.2). For this intent, it is necessary to discuss the subsequent non-homogeneous model

$$D_t^{\alpha}x(t) = \mathscr{A}x(t) + \int_0^t \mathscr{B}(t-s)x(s)ds + \mathscr{F}(t), \quad t \in \mathscr{I}, \tag{2.6}$$

$$x(0) = x_0, \quad x'(0) = 0, \tag{2.7}$$

where $\alpha \in (1,2)$ and $\mathscr{F} \in L^1(\mathscr{I}, \mathbb{X})$. In the follow up, $\mathcal{R}_{\alpha}(\cdot)$ is the operator function characterized by (2.5). Now, we start by presenting the subsequent concept of classical solution.

Definition 2.2 ([23, Definition 2.5]). *A function $x : \mathscr{I} \to \mathbb{X}$, $0 < T$, is called a classical solution of (2.6)-(2.7) on $\mathscr{I}$ if $x \in C(\mathscr{I}, [D(\mathscr{A})]) \cap C(\mathscr{I}, \mathbb{X}), \widetilde{\mu}_{n-\alpha} * x \in C^1(\mathscr{I}, \mathbb{X}), n = 1,2,$ the condition (2.7) holds and the equation (2.6) is verified on $\mathscr{I}$.*

Definition 2.3 ([23, Definition 2.6]). *Let $\alpha \in (1,2)$, we describe the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by*

$$\mathcal{S}_\alpha(t)x := \int_0^t \widetilde{\mu}_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)x ds,$$

for each $t \geq 0$.

Now, just as before, we need to present some additional conventional outcomes from [20].

Lemma 2.2 ([20, Lemma 3.9]). *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}(\mathbb{X})$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}(\mathbb{X})$.*

Lemma 2.3 ([20, Lemma 3.10]). *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}([D(\mathscr{A})])$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}([D(\mathscr{A})])$.*

Theorem 2.3 ([20, Theorem 3.2]). *Let $z \in D(\mathscr{A})$. Assume that $\mathscr{F} \in C(\mathscr{I}, \mathbb{X})$ and $x(\cdot)$ is a classical solution of (2.6)-(2.7) on $\mathscr{I}$. Then*

$$x(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)\, ds, \quad t \in \mathscr{I}. \tag{2.8}$$

It is obvious from the earlier definition that $\mathcal{R}_\alpha(\cdot)z$ is a solution of problem (2.1)-(2.2) on $(0,\infty)$ for $z \in D(\mathscr{A})$.

Definition 2.4 ([23, Definition 2.10]). *Let $\mathscr{F} \in L^1(\mathscr{I}, \mathbb{X})$. A function $x \in C(\mathscr{I}, \mathbb{X})$ is called a mild solution of (2.6)-(2.7) if*

$$x(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)\, ds, \quad t \in \mathscr{I}.$$

Theorem 2.4 ([20, Theorem 3.3]). *Let $z \in D(\mathscr{A})$ and $\mathscr{F} \in C(\mathscr{I}, \mathbb{X})$. If $\mathscr{F} \in L^1(\mathscr{I}, [D(\mathscr{A})])$ then the mild solution of (2.6)-(2.7) is a classical solution.*

Theorem 2.5 ([20, Theorem 3.4]). *Let $z \in D(\mathscr{A})$ and $\mathscr{F} \in C(\mathscr{I}, \mathbb{X})$. If $\mathscr{F} \in W^{1,1}(\mathscr{I}, \mathbb{X})$, then the mild solution of (2.6)-(2.7) is a classical solution.*

In the subsequent result, we signify by $(-\mathscr{A})^\vartheta$ the fractional power of the operator $-\mathscr{A}$, (see [34] for details).

Lemma 2.4 ([23, Lemma 3.1]). *Suppose that the conditions $(P1)-(P3)$ are satisfied. Let $\alpha \in (1,2)$ and $\vartheta \in (0,1)$ such that $\alpha\vartheta \in (0,1)$, then there exists positive number C such that*

$$\|(-\mathscr{A})^\vartheta \mathcal{R}_\alpha(t)\| \leq Ce^{rt}t^{-\alpha\vartheta}, \tag{2.9}$$

$$\|(-\mathscr{A})^\vartheta \mathcal{S}_\alpha(t)\| \leq Ce^{rt}t^{\alpha(1-\vartheta)-1}, \tag{2.10}$$

for all $t > 0$.

Remark 2.1. *The verifications of the above results are excessively standard, subsequently we overlook here. For additional data about this idea, we propose the peruser to allude [20, 23].*

Definition 2.5. *Let $x_T(\varsigma; u)$ be the state value of the model (1.3)-(1.4) at terminal time T corresponding to the control u and the initial value $\varsigma \in \mathscr{B}_h$. Present the set $\mathscr{R}(T, \varsigma) = \{x_T(\varsigma; u)(0) : u(\cdot) \in L^2(\mathscr{I}, U)\}$, which is known as the reachable set of model (1.3)-(1.4) at terminal time T.*

Definition 2.6. *The model (1.3)-(1.4) is said to be exactly controllable on $\mathscr{I}$ if $\mathscr{R}(T, \varsigma) = \mathbb{X}$.*

Assume that the fractional differential control model

$$D^\alpha x(t) = \mathscr{A} x(t) + (\mathcal{C}u)(t), \quad t \in \mathscr{I}, \tag{2.11}$$

$$x_0 = \varsigma \in \mathscr{B}_h, \tag{2.12}$$

is exactly controllable. It is practical at this position to present the controllability operator linked with (2.11)-(2.12) as

$$\Gamma_0^T = \int_0^T \mathcal{S}_\alpha(T-s)\mathcal{C}\mathcal{C}^*\mathcal{S}_\alpha^*(T-s)ds,$$

where $\mathcal{C}^*$ and $\mathcal{S}_\alpha^*(t)$ denotes the adjoints of $\mathcal{C}$ and $\mathcal{S}_\alpha(t)$, accordingly. It is simple that the operator Γ_0^T is a linear bounded operator [35, Theorem 3.2].

Lemma 2.5. *If the linear fractional model (2.11)-(2.12) is exactly controllable if and only then for some $\gamma > 0$ such that $\langle \Gamma_0^T x, x\rangle \geq \gamma \|x\|^2$, for all $x \in \mathbb{X}$ and as a result $\|(\Gamma_0^T)^{-1}\| \leq \frac{1}{\gamma}$.*

Remark 2.2. *Further, we assume that the linear fractional control system (2.11)-(2.12) is exactly controllable.*

3. Existence Results

In this section, we exhibit and demonstrate the existence of solutions for the structure (1.1)-(1.2) under Banach fixed point theorem. In the first place, we present the mild solution for the model (1.1)–(1.2).

Definition 3.1. *A function $x : (-\infty, T] \to \mathbb{X}$, is called a mild solution of (1.1)-(1.2) on $[0, T]$, if $x_0 = \varsigma$; $x|_{[0,T]} \in C([0, T] : \mathbb{X})$; the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)})d\tau\right)$ and $s \to \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)})d\xi\right) d\tau$ is integrable on $[0, t)$ for all $t \in (0, T]$ and for $t \in [0, T]$,*

$$\begin{aligned} x(t) = {} & \mathcal{R}_\alpha(t)\big[\varsigma(0) + \mathscr{G}(0, \varsigma(0), 0)\big] - \mathscr{G}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_1(t, s, x_{\varrho(s,x_s)})ds\right) \\ & - \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)})d\tau\right) ds \\ & - \int_0^t \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)})d\xi\right) d\tau ds \end{aligned} \tag{3.1}$$

$$+ \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_2(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds$$
$$+ \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_3(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds.$$

Presently, we itemizing the subsequent suppositions:

(H1) The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exists a constant $\mathcal{M}$ in a way that $\|\mathcal{R}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})} \leq \mathcal{M}$ and $\|\mathcal{S}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})} \leq \mathcal{M}$ for every $t \in \mathscr{I}$ and
$$\|(-\mathscr{A})^\vartheta \mathcal{S}_\alpha(t)\|_{\mathbb{X}} \leq \mathcal{M}t^{\alpha(1-\vartheta)-1}, \quad 0 < t \leq T.$$

(H2) The subsequent conditions are fulfilled.
 (a) $\mathscr{B}(\cdot)x \in C(\mathscr{I}, \mathbb{X})$ for every $x \in [D((-\mathscr{A})^{1-\vartheta})]$.
 (b) There is a function $\mu(\cdot) \in L^1(\mathscr{I}; \mathbb{R}^+)$, to ensure that
$$\|\mathscr{B}(s)\mathcal{S}_\alpha(t)\|_{\mathscr{L}([D((-\mathscr{A})^\vartheta)],\mathbb{X})} \leq \mathcal{M}\mu(s)t^{\alpha\vartheta-1}, \quad 0 \leq s < t \leq T.$$

(H3) (i) The function $\mathscr{F} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{F}}, \widetilde{L}_{\mathscr{F}} > 0$ and $L^*_{\mathscr{F}} > 0$ in ways that
$$\|\mathscr{F}(t,\psi_1,x) - \mathscr{F}(t,\psi_2,y)\|_{\mathbb{X}} \leq L_{\mathscr{F}}\|\psi_1-\psi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{F}}\|x-y\|_{\mathbb{X}}, \quad t \in \mathscr{I}, \quad x, y \in \mathbb{X},$$
and
$$L^*_{\mathscr{F}} = \max_{t\in\mathscr{I}} \|\mathscr{F}(t,0,0)\|_{\mathbb{X}}.$$
 (ii) The function $\mathscr{H} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{H}}, \widetilde{L}_{\mathscr{H}} > 0$ and $L^*_{\mathscr{H}} > 0$ in ways that
$$\|\mathscr{H}(t,\psi_1,x) - \mathscr{H}(t,\psi_2,y)\|_{\mathbb{X}} \leq L_{\mathscr{H}}\|\psi_1-\psi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{H}}\|x-y\|_{\mathbb{X}}, \quad t \in \mathscr{I}, \quad x, y \in \mathbb{X},$$
and
$$L^*_{\mathscr{H}} = \max_{t\in\mathscr{I}} \|\mathscr{H}(t,0,0)\|_{\mathbb{X}}.$$

(H4) $e_i : \mathscr{D} \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find constants $L_{e_i} > 0, L^*_{e_i} > 0$ to ensure that
$$\big\|e_i(t,s,\varsigma) - e_i(t,s,\psi)\big\|_{\mathbb{X}} \leq L_{e_i}\|\varsigma-\psi\|_{\mathscr{B}_h}, \quad (t,s) \in \mathscr{D}, \quad (\varsigma,\psi) \in \mathscr{B}_h^2,\ i = 1,2,3;$$
and
$$L^*_{e_i} = \max_{t\in\mathscr{I}} \|e_i(t,s,0)\|_{\mathbb{X}}, \quad i = 1,2,3.$$

(H5) The function $\mathscr{G}(\cdot)$ is $(-\mathscr{A})^\vartheta$-valued, $\mathscr{G} : \mathscr{I} \times \mathscr{B}_h \times \mathbb{X} \to [D((-\mathscr{A})^{-\vartheta})]$ is continuous and there exist positive constants $L_{\mathscr{G}}, \widetilde{L}_{\mathscr{G}} > 0$ and $L^*_{\mathscr{G}} > 0$ such that for all $(t,\varsigma_j) \in \mathscr{I} \times \mathscr{B}_h, j = 1,2;$
$$\|(-\mathscr{A})^\vartheta\mathscr{G}(t,\varsigma_1,x) - (-\mathscr{A})^\vartheta\mathscr{G}(t,\varsigma_2,y)\|_{\mathbb{X}} \leq L_{\mathscr{G}}\|\varsigma_1-\varsigma_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{G}}\|x-y\|_{\mathbb{X}},\ x, y \in \mathbb{X},$$
$$\|(-\mathscr{A})^\vartheta\mathscr{G}(t,\varsigma,0)\|_{\mathbb{X}} \leq L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L^*_{\mathscr{G}},$$

where
$$L^*_{\mathscr{G}} = \max_{t\in\mathscr{I}} \|(-\mathscr{A})^{\vartheta}\mathscr{G}(t,0,0)\|_{\mathbb{X}}.$$

(H6) The following inequalities holds:

(i) Let

$$\begin{aligned}
&\mathcal{M}\mathcal{M}_0\big[L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L^*_{\mathscr{G}}\big] + \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L^*_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL^*_{e_1}\right)\\
&+\mathcal{M}T\left\{(L^*_{\mathscr{F}} + L^*_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L^*_{e_2} + \widetilde{L}_{\mathscr{H}}L^*_{e_3})\right\}\\
&+(\mathscr{D}^*_1 r + c_n)\Bigg[\mathcal{M}T\left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L_{e_2} + \widetilde{L}_{\mathscr{H}}L_{e_3})\right)\\
&+\left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg] \le r,
\end{aligned}$$

for some $r > 0$.

(ii) Let

$$\begin{aligned}
\Lambda = \mathscr{D}^*_1 &\Bigg[\mathcal{M}T\left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L_{e_2} + \widetilde{L}_{\mathscr{H}}L_{e_3})\right)\\
&+\left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg] < 1
\end{aligned}$$

be such that $0 \le \Lambda < 1$.

Theorem 3.1. *Assume that the conditions (H1)-(H6) hold. Then the structure (1.1)-(1.2) has a unique mild solution on $\mathscr{I}$.*

Proof. We will transmute the structure (1.1)-(1.2) into a fixed-point problem. Recognize the operator $\Upsilon : \mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon x)(t) = \begin{cases}
\mathcal{R}_\alpha(t)\big[\varsigma(0) + \mathscr{G}(0,\varsigma(0),0)\big] - \mathscr{G}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_1(t,s,x_{\varrho(s,x_s)})ds\right)\\
-\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds\\
-\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau,\xi,x_{\varrho(\xi,x_\xi)})d\xi\right)d\tau ds\\
+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_2(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds\\
+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_3(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds, \quad t\in\mathscr{I}.
\end{cases}$$

It is evident that the fixed points of the operator Υ are mild solutions of the model (1.1)-(1.2). We express the function $y(\cdot):(-\infty,T]\to\mathbb{X}$ by

$$y(t)=\begin{cases}\varsigma(t), & t\le 0;\\ \mathcal{R}_\alpha(t)\varsigma(0), & t\in\mathscr{I},\end{cases}$$

then $y_0=\varsigma$. For every function $z\in C(\mathscr{I},\mathbb{R})$ with $z(0)=0$, we allocate as $\widetilde{z}$ is characterized by

$$\widetilde{z}(t)=\begin{cases}0, & t\le 0;\\ z(t), & t\in\mathscr{I}.\end{cases}$$

If $x(\cdot)$ fulfills (3.1), we are able to split it as $x(t)=y(t)+z(t)$, $t\in\mathscr{I}$, which suggests $x_t=y_t+z_t$, for each $t\in\mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$\begin{aligned}
z(t)&=\mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma,0)\\
&\quad-\mathscr{G}\left(t,z_{\varrho(t,z_t+y_t)}+y_{\varrho(t,z_t+y_t)},\int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)})ds\right)\\
&\quad-\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\\
&\quad(\times)\mathscr{G}\left(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right)ds\\
&\quad-\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\\
&\quad(\times)\mathscr{G}\left(\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)},\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)}+y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\right)d\tau ds\\
&\quad+\int_0^t \mathcal{S}_\alpha(t-s)\\
&\quad(\times)\mathscr{F}\left(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right)ds\\
&\quad+\int_0^t \mathcal{S}_\alpha(t-s)\\
&\quad(\times)\mathscr{H}\left(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},\int_0^s e_3(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right)ds.
\end{aligned}$$

Let $\mathscr{B}_T^0=\{z\in\mathscr{B}_T^0\colon z_0=0\in\mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|z\|_{\mathscr{B}_T^0}=\sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}}+\|z_0\|_{\mathscr{B}_h}=\sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}},\quad z\in\mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. We delimit the operator $\overline{\Upsilon}: \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$\begin{aligned}
&(\overline{\Upsilon}z)(t)\\
&= \mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma,0)\\
&\quad - \mathscr{G}\left(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}, \int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\right)\\
&\quad - \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\\
&\quad (\times)\mathscr{G}\left(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right) ds\\
&\quad - \int_0^t \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\\
&\quad (\times)\mathscr{G}\left(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}, \int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\right) d\tau ds\\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s)\\
&\quad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right) ds\\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s)\\
&\quad (\times)\mathscr{H}\left(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_3(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right) ds.
\end{aligned}$$

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point.

Remark 3.1. *Let $B_r = \{x \in \mathbb{X} : \|x\| \leq r\}$ for some $r > 0$. From the above discussion, we have the subsequent estimates:*

(i)

$$\begin{aligned}
&\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h}\\
&\leq \|z_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h} + \|y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h}\\
&\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_2^* + J^\varsigma)\|z_0\|_{\mathscr{B}_h} + \mathscr{D}_1^*|y(s)| + (\mathscr{D}_2^* + J^\varsigma)\|y_0\|_{\mathscr{B}_h}\\
&\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\|\mathcal{R}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})}|\varsigma(0)| + (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h}\\
&\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\mathcal{M}H\|\varsigma\|_{\mathscr{B}_h} + (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h}\\
&\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_1^*\mathcal{M}H + \mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h}.
\end{aligned}$$

In the event that $\|z\|_{\mathbb{X}} < r$, $r > 0$, *then*

$$\|z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}\|_{\mathscr{B}_h} \le \mathscr{D}_1^* r + c_n,$$

where $c_n = (\mathscr{D}_1^* \mathcal{M} H + \mathscr{D}_2^* + J^{\varsigma})\|\varsigma\|_{\mathscr{B}_h}$.

(ii) *From suppositions (H1) and (H5), we sustain*

$$\|\mathcal{R}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})}\|\mathscr{G}(0,\varsigma,0)\|_{\mathbb{X}} \le \mathcal{M}\mathcal{M}_0\big[L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L_{\mathscr{G}}^*\big],$$

where $\|(-\mathscr{A})^{-\vartheta}\| = \mathcal{M}_0$.

(iii)

$$\begin{aligned}
&\left\|\mathscr{G}\left(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}, \int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\right)\right\|_{\mathbb{X}} \\
&\le \|(-\mathscr{A})^{-\vartheta}\|\Bigg[\Big\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Big(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}, \\
&\quad \int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\Big) - (-\mathscr{A})^{\vartheta}\mathscr{G}(t,0,0)\Big\|_{\mathbb{X}} + \|(-\mathscr{A})^{\vartheta}\mathscr{G}(t,0,0)\|_{\mathbb{X}}\Bigg] \\
&\le \mathcal{M}_0\Bigg[L_{\mathscr{G}}\|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{G}}\Big\|\int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\Big\|_{\mathbb{X}} \\
&\quad + L_{\mathscr{G}}^*\Bigg] \\
&\le \mathcal{M}_0 L_{\mathscr{G}}(\mathscr{D}_1^* r + c_n) + \mathcal{M}_0\widetilde{L}_{\mathscr{G}}\int_0^t\Bigg[\Big\|e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - e_1(t,s,0)\Big\|_{\mathbb{X}} \\
&\quad + \|e_1(t,s,0)\|_{\mathbb{X}}\Bigg]ds + \mathcal{M}_0 L_{\mathscr{G}}^* \\
&\le \mathcal{M}_0 L_{\mathscr{G}}(\mathscr{D}_1^* r + c_n) + \mathcal{M}_0 L_{\mathscr{G}}^* + \mathcal{M}_0\widetilde{L}_{\mathscr{G}}T\big[L_{e_1}\|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{\mathscr{B}_h} + L_{e_1}^*\big] \\
&\le \mathcal{M}_0 L_{\mathscr{G}}(\mathscr{D}_1^* r + c_n) + \mathcal{M}_0 L_{\mathscr{G}}^* + \mathcal{M}_0\widetilde{L}_{\mathscr{G}}T L_{e_1}(\mathscr{D}_1^* r + c_n) + \mathcal{M}_0\widetilde{L}_{\mathscr{G}}T L_{e_1}^*,
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg\|\mathscr{G}\left(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}, \int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\right) \\
&\quad - \mathscr{G}\left(t, \overline{z}_{\varrho(t,\overline{z}_t+y_t)} + y_{\varrho(t,\overline{z}_t+y_t)}, \int_0^t e_1(t,s,\overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)})ds\right)\Bigg\|_{\mathbb{X}} \\
&\le \|(-\mathscr{A})^{-\vartheta}\|\Big\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Big(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)},
\end{aligned}$$

$$\int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)})ds\Big)$$
$$-(-\mathscr{A})^{\vartheta}\mathscr{G}\left(t,\overline{z}_{\varrho(t,\overline{z}_t+y_t)}+y_{\varrho(t,\overline{z}_t+y_t)},\int_0^t e_1(t,s,\overline{z}_{\varrho(s,\overline{z}_s+y_s)}+y_{\varrho(s,\overline{z}_s+y_s)})ds\right)\Big\|_{\mathbb{X}}$$
$$\leq \mathcal{M}_0\Big[L_{\mathscr{G}}\|z_{\varrho(t,z_t+y_t)}-\overline{z}_{\varrho(t,\overline{z}_t+y_t)}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{G}}TL_{e_1}\|z_{\varrho(t,z_t+y_t)}-\overline{z}_{\varrho(t,\overline{z}_t+y_t)}\|_{\mathscr{B}_h}\Big]$$
$$\leq \mathcal{M}_0\mathscr{D}_1^*\Big[L_{\mathscr{G}}+\widetilde{L}_{\mathscr{G}}TL_{e_1}\Big]\|z-\overline{z}\|_{\mathscr{B}_T^0},$$

since

$$\begin{aligned}&\|z_{\varrho(s,z_s+y_s)}-\overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h}\\&\leq \mathscr{D}_1^*|z(s)|+(\mathscr{D}_2^*+J^{\varsigma})\|z_0\|_{\mathscr{B}_h}-\mathscr{D}_1^*|\overline{z}(s)|-(\mathscr{D}_2^*+J^{\varsigma})\|\overline{z}_0\|_{\mathscr{B}_h}\\&\leq \mathscr{D}_1^*|z(s)-\overline{z}(s)|_{\mathbb{X}}\\&\leq \mathscr{D}_1^*\|z-\overline{z}\|_{\mathscr{B}_T^0}.\end{aligned}$$

(iv)

$$\Big\|\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds\Big\|_{\mathbb{X}}$$

$$\leq \int_0^t \|(-\mathscr{A})^{1-\vartheta}\mathcal{S}_\alpha(t-s)\|_{\mathbb{X}}\Big[\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)-(-\mathscr{A})^{\vartheta}\mathscr{G}(s,0,0)\|_{\mathbb{X}}$$
$$+\|(-\mathscr{A})^{\vartheta}\mathscr{G}(s,0,0)\|_{\mathbb{X}}\Big]ds$$
$$\leq \int_0^t \mathcal{M}(t-s)^{\alpha\vartheta-1}\Big[L_{\mathscr{G}}\|z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h}$$
$$+\widetilde{L}_{\mathscr{G}}\int_0^s\Big[\|e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})-e_1(s,\tau,0)\|_{\mathbb{X}}+\|e_1(s,\tau,0)\|_{\mathbb{X}}\Big]d\tau$$
$$+L_{\mathscr{G}}^*\Big]ds$$
$$\leq \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\Big[L_{\mathscr{G}}(\mathscr{D}_1^*r+c_n)+\widetilde{L}_{\mathscr{G}}TL_{e_1}(\mathscr{D}_1^*r+c_n)+\widetilde{L}_{\mathscr{G}}TL_{e_1}^*+L_{\mathscr{G}}^*\Big]$$

$$\leq \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}(\mathscr{D}_1^* r + c_n)\Big\{L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\Big\} + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\widetilde{L}_{\mathscr{G}}TL_{e_1}^* + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}L_{\mathscr{G}}^*,$$

and

$$\begin{aligned}
&\Big\| \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds \\
&\quad - \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(s, \overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}, \\
&\quad \int_0^s e_1(s,\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)})d\tau\Big)ds\Big\|_{\mathbb{X}} \\
&\leq \int_0^t \|(-\mathscr{A})^{1-\vartheta}\mathcal{S}_\alpha(t-s)\|_{\mathbb{X}}\Big[\Big\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big) \\
&\quad - (-\mathscr{A})^{\vartheta}\mathscr{G}\Big(s, \overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}, \\
&\quad \int_0^s e_1(s,\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)})d\tau\Big)\Big\|_{\mathbb{X}}\Big]ds \\
&\leq \int_0^t \mathcal{M}(t-s)^{\alpha\vartheta-1}\Big[L_{\mathscr{G}}\|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \\
&\quad + \widetilde{L}_{\mathscr{G}}TL_{e_1}\|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h}\Big]ds \\
&\leq \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\mathscr{D}_1^*\Big[L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\Big]\|z-\overline{z}\|_{\mathscr{B}_T^0}.
\end{aligned}$$

(v)

$$\begin{aligned}
&\Big\| \int_0^t \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}, \\
&\quad \int_0^\tau e_1(\tau,\xi, z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Big)d\tau ds\Big\|_{\mathbb{X}} \\
&\leq \int_0^t \int_0^s \mu(s-\tau)\mathcal{M}(t-s)^{\alpha\vartheta-1} \\
&\quad (\times)\Big[\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Big(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}, \int_0^\tau e_1(\tau,\xi, z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Big)
\end{aligned}$$

$$- \|(-\mathscr{A})^{\vartheta}\mathscr{G}(\tau,0,0)\|_{\mathbb{X}} + \|(-\mathscr{A})^{\vartheta}\mathscr{G}(\tau,0,0)\|_{\mathbb{X}}\Bigg] d\tau ds$$

$$\leq \left(\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\int_0^T \mu(\tau)d\tau\right)\Bigg[L_{\mathscr{G}}\|z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}\|_{\mathscr{B}_h}$$

$$+ \widetilde{L}_{\mathscr{G}}\int_0^t \Big[\|e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})$$

$$- e_1(t,s,0)\|_{\mathbb{X}} + \|e_1(t,s,0)\|_{\mathbb{X}}\Big] ds + L_{\mathscr{G}}^*\Bigg]$$

$$\leq \left(\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\int_0^T \mu(\tau)d\tau\right)\Bigg[(\mathscr{D}_1^* r + c_n)\{L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\} + \widetilde{L}_{\mathscr{G}}TL_{e_1}^* + L_{\mathscr{G}}^*\Bigg],$$

and

$$\Bigg\|\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\Bigg(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)},$$

$$\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Bigg) d\tau ds$$

$$- \int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\Bigg(\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)},$$

$$\int_0^\tau e_1(\tau,\xi,\overline{z}_{\varrho(\xi,\overline{z}_\xi+y_\xi)} + y_{\varrho(\xi,\overline{z}_\xi+y_\xi)})d\xi\Bigg) d\tau ds\Bigg\|_{\mathbb{X}}$$

$$\leq \int_0^t\int_0^s \mu(s-\tau)\mathcal{M}(t-s)^{\alpha\vartheta-1}\Bigg[\|(-\mathscr{A})^{\vartheta}\mathscr{G}\Bigg(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)},$$

$$\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Bigg)$$

$$- (-\mathscr{A})^{\vartheta}\mathscr{G}\Bigg(\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)},$$

$$\int_0^\tau e_1(\tau,\xi,\overline{z}_{\varrho(\xi,\overline{z}_\xi+y_\xi)} + y_{\varrho(\xi,\overline{z}_\xi+y_\xi)})d\xi\Bigg)\|_{\mathbb{X}}\Bigg] d\tau ds$$

$$\leq \left(\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\int_0^T \mu(\tau)d\tau\right)\Bigg[L_{\mathscr{G}}\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,\overline{z}_t+y_t)}\|_{\mathscr{B}_h}$$

$$+ \widetilde{L}_{\mathscr{G}}TL_{e_1}\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,\overline{z}_t+y_t)}\|_{\mathscr{B}_h}\Bigg]$$

$$\leq \left(\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \int_0^T \mu(\tau)d\tau \right) \mathscr{D}_1^* \Big[L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \Big] \|z - \overline{z}\|_{\mathscr{B}_T^0}.$$

(vi)

$$\begin{aligned}
&\Big\| \int_0^t \mathcal{S}_\alpha(t-s) \mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau \Big) ds \Big\|_{\mathbb{X}} \\
&\leq \int_0^t \|\mathcal{S}_\alpha(t-s)\|_{\mathscr{L}(\mathbb{X})} \Big[\|\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau \Big) - \mathscr{F}(s,0,0)\|_{\mathbb{X}} + \|\mathscr{F}(s,0,0)\|_{\mathbb{X}} \Big] ds \\
&\leq \mathcal{M} \int_0^t \Big[L_{\mathscr{F}} \|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h} \\
&\quad + \widetilde{L}_{\mathscr{F}} \int_0^s \Big[\|e_2(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}) - e_2(s,\tau,0)\|_{\mathbb{X}} + \|e_2(s,\tau,0)\|_{\mathbb{X}} \Big] d\tau \\
&\quad + L_{\mathscr{F}}^* \Big] ds \\
&\leq \mathcal{M}T \Big[(\mathscr{D}_1^* r + c_n) \Big\{ L_{\mathscr{F}} + \widetilde{L}_{\mathscr{F}} T L_{e_2} \Big\} + \widetilde{L}_{\mathscr{F}} T L_{e_2}^* + L_{\mathscr{F}}^* \Big],
\end{aligned}$$

and

$$\begin{aligned}
&\Big\| \int_0^t \mathcal{S}_\alpha(t-s) \mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau \Big) ds \\
&- \int_0^t \mathcal{S}_\alpha(t-s) \mathscr{F}\Big(s, \overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}, \int_0^s e_2(s,\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)})d\tau \Big) ds \Big\|_{\mathbb{X}} \\
&\leq \int_0^t \|\mathcal{S}_\alpha(t-s)\|_{\mathscr{L}(\mathbb{X})} \Big[L_{\mathscr{F}} \|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \\
&\quad + \widetilde{L}_{\mathscr{F}} T L_{e_2} \|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \Big] ds \\
&\leq \mathcal{M}T\mathscr{D}_1^* \Big\{ L_{\mathscr{F}} + \widetilde{L}_{\mathscr{F}} T L_{e_2} \Big\} \|z - \overline{z}\|_{\mathscr{B}_T^0}.
\end{aligned}$$

(vii)

$$\begin{aligned}
&\left\| \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_3(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right)ds\right\|_{\mathbb{X}} \\
&\leq \int_0^t \|\mathcal{S}_\alpha(t-s)\|_{\mathscr{L}(\mathbb{X})}\Bigg[\Big\|\mathscr{H}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s e_3(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big) - \mathscr{H}(s,0,0)\Big\|_{\mathbb{X}} + \|\mathscr{H}(s,0,0)\|_{\mathbb{X}}\Bigg]ds \\
&\leq \mathcal{M}\int_0^t \Bigg[L_{\mathscr{H}}\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}_h} \\
&\quad + \widetilde{L}_{\mathscr{H}}\int_0^s \Big[\|e_3(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}) - e_3(s,\tau,0)\|_{\mathbb{X}} + \|e_3(s,\tau,0)\|_{\mathbb{X}}\Big]d\tau \\
&\quad + L^*_{\mathscr{H}}\Bigg]ds \\
&\leq \mathcal{M}T\left[(\mathscr{D}_1^* r + c_n)\Big\{L_{\mathscr{H}} + \widetilde{L}_{\mathscr{H}}TL_{e_3}\Big\} + \widetilde{L}_{\mathscr{H}}TL^*_{e_3} + L^*_{\mathscr{H}}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg\| \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s e_3(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\right)ds \\
&- \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, \overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}, \int_0^s e_3(s,\tau, \overline{z}_{\varrho(\tau,\overline{z}_\tau+y_\tau)} + y_{\varrho(\tau,\overline{z}_\tau+y_\tau)})d\tau\right)ds\Bigg\|_{\mathbb{X}} \\
&\leq \int_0^t \|\mathcal{S}_\alpha(t-s)\|_{\mathscr{L}(\mathbb{X})}\Bigg[L_{\mathscr{H}}\|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \\
&\quad + \widetilde{L}_{\mathscr{H}}TL_{e_3}\|z_{\varrho(s,z_s+y_s)} - \overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h}\Bigg]ds \\
&\leq \mathcal{M}T\mathscr{D}_1^*\Big\{L_{\mathscr{H}} + \widetilde{L}_{\mathscr{H}}TL_{e_3}\Big\}\|z-\overline{z}\|_{\mathscr{B}_T^0}.
\end{aligned}$$

Now, we enter the main proof of this theorem. Initially, we demonstrate that $\overline{\Upsilon}$ maps $B_r(0,\mathscr{B}_T^0)$ into $B_r(0,\mathscr{B}_T^0)$. For any $z(\cdot)\in\mathscr{B}_T^0$, by employing Remark 3.1, we sustain

$$\begin{aligned}
&\|(\overline{\Upsilon}z)(t)\|_{\mathbb{X}} \\
&\mathcal{M}\mathcal{M}_0\Big[L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L^*_{\mathscr{G}}\Big] + \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^T \mu(\tau)d\tau\right)\right\}\left(L^*_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL^*_{e_1}\right)
\end{aligned}$$

$$+ \mathcal{M}T \left\{ (L^*_{\mathscr{F}} + L^*_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L^*_{e_2} + \widetilde{L}_{\mathscr{H}} L^*_{e_3}) \right\}$$
$$+ (\mathscr{D}^*_1 r + c_n) \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$
$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \le r.$$

Therefore, $\overline{\Upsilon}$ maps the ball $B_r(0, \mathscr{B}^0_T)$ into itself. Finally, we show that $\overline{\Upsilon}$ is a contraction on $B_r(0, \mathscr{B}^0_T)$. For this, let us consider $z, \overline{z} \in B_r(0, \mathscr{B}^0_T)$, then from Remark 3.1, we sustain

$$\|(\overline{\Upsilon} z)(t) - (\overline{\Upsilon}\overline{z})(t)\|_{\mathbb{X}}$$
$$\le \mathscr{D}^*_1 \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$
$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \|z - \overline{z}\|_{\mathscr{B}^0_T}$$
$$\le \Lambda \|z - \overline{z}\|_{\mathscr{B}^0_T}.$$

From the assumption (H6) and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}$ includes a unique fixed point $z \in \mathscr{B}^0_T$ which is a mild solution of the model (1.1)-(1.2) on $(-\infty, T]$. The proof is now completed. □

4. Controllability Results

In this section, we present and prove the controllability of FNIDS with SDD of the structure (1.3)-(1.4) under Banach fixed point theorem. First, we present the mild solution for the model (1.3)-(1.4).

Definition 4.1. *A function $x : (-\infty, T] \to \mathbb{X}$, is called a mild solution of (1.3)-(1.4) on $[0, T]$, if $x_0 = \varsigma$; $x|_{[0,T]} \in C([0, T] : \mathbb{X})$; the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)}) d\tau\right)$ and $s \to \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)}) d\xi\right) d\tau$ is integrable on $[0, t)$ for all $t \in (0, T]$ and for $t \in [0, T]$ and $u \in L^2(\mathscr{I}, U)$,*

$$x(t) = \mathcal{R}_\alpha(t)\big[\varsigma(0) + \mathscr{G}(0, \varsigma(0), 0)\big] - \mathscr{G}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_1(t, s, x_{\varrho(s,x_s)}) ds\right)$$
$$- \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)}) d\tau\right) ds$$
$$- \int_0^t \int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)}) d\xi\right) d\tau ds \qquad (4.1)$$

$$+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_2(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds$$
$$+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_3(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{C}u(s)ds.$$

For the study of the structure (1.3)-(1.4), we report the further right after hypothesis:

(H6)* The following inequalities holds:
(i) Let

$$\left(\frac{1}{\gamma}\mathcal{M}^2\mathcal{M}_\mathcal{C}^2 T\right)\|x_T\| + \left(1+\frac{1}{\gamma}\mathcal{M}^2\mathcal{M}_\mathcal{C}^2 T\right)\Bigg[\mathcal{M}\mathcal{M}_0\Big[L_\mathscr{G}\|\varsigma\|_{\mathscr{B}_h} + L_\mathscr{G}^*\Big]$$
$$+\left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L_\mathscr{G}^* + \widetilde{L}_\mathscr{G} T L_{e_1}^*\right)$$
$$+\mathcal{M}T\left\{(L_\mathscr{F}^* + L_\mathscr{H}^*) + T(\widetilde{L}_\mathscr{F} L_{e_2}^* + \widetilde{L}_\mathscr{H} L_{e_3}^*)\right\}$$
$$+(\mathscr{D}_1^* r + c_n)\Bigg[\mathcal{M}T\left((L_\mathscr{F} + L_\mathscr{H}) + T(\widetilde{L}_\mathscr{F} L_{e_2} + \widetilde{L}_\mathscr{H} L_{e_3})\right)$$
$$+\left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L_\mathscr{G} + \widetilde{L}_\mathscr{G} T L_{e_1}\right)\Bigg]\Bigg] \le r,$$

for some $r > 0$.
(ii) Let

$$\Lambda = \left(1+\frac{1}{\gamma}\mathcal{M}^2\mathcal{M}_\mathcal{C}^2 T\right)\mathscr{D}_1^*\Bigg[\mathcal{M}T\left((L_\mathscr{F} + L_\mathscr{H}) + T(\widetilde{L}_\mathscr{F} L_{e_2} + \widetilde{L}_\mathscr{H} L_{e_3})\right)$$
$$+\left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T \mu(\tau)d\tau\right)\right\}\left(L_\mathscr{G} + \widetilde{L}_\mathscr{G} T L_{e_1}\right)\Bigg] < 1$$

be such that $0 \le \Lambda < 1$.

Theorem 4.1. *Assume that the conditions (H1)-(H5) and (H6)* hold. Then the control system (1.3)-(1.4) is exactly controllable on $\mathscr{I}$.*

Proof. Utilizing the hypothesis, for an arbitrary function $x(\cdot)$, choose the feedback control function as follows:

$$u_x(t) = \begin{cases} \mathcal{C}^*\mathcal{S}_\alpha^*(T-t)(\Gamma_0^T)^{-1}\Bigg[x_T - \mathcal{R}_\alpha(T)\big[\varsigma(0)+\mathscr{G}(0,\varsigma(0),0)\big] \\ +\mathscr{G}\left(T, x_{\varrho(T,x_T)}, \int_0^T e_1(T,s,x_{\varrho(s,x_s)})ds\right) \\ +\int_0^T \mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds \\ +\int_0^T\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(T-s) \\ (\times)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau,\xi,x_{\varrho(\xi,x_\xi)})d\xi\right)d\tau ds \\ -\int_0^T \mathcal{S}_\alpha(T-s)\mathscr{F}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_2(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds \\ -\int_0^T \mathcal{S}_\alpha(T-s)\mathscr{H}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_3(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds\Bigg]. \end{cases} \tag{4.2}$$

Presently, we determine the operator $\Upsilon_1 : \mathscr{B}_T \to \mathscr{B}_T$ by

$$\begin{aligned}(\Upsilon_1 x)(t) &= \mathcal{R}_\alpha(t)\big[\varsigma(0)+\mathscr{G}(0,\varsigma(0),0)\big] - \mathscr{G}\left(t, x_{\varrho(t,x_t)}, \int_0^t e_1(t,s,x_{\varrho(s,x_s)})ds\right) \\ &\quad -\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_1(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds \\ &\quad -\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\left(\tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau,\xi,x_{\varrho(\xi,x_\xi)})d\xi\right)d\tau ds \\ &\quad +\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_2(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds \\ &\quad +\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\left(s, x_{\varrho(s,x_s)}, \int_0^s e_3(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\right)ds \\ &\quad +\int_0^t \mathcal{S}_\alpha(t-s)\mathcal{C}u_x(s)ds, \quad t\in\mathscr{I}.\end{aligned}$$

Observe that the control (4.2) transfers the system (1.3)-(1.4) from the initial state ς to the final state x_T provided that the operator Υ_1 has a fixed point. To confirm the exact controllability outcome, it is adequate to demonstrate that the operator Υ_1 has a fixed point in $\mathscr{B}_T$.
We express the function $y(\cdot) : (-\infty, T] \to \mathbb{X}$ by

$$y(t) = \begin{cases}\varsigma(t), \quad t\le 0; \\ \mathcal{R}_\alpha(t)\varsigma(0), \quad t\in\mathscr{I},\end{cases}$$

then $y_0 = \varsigma$. For every function $z \in C(\mathscr{I}, \mathbb{R})$ with $z(0) = 0$, we allocate as $\widetilde{z}$ is characterized by

$$\widetilde{z}(t) = \begin{cases} 0, & t \le 0; \\ z(t), & t \in \mathscr{I}. \end{cases}$$

If $x(\cdot)$ fulfills (4.1), we are able to split it as $x(t) = y(t) + z(t)$, $t \in \mathscr{I}$, which suggests $x_t = y_t + z_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \begin{cases} \mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma,0) - \mathscr{G}\Bigg(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}, \\ \quad \displaystyle\int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\Bigg) \\ -\displaystyle\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\Bigg(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ \quad \displaystyle\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Bigg)ds - \displaystyle\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s) \\ \times\mathscr{G}\left(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}, \displaystyle\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)} + y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\right)d\tau ds \\ +\displaystyle\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\Bigg(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ \quad \displaystyle\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Bigg)ds \\ +\displaystyle\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\Bigg(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ \quad \displaystyle\int_0^s e_3(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Bigg)ds + \displaystyle\int_0^t \mathcal{S}_\alpha(t-s)\mathcal{C}u_{z+y}(s)ds, \end{cases}$$

where

$$\begin{aligned} u_{z+y}(t) = \mathcal{C}^*\mathcal{S}_\alpha^*(T-t)(\Gamma_0^T)^{-1}\Bigg[& x_T - \mathcal{R}_\alpha(T)\mathscr{G}(0,\varsigma,0) \\ & + \mathscr{G}\left(T, z_{\varrho(T,z_T+y_T)} + y_{\varrho(T,z_T+y_T)}, \int_0^T e_1(T,s,z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\right) \\ & + \int_0^T \mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\Bigg(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ & \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Bigg)ds \\ & + \int_0^T\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(T-s)\mathscr{G}\Bigg(\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)}, \end{aligned}$$

$$\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)}+y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Big)d\tau ds$$
$$-\int_0^T \mathcal{S}_\alpha(T-s)\mathscr{F}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds$$
$$-\int_0^T \mathcal{S}_\alpha(T-s)\mathscr{H}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_3(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds\Big].$$

Let $\mathscr{B}_T^0=\{z\in\mathscr{B}_T^0\colon z_0=0\in\mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|z\|_{\mathscr{B}_T^0}=\sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}}+\|z_0\|_{\mathscr{B}_h}=\sup_{t\in\mathscr{I}}\|z(t)\|_{\mathbb{X}},\quad z\in\mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0,\|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. We delimit the operator $\overline{\Upsilon}_1:\mathscr{B}_T^0\to\mathscr{B}_T^0$ by

$$(\overline{\Upsilon}_1 z)(t)=\mathcal{R}_\alpha(t)\mathscr{G}(0,\varsigma,0)-\mathscr{G}\Big(t,z_{\varrho(t,z_t+y_t)}+y_{\varrho(t,z_t+y_t)},$$
$$\int_0^t e_1(t,s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)})ds\Big)$$
$$-\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds$$
$$-\int_0^t\int_0^s \mathscr{B}(s-\tau)\mathcal{S}_\alpha(t-s)\mathscr{G}\Big(\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)},$$
$$\int_0^\tau e_1(\tau,\xi,z_{\varrho(\xi,z_\xi+y_\xi)}+y_{\varrho(\xi,z_\xi+y_\xi)})d\xi\Big)d\tau ds$$
$$+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds$$
$$+\int_0^t \mathcal{S}_\alpha(t-s)\mathscr{H}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},$$
$$\int_0^s e_3(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds+\int_0^t \mathcal{S}_\alpha(t-s)\mathcal{C}u_{z+y}(s)ds.$$

Remark 4.1. *In addition to Remark 3.1, we have the subsequent estimates:*

(i)

$$\begin{aligned}\|\mathcal{C}u_{z+y}(s)\| \le \left(\frac{1}{\gamma}\mathcal{M}\mathcal{M}_{\mathcal{C}}^2\right)\Bigg[&\|x_T\| + \mathcal{M}\mathcal{M}_0\big[L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L_{\mathscr{G}}^*\big]\\ &+ \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}}^* + \widetilde{L}_{\mathscr{G}}TL_{e_1}^*\right)\\ &+ \mathcal{M}T\left\{(L_{\mathscr{F}}^* + L_{\mathscr{H}}^*) + T(\widetilde{L}_{\mathscr{F}}L_{e_2}^* + \widetilde{L}_{\mathscr{H}}L_{e_3}^*)\right\}\\ &+ (\mathscr{D}_1^*r + c_n)\Bigg[\mathcal{M}T\left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L_{e_2} + \widetilde{L}_{\mathscr{H}}L_{e_3})\right)\\ &+ \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg]\Bigg],\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{C}u_{z+y}(s) - \mathcal{C}u_{\overline{z}+y}(s)\|\\ &\le \left(\frac{1}{\gamma}\mathcal{M}\mathcal{M}_{\mathcal{C}}^2\right)\mathscr{D}_1^*\Bigg[\mathcal{M}T\left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L_{e_2} + \widetilde{L}_{\mathscr{H}}L_{e_3})\right)\\ &\quad + \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg]\|z - \overline{z}\|_{\mathscr{B}_T^0}.\end{aligned}$$

Therefore, we have

$$\left\|\int_0^t \mathcal{S}_\alpha(t-s)\mathcal{C}u_{z+y}(s)ds\right\|_{\mathbb{X}} \le \widetilde{C_1} + \widetilde{C_2},$$

where

$$\begin{aligned}\widetilde{C_1} &= \left(\frac{1}{\gamma}\mathcal{M}^2\mathcal{M}_{\mathcal{C}}^2T\right)\|x_T\|,\\ \widetilde{C_2} &= \left(\frac{1}{\gamma}\mathcal{M}^2\mathcal{M}_{\mathcal{C}}^2T\right)\Bigg[\mathcal{M}\mathcal{M}_0\big[L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + L_{\mathscr{G}}^*\big]\\ &\quad + \left\{\mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^T \mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}}^* + \widetilde{L}_{\mathscr{G}}TL_{e_1}^*\right)\\ &\quad + \mathcal{M}T\left\{(L_{\mathscr{F}}^* + L_{\mathscr{H}}^*) + T(\widetilde{L}_{\mathscr{F}}L_{e_2}^* + \widetilde{L}_{\mathscr{H}}L_{e_3}^*)\right\}\\ &\quad + (\mathscr{D}_1^*r + c_n)\Bigg[\mathcal{M}T\left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}}L_{e_2} + \widetilde{L}_{\mathscr{H}}L_{e_3})\right)\end{aligned}$$

$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \Bigg],$$

and

$$\left\| \int_0^t \mathcal{S}_\alpha(t-s) \Big[\mathcal{C}u_{z+y}(s) - \mathcal{C}u_{\overline{z}+y}(s) \Big] ds \right\|_{\mathbb{X}}$$
$$\leq \left(\frac{1}{\gamma} \mathcal{M}^2 \mathcal{M}_{\mathcal{C}}^2 T \right) \mathscr{D}_1^* \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$
$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \|z - \overline{z}\|_{\mathscr{B}_T^0}.$$

Now, we enter the main proof of this theorem. Initially, we demonstrate that $\overline{\Upsilon}_1$ maps $B_r(0, \mathscr{B}_T^0)$ into $B_r(0, \mathscr{B}_T^0)$. For any $z(\cdot) \in \mathscr{B}_T^0$, by employing Remark 3.1 and Remark 4.1, we sustain

$$\|(\overline{\Upsilon}_1 z)(t)\|_{\mathbb{X}} \leq \widetilde{C_1} + \widetilde{C_3}$$
$$\leq r,$$

where

$$\widetilde{C_3} = \left(1 + \frac{1}{\gamma} \mathcal{M}^2 \mathcal{M}_{\mathcal{C}}^2 T \right) \Bigg[\mathcal{M}\mathcal{M}_0 \big[L_{\mathscr{G}} \|\varsigma\|_{\mathscr{B}_h} + L_{\mathscr{G}}^* \big]$$
$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}}^* + \widetilde{L}_{\mathscr{G}} T L_{e_1}^* \right)$$
$$+ \mathcal{M}T \left\{ (L_{\mathscr{F}}^* + L_{\mathscr{H}}^*) + T(\widetilde{L}_{\mathscr{F}} L_{e_2}^* + \widetilde{L}_{\mathscr{H}} L_{e_3}^*) \right\}$$
$$+ (\mathscr{D}_1^* r + c_n) \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$
$$+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \Bigg].$$

Therefore, $\overline{\Upsilon}_1$ maps the ball $B_r(0, \mathscr{B}_T^0)$ into itself. Finally, we show that $\overline{\Upsilon}_1$ is a contraction on $B_r(0, \mathscr{B}_T^0)$. For this, let us consider $z, \overline{z} \in B_r(0, \mathscr{B}_T^0)$, then from Remark 3.1 and Remark 4.1, we sustain

$$\|(\overline{\Upsilon}_1 z)(t) - (\overline{\Upsilon}_1 \overline{z})(t)\|_{\mathbb{X}}$$
$$\leq \left(1 + \frac{1}{\gamma} \mathcal{M}^2 \mathcal{M}_{\mathcal{C}}^2 T \right) \mathscr{D}_1^* \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$

$$+\left\{\mathcal{M}_0+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T\mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}}+\widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg]$$
$$\leq\Lambda\|z-\overline{z}\|_{\mathscr{B}_T^0}.$$

From the assumption (H6)* and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}_1$ includes a unique fixed point $z\in\mathscr{B}_T^0$. Thus, the model (1.3)-(1.4) is exactly controllable on $\mathscr{I}$. The proof is now completed. □

5. APPLICATIONS

Example 5.1:

To exemplify our theoretical outcomes, first we treat the FNIDS with SDD of the model

$$\begin{aligned}
&D_t^\alpha\Bigg[u(t,x)+\int_{-\infty}^t e^{2(s-t)}\frac{u(s-\varrho_1(s)\varrho_2(\|u(s)\|),x)}{49}ds\\
&+\int_0^t\sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)}{36}d\tau ds\Bigg]=\frac{\partial^2}{\partial x^2}u(t,x)\\
&+\int_0^t(t-s)^\delta e^{-\overline{\gamma}(t-s)}\frac{\partial^2}{\partial x^2}u(s,x)ds+\int_{-\infty}^t e^{2(s-t)}\frac{u(s-\varrho_1(s)\varrho_2(\|u(s)\|),x)}{9}ds\\
&+\int_0^t\sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)}{25}d\tau ds\\
&+\int_{-\infty}^t e^{2(s-t)}\frac{u(s-\varrho_1(s)\varrho_2(\|u(s)\|),x)}{64}ds\\
&+\int_0^t\sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}\frac{u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)}{16}d\tau ds,
\end{aligned}\tag{5.1}$$

$$u(t,0)=0=u(t,\pi),\quad t\in[0,T],\tag{5.2}$$

$$u(t,x)=\varsigma(t,x),\quad t\leq 0,\quad x\in[0,\pi],\tag{5.3}$$

where D_t^α is Caputo's fractional derivative of order $\alpha\in(1,2),\delta$ and $\overline{\gamma}$ are positive numbers and $\varsigma\in\mathscr{B}_h$. We consider $\mathbb{X}=L^2[0,\pi]$ having the norm $|\cdot|_{L^2}$ and determine the operator $\mathscr{A}:D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ by $\mathscr{A}w=w''$ with the domain

$$D(\mathscr{A})=\{w\in\mathbb{X}:w,w'\quad\text{are absolutely continuous},\quad w''\in\mathbb{X},\,w(0)=w(\pi)=0\}.$$

Then

$$\mathscr{A}w=\sum_{n=1}^\infty n^2\langle w,w_n\rangle w_n,\quad w\in D(\mathscr{A}),$$

in which $w_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns)$, $n=1,2,\ldots,$. is the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq0}$ in $\mathbb{X}$

and is provided by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}\langle w, w_n\rangle w_n, \quad \text{for all} \quad w \in \mathbb{X}, \quad \text{and every} \quad t > 0.$$

Hence (H1) is fulfilled. If we fix $\vartheta = \frac{1}{2}$, then the operator $(-\mathscr{A})^{\frac{1}{2}}$ is given by

$$(-\mathscr{A})^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n\langle w, w_n\rangle w_n, \ \ w \in (D(-\mathscr{A})^{\frac{1}{2}}),$$

in which $(D(-\mathscr{A})^{\frac{1}{2}}) = \left\{ \omega(\cdot) \in \mathbb{X} : \sum_{n=1}^{\infty} n\langle \omega, w_n\rangle w_n \in \mathbb{X} \right\}$ and $\|(-\mathscr{A})^{-\frac{1}{2}}\| = 1$. Therefore, $\mathscr{A}$ is sectorial of type and the properties (P1) hold. We also take into account the operator $\mathscr{B}(t) : D(\mathscr{A}) \subseteq \mathbb{X} \to \mathbb{X}$, $t \geq 0$, $\mathscr{B}(t)x = t^{\delta} e^{-\overline{\gamma} t}\mathscr{A}x$ for $x \in D(\mathscr{A})$. In addition, it is simple to see that conditions (P2)-(P3)[for more details, refer [23]] are fulfilled with $b(t) = t^{\delta} e^{-\overline{\gamma} t}$ and $D = C_0^{\infty}([0,\pi])$, where $C_0^{\infty}([0,\pi])$ is the space of infinitely differentiable functions that vanish at $x = 0$ and $x = \pi$. From the Lemma 2.4, it is simple to see that condition (H2) is fulfills. For the phase space, we choose $h = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \dfrac{1}{2} < \infty$, for $t \leq 0$ and determine

$$\|\varsigma\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\varsigma(\theta)\|_{L^2} ds.$$

Hence, for $(t, \varsigma) \in [0, T] \times \mathscr{B}_h$, where $\varsigma(\theta)(x) = \varsigma(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$u(t)(x) = u(t, x), \quad \varrho(t, \varsigma) = \varrho_1(t)\varrho_2(\|\varsigma(0)\|),$$

we have

$$\mathscr{G}(t, \varsigma, \overline{\mathscr{H}}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{\varsigma}{49} ds + (\overline{\mathscr{H}}\varsigma)(x),$$

$$\mathscr{F}(t, \varsigma, \widetilde{\mathscr{H}}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{\varsigma}{9} ds + (\widetilde{\mathscr{H}}\varsigma)(x),$$

$$\mathscr{H}(t, \varsigma, \widehat{\mathscr{H}}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{\varsigma}{64} ds + (\widehat{\mathscr{H}}\varsigma)(x),$$

where

$$(\overline{\mathscr{H}}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{36} d\tau ds,$$

$$(\widetilde{\mathscr{H}}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{25} d\tau ds,$$

$$(\widehat{\mathscr{H}}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{\varsigma}{16} d\tau ds,$$

then using these configurations, the system (5.1)-(5.3) is usually written in the theoretical form of design (1.1)-(1.2).

To treat this system we assume that $\varrho_i : [0,\infty) \to [0,\infty),\ i = 1,2$ are continuous. Now, we can see that for $t \in [0,T], \varsigma, \overline{\varsigma} \in \mathscr{B}_h$, we have

$$\begin{aligned}
&\|(-\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\varsigma,\overline{\mathscr{H}}\varsigma)\|_{\mathbb{X}} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{49}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{36}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma\| ds + \frac{1}{36}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{49}\|\varsigma\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{36}\|\varsigma\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{G}}\|\varsigma\|_{\mathscr{B}_h},
\end{aligned}$$

where $L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} = \frac{85\sqrt{\pi}}{1764}$, and

$$\begin{aligned}
&\|(-\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\varsigma,\overline{\mathscr{H}}\varsigma) - (-\mathscr{A})^{\frac{1}{2}}\mathscr{G}(t,\overline{\varsigma},\overline{\mathscr{H}}\overline{\varsigma})\|_{\mathbb{X}} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{49} - \frac{\overline{\varsigma}}{49}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{36} - \frac{\overline{\varsigma}}{36}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{49}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma - \overline{\varsigma}\| ds + \frac{1}{36}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma - \overline{\varsigma}\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{49}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{36}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{G}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{G}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}.
\end{aligned}$$

Similarly, we conclude

$$\begin{aligned}
&\|\mathscr{F}(t,\varsigma,\widetilde{\mathscr{H}}\varsigma)\|_{L^2} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varsigma}{9}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varsigma}{25}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma\| ds + \frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\varsigma\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{9}\|\varsigma\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{25}\|\varsigma\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{F}}\|\varsigma\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{F}}\|\varsigma\|_{\mathscr{B}_h},
\end{aligned}$$

where $L_{\mathscr{F}} + \widetilde{L}_{\mathscr{F}} = \frac{34\sqrt{\pi}}{225}$, and

$$\begin{aligned}
&\|\mathscr{F}(t,\varsigma,\widetilde{\mathscr{H}\varsigma}) - \mathscr{F}(t,\overline{\varsigma},\widetilde{\mathscr{H}\overline{\varsigma}})\|_{L^2} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)} \left\|\frac{\varsigma}{9} - \frac{\overline{\varsigma}}{9}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)} \left\|\frac{\varsigma}{25} - \frac{\overline{\varsigma}}{25}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds + \frac{1}{25}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{9}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{25}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{F}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{F}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}.
\end{aligned}$$

Correspondingly, we have

$$\begin{aligned}
&\|\mathscr{H}(t,\varsigma,\widehat{\mathscr{H}\varsigma})\|_{L^2} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)} \left\|\frac{\varsigma}{64}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)} \left\|\frac{\varsigma}{16}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds + \frac{1}{16}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{64}\|\varsigma\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{16}\|\varsigma\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{H}}\|\varsigma\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{H}}\|\varsigma\|_{\mathscr{B}_h},
\end{aligned}$$

where $L_{\mathscr{H}} + \widetilde{L}_{\mathscr{H}} = \frac{80\sqrt{\pi}}{1024}$, and

$$\begin{aligned}
&\|\mathscr{H}(t,\varsigma,\widehat{\mathscr{H}\varsigma}) - \mathscr{H}(t,\overline{\varsigma},\widehat{\mathscr{H}\overline{\varsigma}})\|_{L^2} \\
&\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(s)} \left\|\frac{\varsigma}{64} - \frac{\overline{\varsigma}}{64}\right\| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)} \left\|\frac{\varsigma}{16} - \frac{\overline{\varsigma}}{16}\right\| d\tau ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds + \frac{1}{16}\int_{-\infty}^0 e^{2(s)} \sup\|\varsigma - \overline{\varsigma}\| ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{64}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \frac{\sqrt{\pi}}{16}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} \\
&\leq L_{\mathscr{H}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{H}}\|\varsigma - \overline{\varsigma}\|_{\mathscr{B}_h}.
\end{aligned}$$

Therefore the conditions (H3) and (H5) are all fulfilled. Furthermore, we assume that $\mathscr{D}_1^* = 1, \mathcal{M}_0 = 1, \mathcal{M} = 1, T = 1, \alpha = \frac{3}{2}, L_{e_1} = 1, L_{e_2} = 1, L_{e_3} = 1$ and $\int_0^1 \mu(\tau)d\tau = 1$. Then

$$\mathscr{D}_1^* \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right) + \left\{ \mathcal{M}_0 + \frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left(L_{\mathscr{G}} + \widetilde{L}_{\mathscr{G}} T L_{e_1} \right) \Bigg] \approx 0.71941 < 1.$$

Thus the condition (H6) holds. Hence by Theorem 3.1, we realize that the system (5.1)–(5.3) has a unique mild solution on $[0, 1]$.

Example 5.2:
In this section, as an application of Theorem 4.1, we treat the FNIDS with SDD of the model

$$\begin{aligned}
D_t^\alpha \Bigg[& u(t,x) + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s)\varrho_2(\|u(s)\|), x)}{49} ds \\
& + \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau - s)} \frac{u(\tau - \varrho_1(\tau)\varrho_2(\|u(\tau)\|), x)}{36} d\tau ds \Bigg] = \frac{\partial^2}{\partial x^2} u(t,x) \\
& + \int_0^t (t-s)^\delta e^{-\overline{\gamma}(t-s)} \frac{\partial^2}{\partial x^2} u(s,x) ds + \nu(t,x) + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s)\varrho_2(\|u(s)\|), x)}{9} ds \\
& + \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau - s)} \frac{u(\tau - \varrho_1(\tau)\varrho_2(\|u(\tau)\|), x)}{25} d\tau ds \\
& + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s)\varrho_2(\|u(s)\|), x)}{64} ds \\
& + \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau - s)} \frac{u(\tau - \varrho_1(\tau)\varrho_2(\|u(\tau)\|), x)}{16} d\tau ds, \qquad (5.4)
\end{aligned}$$

with the conditions (5.2)-(5.3), $D_t^\alpha, \alpha, \delta$ and $\overline{\gamma}$ are same as defined in Example 5.1. Further, we define the operator $\mathcal{C} : U \to \mathbb{X}$ by $\mathcal{C}u(t,x) = \nu(t,x), 0 < x < \pi, u \in U$, where $\nu : [0,1] \times [0,\pi] \to [0,\pi]$. In perspective of Example 5.1 and using these configurations, the system (5.4) with the conditions (5.2)-(5.3) is usually written in the theoretical form of design (1.3)-(1.4).

Furthermore, we assume that $\mathscr{D}_1^* = \frac{1}{2}, \mathcal{M}_0 = 1, \mathcal{M} = 1, \mathcal{M}_{\mathcal{C}} = 1, \gamma = 1, T = 1, \alpha = \frac{3}{2}, L_{e_1} = 1, L_{e_2} = 1, L_{e_3} = 1$ and $\int_0^1 \mu(\tau)d\tau = 1$. Then

$$\left(1 + \frac{1}{\gamma} \mathcal{M}^2 \mathcal{M}_{\mathcal{C}}^2 T \right) \mathscr{D}_1^* \Bigg[\mathcal{M}T \left((L_{\mathscr{F}} + L_{\mathscr{H}}) + T(\widetilde{L}_{\mathscr{F}} L_{e_2} + \widetilde{L}_{\mathscr{H}} L_{e_3}) \right)$$

$$+\left\{\mathcal{M}_0+\frac{\mathcal{M}T^{\alpha\vartheta}}{\alpha\vartheta}\left(1+\int_0^T\mu(\tau)d\tau\right)\right\}\left(L_{\mathscr{G}}+\widetilde{L}_{\mathscr{G}}TL_{e_1}\right)\Bigg]\approx 0.71941<1.$$

Thus the condition (H6)* holds. Hence by Theorem 4.1, we realize that the system (5.4) with the conditions (5.2)–(5.3) has a unique mild solution on $[0, 1]$.

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REFERENCES

[1] D. Baleanu, J.A.T. Machado, and A.C.J. Luo, *Fractional Dynamics and Control*, Springer, New York, USA, (2012).

[2] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, (2006).

[3] G. Bonanno, R. Rodriguez-Lopez, and S. Tersian, *Existence of solutions to boundary value problem for impulsive fractional differential equations*, Fractional Calculus and Applied Analysis, **17**(3) (2014), 717–744.

[4] R. Rodriguez-Lopez and S. Tersian, *Multiple solutions to boundary value problem for impulsive fractional differential equations*, Fractional Calculus and Applied Analysis, **17**(4) (2014), 1016–1038.

[5] R.P. Agarwal, V. Lupulescu, D. O'Regan, and G. Rahman, *Fractional calculus and fractional differential equations in nonreflexive Banach spaces*, Communications in Nonlinear Science and Numerical Simulation, **20**(1) (2015), 59–73.

[6] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, *Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations*, Applied Mathematical Modelling, **38** (2014), 6038–6051.

[7] Z. Lv and B. Chen, *Existence and uniqueness of positive solutions for a fractional switched system*, Abstract and Applied Analysis, **2014** (2014), Article ID 828721, 7 pages.

[8] Y. Wang, L. Liu, and Y. Wu, *Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters*, Advances in Difference Equations, **2014**(268), 1–24.

[9] B. Ahmad, S.K. Ntouyas, and A. Alsaed, *Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions*, Advances in Difference Equations, **2014**(257), 1–18.

[10] R.P. Agarwal and B.D. Andrade, *On fractional integro-differential equations with state-dependent delay*, Comp. Math. App., **62** (2011), 1143–1149.

[11] M. Benchohra and F. Berhoun, *Impulsive fractional differential equations with state-dependent delay*, Commun. Appl. Anal., **14**(2) (2010), 213–224.

[12] K. Aissani and M. Benchohra, *Fractional integro-differential equations with state-dependent delay*, Advances in Dynamical Systems and Applications, **9**(1) (2014), 17–30.

[13] J. Dabas and G.R. Gautam, *Impulsive neutral fractional integro-differential equation with state-dependent delay and integral boundary condition*, Electronic Journal of Differential Equations, **2013**(273) (2013), 1–13.

[14] S. Suganya, M. Mallika Arjunan, and J.J. Trujillo, *Existence results for an impulsive fractional integro-differential equation with state-dependent delay*, Applied Mathematics and Computation, **266** (2015), 54–69.

[15] J.P.C. Dos Santos, C. Cuevas, and B. de Andrade, *Existence results for a fractional equations with state-dependent delay*, Advances in Difference Equations, (2011), 1–15.

[16] J.P.C. Dos Santos, M. Mallika Arjunan, and C. Cuevas, *Existence results for fractional neutral integrodifferential equations with state-dependent delay*, Comput. Math. Appl., **62** (2011), 1275–1283.

[17] Z. Yan, *Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces*, IMA Journal of Mathematical Control and Information, **30** (2013), 443–462.

[18] Z. Yan and X. Jia, *Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay*, Collect. Math., **66** (2015), 93–124.

[19] V. Vijayakumar, C. Ravichandran, and R. Murugesu, *Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay*, Nonlinear Studies, **20**(4) (2013), 513–532.

[20] R.P. Agarwal, J.P.C. Dos Santos, and C. Cuevas, *Analytic resolvent operator and existence results for fractional integro-differential equations*, J. Abstr. Differ. Equ. Appl., **2**(2) (2012), 26–47.

[21] V. Vijayakumar, A. Selvakumar, and R. Murugesu, *Controllability for a class of fractional neutral integro-differential equations with unbounded delay*, Applied Mathematics and Computation, **232** (2014), 303–312.

[22] C. Rajivganthi, P. Muthukumar, and B. Ganesh Priya, *Approximate controllability of fractional stochastic integro-differential equations with infinite delay of order* $1 < \alpha < 2$, IMA Journal of Mathematical Control and Information, (2015), 1–15, Available Online.

[23] B.D. Andrade and J.P.C. Dos Santos, *Existence of solutions for a fractional neutral integro-differential equation with unbounded delay*, Electronic Journal of Differential Equations, **2012**(90) (2012), 1–13.

[24] J.P.C. Dos Santos, V. Vijayakumar, and R. Murugesu, *Existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro-differential equation with unbounded delay*, Communications in Mathematical Analysis, **X** (2011), 1–13.

[25] X. Shu and Q. Wang, *The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order* $1 < \alpha < 2$, Comput. Math. Appl., **64** (2012), 2100–2110.

[26] Z. Yan and X. Jia, *Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators*, Mediterr. Math., **11** (2014), 393–428.

[27] Z. Yan and F. Lu, *On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay*, Applicable Analysis, (2014), 1235–1258.

[28] L. Kexue, P. Jigen, and G. Jinghuai, *Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces*, Rep. Math. Phys., **71** (2013), 33–43.

[29] R. Sakthivel, R. Ganesh, Y. Ren, and S. Marshal Anthoni, *Approximate controllability of nonlinear fractional dynamical systems*, Commun. Nonlinear Sci. Numer. Simulat., **18** (2013), 3498–3508.

[30] J. Dabas, A. Chauhan, and M. Kumar, *Existence of the mild solutions for impulsive fractional equations with infinite delay*, International Journal of Differential Equations, **2011**, Article ID 793023, 20 pages.

[31] J. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21** (1978), 11–41.

[32] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Unbounded Delay,* Springer-Verlag, Berlin, (1991).

[33] X. Fu and R. Huang, *Existence of solutions for neutral integro-differential equations with state-dependent delay*, Appl. Math. Comp., **224** (2013), 743–759.

[34] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, (1983).

[35] N.I. Mahmudov and A. Denker, *On controllability of linear stochastic systems*, Int. J. Control, **73** (2000), 144–151.