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ABSTRACT. The continuous analysis, such as smoothness and uniform convergence, for polynomials and polynomial-like functions using differential operators have been studied considerably, parallel to the study of discrete analysis for these functions, using difference operators.

In this work, for the difference operator ∇_h with size $h > 0$, we verify that for an integer $m \geq 0$ and a strictly decreasing sequence h_n converging to zero, a continuous function $f(x)$ satisfying

$$\nabla_{h_n}^{m+1} f(kh_n) = 0, \quad \text{for every } n \geq 1 \text{ and } k \in \mathbb{Z},$$

turns to be a polynomial of degree $\leq m$. The proof used the polynomial convergence, and additionally, we investigated several conditions on convergence to polynomials.

1. INTRODUCTION

In recent decades, the problem of approximation by a linear combination of integer translates of one or more basis functions has arisen, especially in the study of Wavelet and Computer Aided Geometric Design (CAGD). In CAGD, polynomials and polynomial-like functions are used as basis functions, for example, Bernstein polynomials, B-spline polynomials, Box splines and so on. In drawing curves and surfaces using computers, polynomials and polynomial-like functions are playing much important roles. In turn, the continuous analysis, such as smoothness and uniform convergence, for these functions have been studied considerably, in harmony with the study of discrete analogue for these functions. For more details, we refer to [1] and [2].

In polynomial applications, we come across the difference operators and the polynomial (point-wise) convergence in analyzing the smoothness of the curves or surfaces. The difference operators and the convergence of polynomial sequences provide efficient implements in

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estimating whether a subdivision scheme generates polynomials. Just as the differential operator is to the continuous analysis, so is the difference operator to the discrete analysis.

In this work, we consider the divided difference equations and several conditions on the convergence to polynomial. The backward and forward (divided) difference operators with size $h > 0$ are defined as

$$\nabla_h f(x) := f(x) - f(x - h) \quad \text{and} \quad \Delta_h f(x) := f(x + h) - f(x)$$

for a function $f(x)$. The iterated operators are written as

$$\nabla_h^m f(x) := \nabla_h (\nabla_h^{m-1} f(x)) \quad \text{and} \quad \Delta_h^m f(x) := \Delta_h (\Delta_h^{m-1} f(x)).$$

Note that we have

$$\nabla_h^m f(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(x - kh) \quad \text{and} \quad \Delta_h^m f(x) = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} f(x + kh).$$

Then for a polynomial $p(x)$ of degree $\leq m$, we easily see, like differential equations, that $\nabla_h^{m+1} p \equiv 0$ and $\Delta_h^{m+1} p \equiv 0$ for any h .

Conversely, we verify that for a strictly decreasing sequence $\{h_n\}_{n=1}^{\infty}$ convergent to zero, a continuous function f satisfying the divided difference equations of order $m + 1$

$$\nabla_{h_n}^{m+1} f(kh_n) = 0, \quad \text{for every } n \geq 1 \text{ and } k \in \mathbb{Z},$$

is also a polynomial of degree $\leq m$. However, the continuity condition on solutions of the divided difference equations is necessary. For example, a real valued function f defined on the whole real line \mathbb{R} satisfying the linearity :

$$f(x + y) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{R},$$

which yields f to satisfy the differences

$$\nabla_h^2 f \equiv 0 \quad \text{and} \quad \Delta_h^2 f \equiv 0 \quad \text{for every } h > 0.$$

Such a function f seems to be a line. Surprisingly, there are functions which satisfy the linearity but are not continuous at any point. We construct a function satisfying the linearity but are not continuous at any point (see Theorem 2.1). And we study several conditions on the convergence to polynomial; we investigate the convergence in the dyadic set (Theorem 3.1 and Corollary 3.2), and we consider general cases (Theorem 3.3 and Remark 3.4). In Section 4, we generalize Theorem 2.1 to Theorem 4.2 and Corollary 4.3 using the polynomial convergence based on the results in Section 3, and give some analogy of a differential equation with respect to a linear combination of discrete difference operators. Even though the results have been widely used in Approximation theory and Functional Analysis, their rigorous proofs have not been given yet as far as we know.

2. CHARACTERIZATION OF SOLUTIONS TO DIFFERENCE EQUATIONS

In the section, we investigate the solution to the backward difference equations

$$\nabla_h^{m+1} f \equiv 0, \quad \text{for every } h = 1/2^\ell, \ell = 0, 1, 2, \dots \quad (2.1)$$

To this end, we introduce some notations and terminologies. By \mathbb{N}_0 we denote the set of all non-negative integers and by K the dyadic set defined as

$$K = \{i/2^j : i \in \mathbb{Z}, j \in \mathbb{N}_0\},$$

where \mathbb{Z} stands for the set of all integers. For given $m + 1$ distinct (real or complex) numbers $\{x_k\}_{k=1}^{m+1}$, we introduce the Lagrange polynomials $\{\ell_k(x)\}_{k=1}^{m+1}$ defined by

$$\ell_k(x) = \frac{w(x)}{w'(x_k)(x - x_k)}, \quad k = 1, 2, \dots, m + 1,$$

where $w(x) = (x - x_1)(x - x_2) \cdots (x - x_{m+1})$. Then it is well-known that $\ell_k(x)$ is a unique polynomial of degree $\leq m$ having the properties :

$$\ell_k(x_j) = \delta_{kj} = \begin{cases} 0, & \text{if } k \neq j, \\ 1, & \text{if } k = j. \end{cases}$$

For a given function f , the interpolation polynomial $p_m(f; x)$ of degree $\leq m$ satisfying the conditions:

$$p_m(f; x_k) = f(x_k), \quad k = 1, 2, \dots, m + 1$$

is written as

$$p_m(f; x) = \sum_{k=1}^{m+1} f(x_k) \ell_k(x). \quad (2.2)$$

In particular, if f is a polynomial of degree $\leq m$, then Equation (2.2) implies that $f(x)$ is identically equal to $p_m(f; x)$.

Theorem 2.1. *Let f be a function defined on the whole real line \mathbb{R} . Assume that there is an integer $m \geq 0$ such that*

$$\nabla_h^{m+1} f(jh) = 0, \quad \text{for every } h = 1/2^\ell, \ell \in \mathbb{N}_0, \text{ and } j \in \mathbb{Z}. \quad (2.3)$$

If f is continuous, then f is a polynomial of degree $\leq m$. Moreover, there is a discontinuous function that satisfies

$$\nabla_h^{m+1} f \equiv 0$$

for any size $h > 0$.

Proof. Assume that f is continuous on \mathbb{R} . Fix ℓ to be a non-negative integer. Then Equation (2.3) implies that for every $j \in \mathbb{Z}$, we have

$$\nabla_h^{m+1} f(j/2^\ell) = \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} f\left(\frac{j-k}{2^\ell}\right) = 0, \quad (2.4)$$

where $h = 1/2^\ell$. Let $p_\ell(x)$ be the polynomial of degree $\leq m$ interpolating to $f(x)$ at $x = j/2^\ell$, $j = 0, 1, 2, \dots, m$. Then $p_\ell(x)$ satisfies the difference equation (2.3). Since the equation

$$\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} y_{i-k} = 0, \quad i \in \mathbb{Z}$$

has a unique solution $\{y_i\}_{i \in \mathbb{Z}}$ with initial values y_0, y_1, \dots, y_m , we induce that

$$f\left(\frac{i}{2^\ell}\right) = p_\ell\left(\frac{i}{2^\ell}\right), \quad \text{for every } i \in \mathbb{Z}. \quad (2.5)$$

By the same argument with $h = 1/2^{\ell+1}$, we can see that there is a polynomial $p_{\ell+1}(x)$ of degree $\leq m$ such that

$$f\left(\frac{i}{2^{\ell+1}}\right) = p_{\ell+1}\left(\frac{i}{2^{\ell+1}}\right), \quad \text{for every } i \in \mathbb{Z}. \quad (2.6)$$

Equations (2.5) and (2.6) show that the polynomials $p_\ell(x)$ and $p_{\ell+1}(x)$ have the same values at all the points $x = i/2^\ell, i \in \mathbb{Z}$. Thus p_ℓ and $p_{\ell+1}$ are the same. Continuing this process, we have that there is a polynomial $p(x)$ of degree $\leq m$ such that

$$f(x) = p(x), \quad \text{for every } x \in K.$$

Since the set K is dense in \mathbb{R} , for every $x \in \mathbb{R}$, there is a sequence $\{y_k\}_{k=0}^\infty$ in K converging to x . The continuity of f at x implies that

$$f(x) = \lim_{k \rightarrow \infty} f(y_k) = \lim_{k \rightarrow \infty} p(y_k) = p(x).$$

Thus we have that $f \equiv p$, which proves the first claim.

Now, we shall construct a discontinuous function satisfying (2.3), which is somewhat in the abstract.

We regard the space of real numbers \mathbb{R} as an infinite dimensional vector space over the rational field \mathbb{Q} , the set of all rational numbers. Using Zorn's lemma ([3, Chapter 4]), we deduce that \mathbb{R} has a (Hamel) basis $= \{e_\alpha\}_{\alpha \in I}$ including $1, \sqrt{2}$. Let B denote the basis. Then every real number x has a unique representation as a linear combination of finitely many elements of B ,

$$x = c_1 + c_2\sqrt{2} + \sum_{k=1}^n c_k e_{\alpha_k}, \quad (2.7)$$

where c_1, c_2 , and c_k are nonzero rational numbers depending on x . Now, define a function f on \mathbb{R} as follows. First, define the value of f on B as

$$f(1) = 0, f(\sqrt{2}) = 1, \quad \text{and} \quad f(e_\alpha) = 0, \quad \forall \alpha \in I,$$

and then for every $x \in \mathbb{R}$ in the form (2.7), $f(x)$ is defined as

$$f(x) = f\left(c_1 + c_2\sqrt{2} + \sum_{k=1}^n c_k e_{\alpha_k}\right) := c_1 f(1) + c_2 f(\sqrt{2}) + \sum_{k=1}^n c_k f(e_{\alpha_k}).$$

By the definition of f , we find that f satisfies the linearity:

- (i) $f(rx) := rf(x)$ for every $r \in \mathbb{Q}$ and $x \in \mathbb{R}$;
- (ii) $f(x+y) := f(x) + f(y)$ for every $x, y \in \mathbb{R}$.

In other words, $f(x)$ is the \mathbb{Q} -linear function from \mathbb{R} to \mathbb{Q} sending a real number to its $\sqrt{2}$ -coordinate with respect to the basis. Let y be a fixed real number. We can also find sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of rational numbers converging to $\sqrt{2}$ and y , respectively. We have from the definition of f that

$$f(y_n) = 0, \quad \text{for all } n \geq 1.$$

On the other hand, the sequence $\{x_n^{-1}y_n\sqrt{2}\}_{n=1}^{\infty}$ converges to y and

$$\lim_{n \rightarrow \infty} f\left(\frac{y_n}{x_n}\sqrt{2}\right) = \lim_{n \rightarrow \infty} \frac{y_n}{x_n} = \frac{y}{\sqrt{2}},$$

which shows the discontinuity of f at $y \neq 0$. Also, we see that sequence $\{\frac{\sqrt{2}}{x_n} - 1\}$ converges to zero as n tends to ∞ , but

$$\lim_{n \rightarrow \infty} f\left(\frac{\sqrt{2}}{x_n} - 1\right) = \lim_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{\sqrt{2}} \neq f(0).$$

Hence, we have shown that f is discontinuous at every point in \mathbb{R} .

Note that the function f satisfies

$$\nabla_h^2 f \equiv 0, \quad \forall h > 0 \quad \text{and} \quad \nabla_h f \equiv 0, \quad \forall h = \frac{1}{2^\ell}, \ell \in \mathbb{N}_0.$$

We define f_m by $f_m(x) := (f(x))^m$. Using the identities

$$\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} k^j = \begin{cases} 0 & \text{if } 0 \leq j < m+1, \\ (-1)^{m+1} (m+1)! & \text{if } j = m+1, \end{cases}$$

we can see that

$$\nabla_h^{m+1} f_m \equiv 0 \quad \forall h > 0$$

but for any real number x , there exists at least one (and therefore infinitely many) $h > 0$ such that

$$\nabla_h^m f_m(x) \neq 0.$$

□

Definition 2.2. We call the function f in the proof of the second argument of Theorem 2.1 as the $\sqrt{2}$ -coordinate function, and f_m as the m th-power $\sqrt{2}$ -coordinate function.

A subtle change of expression of Theorem 2.1 makes the following corollary.

Corollary 2.3. Let f be a continuous function on the set of real numbers. Then there is an integer $m \geq 0$ such that

$$\nabla_h^{m+1} f \equiv 0, \quad \text{for every } h = 1/2^\ell, \ell \in \mathbb{N}_0$$

if and only if f is a polynomial of degree $\leq m$.

The property for the forward difference follows from the same argument of proof of Theorem 2.1:

Corollary 2.4. *Let f be a function defined on the whole real line. Assume that there is an integer $m \geq 0$ such that*

$$\Delta_h^{m+1} f(jh) = 0, \quad \text{for every } h = 1/2^\ell, \ell \in \mathbb{N}_0, \text{ and } j \in \mathbb{Z}.$$

If f is continuous, then f is a polynomial of degree $\leq m$.

Now we give some generalization for Theorem 2.1. We write the condition (2.3) for Theorem 2.1 as follows:

$$\nabla_h^{m+1} f = 0 \text{ in } K \text{ where } h = 1/2^\ell \text{ for infinitely many } \ell \in \mathbb{N}_0. \quad (2.8)$$

By changing K into some dense subgroup D of \mathbb{R} , we have the following.

Theorem 2.5. *Let D be a subgroup of the additive group \mathbb{R} which is dense in \mathbb{R} . Suppose D has a set of generators $B = \{h_1, h_2, h_3, \dots\}$ such that for all $i \in \mathbb{N}$, $h_{i-1} = n_i h_i$ for some integer n_i . Assume that for a real valued function f defined on \mathbb{R} , there is an integer $m \geq 0$ such that*

$$\nabla_{h_n}^{m+1} f = 0 \text{ in } D, \quad \text{for all } n \in \mathbb{N}. \quad (2.9)$$

Then f is a polynomial of degree $\leq m$ if and only if f is continuous.

Proof. The proof is the same as that of the first part of Theorem 2.1 by replacing $1/2^\ell$ into h_n with $n = \ell + 1$. \square

Remark 2.6. *Let D be a dense subgroup of \mathbb{R} . The followings are equivalent:*

- (1) *D has a set of generators $B = \{b_1, b_2, b_3, \dots\}$ such that any two elements b_i, b_j in B has a rational ratio, that is, $b_i = r_{i,j} b_j$ for some rational number $r_{i,j}$.*
- (2) *D has a set of generators $C = \{h_1, h_2, h_3, \dots\}$ with the property $h_{i-1} = n_i h_i$ for some integer n_i .*

Proof. (2) trivially implies (1). Suppose that (1) holds. First we notice that there is a real number α such that any element of D can be written as $q\alpha$ for some rational number q . Take $h_1 = b_1$. Inductively, suppose we have elements h_1, h_2, \dots, h_i of D such that $h_{j-1} = n_j h_j$ for $j = 2, 3, \dots, i$, and the subgroup $\langle h_1, h_2, \dots, h_i \rangle = \langle h_i \rangle = \{k h_i : k \in \mathbb{Z}\}$ generated by those elements is equal to the subgroup $\langle b_1, b_2, \dots, b_i \rangle$. The subgroup generated by h_i and b_{i+1} has one generator. Indeed, if we denote $h_i = (k_1/k_2)\alpha$ and $b_{i+1} = (k_3/k_4)\alpha$ for integers k_1, k_2, k_3 , and k_4 , then

$$\langle h_i, b_{i+1} \rangle = \left\langle \frac{\gcd(k_1 k_4, k_3 k_2)}{k_2 k_4} \alpha \right\rangle.$$

Take $h_{i+1} \in D$ the generator. Then $\langle h_{i+1} \rangle = \langle h_1, h_2, \dots, h_{i+1} \rangle = \langle b_1, b_2, \dots, b_{i+1} \rangle$ and there is an integer n_{i+1} such that $h_i = n_{i+1} h_{i+1}$. \square

If a dense subgroup is not of type in Remark 2.6, then there exist at least two elements with irrational ratio. But a subgroup containing two elements with irrational ratio is dense in \mathbb{R} , e.g.

$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\} = \langle 1, \sqrt{2} \rangle = \mathbb{Z}[\sqrt{2}]$ is dense. Indeed, let α and β be two nonzero (we may assume positive) real numbers such that α/β is not a rational number. To show that the group $\langle \alpha, \beta \rangle$ is dense, it is enough to verify that for any positive $\gamma \in \langle \alpha, \beta \rangle$, there exists a positive element in $\langle \alpha, \beta \rangle$ less than $\gamma/2$. Let $\gamma = a\alpha + b\beta \in \langle \alpha, \beta \rangle$ be positive and suppose $b \neq 0$ where a, b are integers. We may assume $0 < \gamma < \alpha$ by subtracting $k\alpha$ for a suitable integer k . Take a positive integer m such that $m\gamma < \alpha < (m+1)\gamma$, that is, $m = \lfloor \frac{\alpha}{\gamma} \rfloor$. Then one of the positive differences $(m+1)\gamma - \alpha$ or $\alpha - m\gamma$ should be less than $\gamma/2$, and it is in the subgroup $\langle \alpha, \beta \rangle$, with coefficient of β equal to $(m+1)b$ or $-mb$, any of which is not zero. For the case $\gamma = a\alpha$, change the roles of α and β . Hence the subgroup $\langle \alpha, \beta \rangle$ is dense subgroup of \mathbb{R} .

Now, set $h_1 = \beta - k\alpha$ where the integer k satisfying $0 < h_1 < \alpha$ and make the sequence $\{h_n\}_{n \in \mathbb{N}}$ by the above process: for example, we get a decreasing sequence

$$\{\sqrt{2} - 1 \approx 0.414, 3 - 2\sqrt{2} \approx 0.172, 17 - 12\sqrt{2} \approx 0.0294, \dots\}$$

for the group $\langle 1, \sqrt{2} \rangle = \mathbb{Z}[\sqrt{2}]$. The obtained sequence generates the subgroup $\langle \alpha, \beta \rangle$. Indeed, write $h_n = a_n\alpha + b_n\beta$ for $a_n, b_n \in \mathbb{Z}$. Since $h_{n+1} = (m_n + 1)h_n - \alpha$ or $h_{n+1} = \alpha - m_n h_n$ for the chosen integer m_n , we have $\alpha \in \langle h_n, h_{n+1} \rangle$ and the coefficient b_{n+1} of β in h_{n+1} is a multiple of b_n . Therefore, we have

$$\langle h_n, h_{n+1} \rangle = \langle \alpha, b_n\beta \rangle,$$

and, since $b_1 = 1$, the decreasing sequence $\{h_n\}$ is a generating set of the group $\langle \alpha, \beta \rangle$.

In summary, for every dense group we can find a dense subgroup with a constructible set of generators as a sequence $\{h_n\}_{n \in \mathbb{N}}$ of positive numbers decreasing and converging to zero. Then, with such a sequence $\{h_n\}_{n=1}^{\infty}$, it is enough to consider the union of subgroups generated by each h_n : $\{kh_n : k \in \mathbb{Z}, n \in \mathbb{N}\}$, in order to obtain the results given here in the work, as may be seen in Theorem 2.5. For any decreasing sequence $\{h_n\}_{n \in \mathbb{N}}$ converging to zero, the union of subgroups generated by each h_n is dense in \mathbb{R} . With such a decreasing sequence, Theorem 2.1 will be generalized further (Theorem 4.2 and Corollary 4.3).

3. POINT-WISE CONVERGENCE TO POLYNOMIAL

In this section, we investigate conditions on the point-wise convergence to a polynomial of degree $\leq m$ on the dyadic set K . And we consider general cases: the set of randomly chosen points (see Theorem 3.3 and Remark 3.4.) So that Theorem 3.1 is improved to Theorem 3.3.

Theorem 3.1. *Let f be a continuous function defined on the whole real line and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials p_n of degree $\leq m$ for an integer $m \geq 0$. If $\{p_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ in K point-wise:*

$$\lim_{n \rightarrow \infty} p_n(x) = f(x), \quad \text{for all } x \in K.$$

Then f is a polynomial of degree $\leq m$ and for each $j \geq 0$,

$$\lim_{n \rightarrow \infty} p_n^{(j)}(x) = f^{(j)}(x)$$

uniformly on every bounded subset of the real line.

Proof. For each $n \geq 0$, let

$$p_n(x) = a_{n0} + a_{n1}x + \cdots + a_{nm}x^m.$$

We shall show that if $\{p_n(x)\}_{n=1}^\infty$ converges to $f(x)$ at every $x \in K$, then for each $j = 0, 1, 2, \dots$, $\{a_{nj}\}_{n=1}^\infty$ converges to a real number, say a_j ;

$$\lim_{n \rightarrow \infty} a_{nj} = a_j, \quad j = 0, 1, \dots, m.$$

By linearity of the difference operator and the assumption that $\{p_n(x)\}$ converges to $f(x)$ for every $x \in K$ which contains all integers, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \nabla^m p_n(0) &= \lim_{n \rightarrow \infty} \sum_{k=0}^m (-1)^k \binom{m}{k} p_n(-k) \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} f(-k) \\ &= \nabla^m f(0), \end{aligned}$$

where ∇ is the backward difference with size $h = 1$. On the other hand, using the identities

$$\sum_{k=0}^m (-1)^k \binom{m}{k} k^j = \begin{cases} 0 & \text{if } 0 \leq j < m, \\ (-1)^m m! & \text{if } j = m \end{cases} \quad (3.1)$$

we have that

$$\nabla^m p_n(0) = \sum_{k=0}^m a_{nk} \nabla^m x^k|_{x=0} = a_{nm} m!.$$

Thus the sequence $\{a_{nm}\}_{n=0}^\infty$ converges to a_m ,

$$a_m := \lim_{n \rightarrow \infty} a_{nm} = \nabla^m f(0)/m!.$$

By induction on $j = m, m-1, \dots, 0$, we assume that there exists a positive integer $k \leq m$ such that for $j = k, k+1, \dots, m$, the sequence $\{a_{nj}\}_{n=0}^\infty$ converges to a_j , that is,

$$a_j := \lim_{n \rightarrow \infty} a_{nj}, \quad \text{for } j = k, k+1, \dots, m.$$

Then by the assumption and linearity, we have that $\lim_{n \rightarrow \infty} \nabla^{k-1} p_n(0) = \nabla^{k-1} f(0)$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \nabla^{k-1} p_n(0) &= \lim_{n \rightarrow \infty} \sum_{i=0}^m a_{ni} \nabla^{k-1} x^i|_{x=0} \\ &= \lim_{n \rightarrow \infty} \sum_{i=k-1}^m a_{ni} \nabla^{k-1} x^i|_{x=0} \\ &= \lim_{n \rightarrow \infty} \left[(k-1)! a_{n,k-1} + \sum_{i=k}^m a_{ni} \nabla^{k-1} x^i|_{x=0} \right] \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} a_{n,k-1} := (\nabla^{k-1} f(0) - \sum_{i=k}^m a_i \nabla^{k-1} x^i|_{x=0}) / (k-1)!.$$

So we have proved that there are numbers $\{a_i\}_{i=0}^m$ such that

$$\lim_{n \rightarrow \infty} a_{ni} = a_i, \quad i = 0, 1, 2, \dots, m.$$

Define a polynomial $p(x)$ by

$$p(x) = a_0 + a_1 x + \dots + a_m x^m,$$

then from the convergence of coefficients $\{a_{ni}\}_{n=0, i=0}^{\infty, m}$, we can see that for each $j \geq 0$,

$$\lim_{n \rightarrow \infty} p_n^{(j)}(x) = p^{(j)}(x)$$

point-wisely for every real number $x \in \mathbb{R}$ and uniformly on every bounded subset of \mathbb{R} . Also since the set K is dense in \mathbb{R} and f is continuous on \mathbb{R} , $f(x) = p(x)$ for all the real number x , which proves the theorem. \square

Corollary 3.2. *Let f be a function defined on the whole real line and let $\{p_n\}$ be a sequence of polynomials p_n of degree $\leq m$ for an integer $m \geq 0$. If $\{p_n(x)\}$ converges point-wisely to $f(x)$ at every real number x :*

$$\lim_{n \rightarrow \infty} p_n(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

then f is a polynomial of degree $\leq m$ and for each $j \geq 0$,

$$\lim_{n \rightarrow \infty} p_n^{(j)}(x) = f^{(j)}(x),$$

uniformly on every bounded subset of the real line.

Now we consider a more general case of polynomial convergence.

Theorem 3.3. *Let $\{p_n(x)\}_{n=1}^{\infty}$ be a sequence of polynomials of degree $\leq m$ for some non-negative integer m . Suppose that there are $m+1$ distinct numbers $\{x_k\}_{k=1}^{m+1}$ at which $\{p_n(x)\}_{n=1}^{\infty}$ converges, say,*

$$\lim_{n \rightarrow \infty} p_n(x_k) = y_k, \quad k = 1, 2, \dots, m+1,$$

then for each $j \geq 0$ and any bounded subset Ω of \mathbb{R} or \mathbb{C} :

$$\lim_{n \rightarrow \infty} p_n^{(j)}(x) = p^{(j)}(x) \quad \text{uniformly on } \Omega, \quad (3.2)$$

where $p(x)$ is the polynomial of degree $\leq m$ given by

$$p(x) = \sum_{k=1}^{m+1} y_k \ell_k(x) \quad (3.3)$$

for the Lagrange polynomials $\{\ell_k(x)\}_{k=1}^{m+1}$ for the numbers $\{x_k\}$.

Proof. The uniqueness of interpolation polynomial guarantees that each polynomial $p_n(x)$ is written as

$$p_n(x) = \sum_{k=1}^{m+1} p_n(x_k) \ell_k(x), \quad n = 1, 2, \dots, \quad (3.4)$$

for the Lagrange polynomials $\{\ell_k(x)\}_{k=1}^{m+1}$. Let $j \geq 0$ be an integer. Differentiating j -times the both sides of Equation (3.4) with respect to the variable of x , we have

$$p_n^{(j)}(x) = \sum_{k=1}^{m+1} p_n(x_k) \ell_k^{(j)}(x), \quad n = 1, 2, \dots. \quad (3.5)$$

Since $\{\ell_k(x)\}_{k=1}^{m+1}$ are uniformly bounded on any bounded subset Ω , it is easily shown that

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n^{(j)}(x) &= \lim_{n \rightarrow \infty} \sum_{k=1}^{m+1} p_n(x_k) \ell_k^{(j)}(x) \\ &= \sum_{k=1}^{m+1} y_k \ell_k^{(j)}(x) \\ &= p^{(j)}(x) \quad \text{uniformly on } \Omega \end{aligned}$$

where $p(x)$ is the polynomial of degree $\leq m$ as in (3.3), which completes the proof. \square
Note that relations (3.5) show that for $j = 0, 1, \dots, m$, the sequences of the coefficients of x^j in $p_n(x)$ also converge.

Remark 3.4. *The convergence holds even if we replace the point-wise convergence by the condition that there is a linearly independent set $\{L_i\}_{i=1}^{m+1}$ of linear functionals on \mathcal{P}_m , the space of all polynomials of degree $\leq m$, such that for each k , $\{L_k(P_n)\}_{n=1}^{\infty}$ converges to y_k . Then we have*

$$\lim_{n \rightarrow \infty} p_n^{(j)}(x) = \sum_{k=1}^{m+1} y_k \tilde{\ell}_k^{(j)}(x) \quad \text{uniformly on } \Omega,$$

where $\{\tilde{\ell}_k(x)\}_{k=1}^{m+1}$ is a linearly independent set in \mathcal{P}_m which is biorthonormal with respect to $\{L_i\}_{i=1}^{m+1}$.

$$L_i(\tilde{\ell}_j) = \delta_{ij}.$$

We may consult [4] for more details.

Note that $m + 1$ is the possible least number on the conditions which guarantee the results obtained in Theorem 3.3 and Remark 3.4.

Corollary 3.5. *Let Ω be a bounded subset of \mathbb{R} or \mathbb{C} consisting of at least $m + 1$ elements. Then the space of all polynomials of degree $\leq m$ is a Banach space with the norm $\|\cdot\|$ defined by*

$$\|p\| = \sup_{x \in \Omega} |p(x)|.$$

4. GENERALIZATIONS

Now, we consider more general cases for divided difference operators treated in Section 2, using the polynomial convergence. From now on, let $I = (a, b)$ be an interval of positive measure and we denote $\{h_n\}_{n=1}^{\infty}$ to be a decreasing sequence of positive numbers h_n converging to zero; for each $n \geq 1$, $h_n > h_{n+1}$ and

$$\lim_{n \rightarrow \infty} h_n = 0.$$

First, we characterize continuous functions f satisfying

$$\nabla_{h_n} f(kh_n) = 0, \quad (4.1)$$

for every $n \geq 1$ and $k \in \mathbb{Z}$ such that $kh_n \in I$.

Lemma 4.1. *Let I be an interval of positive measure and $x_1 < x_2 < \dots < x_{m+1}$ distinct points in I . For $i = 1, \dots, m+1$, let $\{x_{i,k}\}_{k=1}^{\infty} \subset I$ and $\{y_{i,k}\}_{k=1}^{\infty}$ be sequences such that*

$$\lim_{k \rightarrow \infty} x_{i,k} = x_i \quad \text{and} \quad \lim_{k \rightarrow \infty} y_{i,k} = y_i \quad \text{for each } i = 1, 2, \dots, m+1$$

for some constants y_k . Then the polynomials $p_{m,k}(x)$ of degree m interpolating to $y_{i,k}$ at $x_{i,k}$ for $i = 1, \dots, m+1$ converges uniformly on I to $p_m(x)$ of degree m ,

$$\lim_{k \rightarrow \infty} p_{m,k}(x) = p_m(x) \quad \text{uniformly on } I$$

where $p_m(x)$ is the interpolation polynomial satisfying $p_m(x_j) = y_j$ for $j = 1, 2, \dots, m+1$.

Proof. For each $k \geq 1$, we may write the interpolation polynomials $p_{m,k}(x)$ as

$$p_{m,k}(x) = \sum_{j=1}^{m+1} y_{k,j} \ell_{k,j}(x)$$

for the Lagrange polynomials $\ell_{k,j}(x)$ satisfying

$$\ell_{k,j}(x_{k,i}) = \delta_{i,j} \quad \text{for } i, j = 1, 2, \dots, m+1.$$

In Section 2, we have shown that $\ell_{k,j}$ is given by

$$\ell_{k,j}(x) = \frac{w_k(x)}{w'_k(x_{k,j})(x - x_{k,j})}, \quad w_k(x) = (x - x_{k,1}) \cdots (x - x_{k,m+1}). \quad (4.2)$$

Fix j , we write the polynomials $\ell_{k,j}$ by

$$\ell_{k,j}(x) = \sum_{r=0}^m a_{k,r} x^r.$$

Now we consider the polynomials $w'_k(x_{k,j})\ell_{k,j}$ by

$$w'_k(x_{k,j})\ell_{k,j}(x) = \sum_{r=0}^m b_{k,r} x^r.$$

Since the coefficients $b_{k,r}$ are linear combinations of multiplications of $x_{k,1}, x_{k,2}, \dots, x_{m+1}$, as we may see in (4.2), and the constants $x_{k,i}$ converge to x_i as k tends to ∞ , we have

$$\lim_{k \rightarrow \infty} b_{k,r} = b_r, \quad r = 0, 1, \dots, m,$$

for some constants b_r . Consequently, with the limit

$$\lim_{k \rightarrow \infty} w'_k(x_{k,j}) = \lim_{k \rightarrow \infty} \prod_{i \neq j} (x_{k,j} - x_{k,i}) = \prod_{i \neq j} (x_j - x_i),$$

we can see that

$$\lim_{k \rightarrow \infty} a_{k,r} = a_r, \quad r = 0, 1, \dots, m,$$

here a_r is the coefficient of x^r in the Lagrange polynomial $\ell_j(x)$

$$\ell_j(x) = \frac{w(x)}{w'(x_j)(x - x_j)}, \quad w(x) = (x - x_1) \cdots (x - x_{m+1}).$$

Since the interval I is bounded, we show that the interpolation polynomials $p_{m,k}(x)$ of degree m interpolating to $y_{i,k}$ at $x_{i,k}$ for $i = 1, \dots, m+1$,

$$p_{m,k}(x) = \sum_{j=1}^{m+1} y_{k,j} \ell_{k,j}(x),$$

converges uniformly on I to $p_m(x)$ of degree m ,

$$\lim_{k \rightarrow \infty} p_{m,k}(x) = \sum_{j=1}^{m+1} y_j \ell_j(x), \quad \text{uniformly on } I$$

where $p_m(x) := \sum_{j=1}^{m+1} y_j \ell_j(x)$ is the interpolation polynomial satisfying $p_m(x_j) = y_j$ for $j = 1, 2, \dots, m+1$. This completes the proof. \square

Now, we are ready to characterize continuous functions satisfying the equation (4.1)

Theorem 4.2. *Let I be an interval of positive measure and let $\{h_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers which converges to zero. Let f be a function defined on a set containing I such that for all $x \in I$ and $n \geq 1$, $\nabla_{h_n}^{m+1} f(x)$ is well-defined. Assume that f satisfies the equations*

$$\nabla_{h_n}^{m+1} f(kh_n) = 0$$

for every $n \geq 1$ and $k \in \mathbb{Z}$ such that $kh_n \in I$. If f is continuous, then f is a polynomial of degree $\leq m$ in I .

Proof. For each $n \geq 1$, let I_n be the subset of I given by

$$I_n = \{kh_n : kh_n \in I \text{ for some integer } k\} = I \cap \langle h_n \rangle.$$

Here, we may assume without lose of generality that each I_n contains at least $m+1$ elements. Now choose $m+1$ points $x_1 < x_2 < \dots < x_{m+1}$ in I such that x_1 and x_{m+1} are not boundary

points of I , and define $\xi := \min_{i=1,2,\dots,m} |x_i - x_{i+1}|$ and N_0 be an integer such that $2h_{N_0} < \xi$. Then, for each $n \geq N_0$, there exists $m + 1$ points $x_{n,1} < x_{n,2} < \dots < x_{n,m+1}$ in I_n such that

$$|x_{n,i} - x_i| < h_n, \quad i = 1, \dots, m + 1.$$

This choice shows that for each $i = 1, \dots, m + 1$, $x_{n,i}$ converges to x_i as n tends to ∞ . Since $\nabla_{h_n}^{m+1} f \equiv 0$ in I_n , we can see that there exists a polynomial $p_n(x)$ of degree $\leq m$ such that

$$p_n(x) = f(x), \quad \text{for all } x \in I_n. \quad (4.3)$$

In particular, p_n is the polynomial interpolating to $f(x_{n,i})$ at $x_{n,i}$ for $i = 1, \dots, m + 1$. The continuity of f shows that for each $i = 1, \dots, m + 1$, $\lim_{n \rightarrow \infty} f(x_{n,i}) = f(x_i)$. Now, Lemma 4.1 implies that $p_n(x)$ converges uniformly to $p(x)$, the polynomial of degree $\leq m$ interpolating to $f(x_i)$ for $i = 1, \dots, m + 1$,

$$\lim_{n \rightarrow \infty} p_n(x) = p(x), \quad \text{uniformly for } x \in I.$$

Let $t \in I$ be an arbitrary point. If $t = x_i$ for some $i = 1, \dots, m + 1$, then we have shown that $f(t) = p(t)$.

Now, we consider the case where $t \neq x_i$ for any $i = 1, \dots, m + 1$. Let $\varepsilon > 0$ be arbitrarily given. Since f and p are continuous at $x = t$, there exists $\delta > 0$ such that for every $x \in I$ satisfying $|x - t| \leq \delta$, we have

$$|f(x) - f(t)| \leq \varepsilon/3 \quad \text{and} \quad |p(x) - p(t)| \leq \varepsilon/3.$$

And the uniform convergence of p_n to p in I implies that there exists an integer N_1 such that

$$\sup_{x \in I} |p_n(x) - p(x)| \leq \frac{\varepsilon}{3}, \quad \text{for } n \geq N_1.$$

Then for a sufficiently large $n \geq N_1$, there exists a point $t_n \in I_n$ such that $|t - t_n| \leq \delta$. In this case, we have $f(t_n) = p_n(t_n)$ (see (4.3)) so that we obtain

$$\begin{aligned} |f(t) - p(t)| &\leq |f(t) - f(t_n)| + |p_n(t_n) - p(t_n)| + |p(t_n) - p(t)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Since ε is arbitrary, therefore, we have that $f(t) = p(t)$. This completes the proof. \square

Corollary 4.3. *Let $\{h_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers which is converging to zero. Let f be a continuous function defined on \mathbb{R} and assume that f satisfies the equations*

$$\nabla_{h_n} f(kh_n) = 0 \quad \text{for every } n \geq 1 \text{ and } k \in \mathbb{Z}.$$

Then f is a polynomial of degree $\leq m$.

Proof. We cover \mathbb{R} by a chain of intersecting intervals $\{J_k\}_{k \in \mathbb{Z}}$ such that $\nabla_{h_n}^{m+1} f(x)$ is well-defined in J_k for all $k \in \mathbb{Z}$. In each interval, f is equal to a polynomial $p_k(x)$ of degree $\leq m$ by Theorem 4.2. Since polynomials $p_{k-1}(x)$ and $p_k(x)$ of degree $\leq m$ have the same values at the all points in the intersection of J_{k-1} and J_k , the two polynomials are equal. As a consequence, $f(x)$ is equal to one polynomial on the set of all real numbers. \square

In the following, we relate divided differences to derivatives, which is a mean value theorem for equally spaced points.

Theorem 4.4. *Let $f(x) \in C[a, b]$ and suppose that $f^{(n)}(x)$ exists at each point of (a, b) , then for $b \geq x_0 > x_1 = x_0 - h > \cdots > x_n = x_0 - nh \geq a$, there exists a point $\xi \in (x_n, x_0)$ satisfying*

$$\nabla_h^n f(x_0) = h^n f^{(n)}(\xi), \quad x_n < \xi < x_0.$$

Proof. From the definition of the forward and backward operators, we can see that

$$\nabla_h^n f = \Delta_h^n f(\cdot - nh).$$

Then the theorem follows directly from Corollary 3.4.4 in [4]. □

A polynomial $p(x)$ of degree $\leq m$ satisfies the difference equation

$$\nabla_h^{m+1} p(x) = 0$$

for any $h > 0$ and the differential equation

$$\frac{d^{m+1} p(x)}{dx^{m+1}} = 0$$

as well. That is, we may regard the difference operator ∇_h^n as the discretization of the differential operator \mathcal{D}^n ($\mathcal{D} = d/dx$). In the following we extend to differential operators of the form

$$\sum_{k=0}^m a_k \mathcal{D}^k,$$

where a_k are real numbers.

Theorem 4.5. *Let $\{h_n\}_{n=1}^\infty$ be a decreasing sequence of positive numbers which converges to zero and let $f \in C^m(I)$ be a function such that for all $x \in I$ and $n \geq 1$, $\nabla_{h_n}^{m+1} f(x)$ is well-defined. Assume that*

$$\lim_{n \rightarrow \infty} \sum_{k=0}^m a_k h_n^{-k} \nabla_{h_n}^k f(x) = 0, \quad \text{for all } x \in I, \quad (4.4)$$

then

$$\sum_{k=0}^m a_k f^{(k)}(x) = 0, \quad \text{for all } x \in I. \quad (4.5)$$

Proof. Let x be chosen arbitrarily in (a, b) and fixed. Since $\{h_n\}_{n=1}^\infty$ is a decreasing sequence convergent to zero, we may assume that for all $n \geq 1$, $x_n > a$. From Theorem 4.4, we have the relation

$$\nabla_{h_n}^k f(x) = h_n^k f^{(k)}(\xi_{k,n}), \quad x - kh_n < \xi_{k,n} < x.$$

By the assumption that $f \in C^m(I)$ and f satisfies the condition (4.4), we have that $\lim_{n \rightarrow \infty} f^{(k)}(\xi_{k,n}) = f^{(k)}(x)$ and

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \sum_{k=0}^m a_k h_n^{-k} \nabla_{h_n}^k f(x) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^m a_k h_n^{-k} \nabla_{h_n}^k f(\xi_{k,n}) \quad (x - kh_n < \xi_{k,n} < x) \\ &= \sum_{k=0}^m a_k \left(\lim_{n \rightarrow \infty} f^{(k)}(\xi_{k,n}) \right) \\ &= \sum_{k=0}^m a_k f^{(k)}(x), \end{aligned}$$

which implies that

$$\sum_{k=0}^m a_k f^{(k)}(x) = 0.$$

Since x is chosen arbitrarily, f satisfies the equation (4.5) for all $x \in I$. \square

The converse of Theorem 4.5 is obvious. But we can see that a function f satisfying (4.5) may not satisfy the difference equation

$$\sum_{k=0}^m a_k h_n^{-k} \nabla_{h_n}^k f(x) = 0$$

in general. The functions satisfying the differential equation (4.5) play fundamental roles for the construction of a subdivision scheme for C^{m-2} -exponential B-splines, whose pieces are solutions to the differential equation (4.5). For details, we refer to [5] and [2].

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