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Designing Statistical Test for Mean of Random Profiles

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ABSTRACT

A random profile is the result of a process, the output of which is a function instead of a scalar or vector quantity. In the nature of these objects, two main dimensions of "functionality" and "randomness" can be recognized. Valuable researches have been conducted to present control charts for monitoring such processes in which a regression approach has been applied by focusing on "randomness" of profiles. Performing other statistical techniques such as hypothesis testing for different parameters, comparing parameters of two populations, ANOVA, DOE, etc. has been postponed thus far, because the "functional" nature of profiles is ignored. In this paper, first, some needed theorems are proven with an applied approach, so that be understandable for an engineer which is unfamiliar with advanced mathematical analysis. Then, as an application of that, a statistical test is designed for mean of continuous random profiles. Finally, using experimental operating characteristic curves obtained in computer simulation, it is demonstrated that the presented tests are properly able to recognize deviations in the null hypothesis.

Keywords: Random Profile, Profile Monitoring, Functional Approach, Designing Statistical Test, Operating Characteristic Curve

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1. INTRODUCTION

Since the introduction of statistical quality control technique, many developments have been made in this field and this technique has been successfully applied in broad fields of industries (Subramani and Balamurali, 2016). Introducing the concept of profile and presenting methods to control it are regarded as a milestone for development of statistical quality control methods. Before introduction of a profile concept, output or property of a process has been always considered as a scalar quantity or vector quantity (multiple variables), and attempts have been made to ensure if the process is under control by its observation. But there are some processes in which the output, instead of a scalar or vector quantity, is a profile or function, which is a relationship between one independent variable and one or several dependent variables. An example of such processes during an etch step in manufacturing semi-conductor proposed by Kang and Albin (2000). In this process, a critical device is a mass flow controller (MFC) which measures gas pressure. In MFC calibration, at first, the numbers shown by MFC for different gas pressures are recorded. In each observation, the result can be considered a near-linear profile, which is in fact a relationship between real gas pressure and the number shown by MFC. Closeness of the observed profile to $y = x$ line indicates soundness of MFC; otherwise, it should be repaired or replaced. Other examples of the processes with profile output can be found in (Mestek *et al*., 1994; Stover and Brill, 1998). Besides, Woodall (2007) considered some processes with profile output in lumber manufacturing, monitoring of shapes and public health surveillance. The term profile monitoring refers to control of such processes, which is performed during two phases. In phase I, while the process is under control, output profiles are recorded, and the shape and limits of the ideal output, called, "reference profile" are determined. Phase II is related to the time during which the system is working. At this time, observing output of the process and comparing it with "reference profile" lead to discussing whether the process is under or out of control. Kang and Albin (2000) considered output profiles in each observation according to a model as:

$$y = A_0 + A_1 x + \varepsilon \quad (1)$$

To sample the process, they selected n points, $x_1, \cdots, x_n$, in the operating range, and recorded process outputs, $y_{1j}, \cdots, y_{nj}$, at these points for sample j. They assumed that points $(x_1, y_{1j}), \cdots, (x_n, y_{nj})$ were almost located on a line. They estimated values of A_0, A_1 and ε in phase I using regression method, and formed exponentially weighted moving average (EWMA) and Range (R) charts. In phase II, mean of residuals of each observed profile (defined as $e_{ij} = y_{ij} - a_0 - a_1 x_j$) is used to investigate whether the process is under control. Kim *et al.* (2003) considered the base model as (1), and converted mean into zero by coding variables X to make the estimated least squares of intercepts and slope independent. This issue allowed two single-variable EWMA charts to be presented for these two parameters. As shown in that research and in (Gupta *et al.*, 2006), their application is preferred to usage of EWMA and R charts method presented in (Kang and Albin, 2000), and they particularly have a better performance in average run length (ARL).

Woodall *et al.* (2004) discussed general issues in profile monitoring. Using indicator variables in multivariate regression model,Mahmoud and Woodall (2004) presented an F-test based method for phase I. In that research, base model (1) is also considered for describing variability in profiles. Ding *et al.* (2006) studied nonlinear profiles and investigated its applications in principal component analysis (PCA).

Some researchers have considered the concept of change point in their studies, which refers to a point at which a change is made in one or more model parameters compared with the reference model. For example, Zou *et al.* (2006) presented control charts based on the change point when all process parameters were unknown. A model based on segmented regression technique is also suggested by Mahmoud *et al.* (2007) by applying the concept of change point. Zou *et al.* (2007) developed multivariate EWMA control charts (MEWMA) for multivariate linear profiles. The considered base model is as:

$$Y_j = X_j \beta + \varepsilon_j \quad (2)$$

where Y_j is an n_j dimensional vector, X_j is an $n_j \times p$ dimensional matrix (where $n_j > p$), β is a p dimensional vector and ε_js are n_j dimensional independent random vectors with normal distribution, with mean of zero and covariance matrix of $\sigma^2 I$. This model is in fact the expansion of model (1). Recently, Mahmoud (2008), Jensen *et al.* (2008) and Noorossana *et al.* (2010) also presented different control charts for base model (2). In many other studies, nonparametric control charts have been also presented for profile monitoring, in which base model is:

$$y_{ij} = g(x_{ij}) + \varepsilon_{ij}, i = 1, \cdots, n_j, j = 1, \cdots \quad (3)$$

where $(x_{1j}, y_{1j}), \cdots, (x_{n_j j}, y_{n_j j})$ form j[th] random sample, g is a nonparametric profile and ε_{ij} are independent random variables with mean of zero and variance of σ^2. Base model (3) with these different as sumptionsin which g is a nonlinear function and ε_{ij}s are normal independent random variables with mean of zero and variance of σ^2 has been considered in some researches for presenting control charts for nonlinear profiles e.g., (Chicken *et al.*, 2009; Vaghefi *et al.*, 2009; Zhang and Albin, 2009).

Another dimension which has attracted the attention of researchers in profile monitoring is fixed or variable sampling interval, sampling size and sampling rate of profiles. These dimensions have been evaluated in different papers, one of which is the method presented by Li and Wang (2010); in all of these papers, base models are not different from what has been mentioned before.

Assume the developed base model as follows:

$$Y_j = f(X_j) + \varepsilon_j \quad (4)$$

where Y_j is a n dimensional vector, X_j is a m dimensional vector ($m \ge n$), $f: R^m \to R^n$ is a smooth function and ε_js are independent random n dimensional vectors with mean of zero and covariance matrix of $\Sigma_{n \times n}$. In this case, most of the conducted researches have considered a special state of the developed model (4) and have tried to study whether the process is under control by presenting control charts in a regression approach. Considering that outputs of the mentioned processes are mainly continuous profiles, can this model describe space of profiles in the best manner? To clarify the problem, the MFC investigated by Kang and Albin (2000) can be considered again. If it is possible to gradually and continuously increase gas pressure from x_l (minimum value) to x_h (maximum value) during system calibration and concurrently record the numbers shown by the controller as gas pressure in a continuous way, one continuous profile will be obtained in each sample. Probably, it is not possible to record profiles continuously in all applications; however, it is possible in some. An example of such a process can be observed in failure test of K type thermocouples of R-400 reactor in production line of linear light polyethylene in petrochemical industries. Due to very serious problems and high costs caused by failure of these thermocouples, inspectors of maintenance instrumentation section first remove them from the mother device as soon as suspecting failure in these parts and then put them in an oven (DRUCK limited model),which is connected to a distribution control system (DCS) (CS-3000 model). Then, the temperature sensed by the thermocouple in DCS is drawn as a graph with gradual increase of oven temperature in the operating range. If the drawn graph has significant deviation from y = x line, it will indicate failure of the thermocouple, which should be immediately replaced.

Now suppose a profile, which can be continuously recorded. The question is that should this profile be controlled by applying model (4) with selecting some points and assessing the residuals? How many points are desirable? If a better result is obtained by increasing these points, is there a method for limit state including involvement of all interval points? This question is a motivation for presenting a base model which is more compatible with continuous nature of profiles; a model in which profiles are considered as random functions which slide from the beginning to the end of their domain by following a random pattern (not some points which have been randomly scattered in the plane).

Considering that statistical tests are basis of different quality control methods, the goal of this research is to generalize and develop statistical tests of random scalar variables to space of random profiles. In this case, to present a method for performing other quality control techniques such as acceptance sampling, comparison of two populations of profiles, design of experiments, etc. can be possible in addition to control charts.

The rest of this paper is organized as follows. Theory of random profiles with a functional approach based on their random nature is presented in Section 2.This is done by developing concepts of classic statistics and probability of scalar random variables to space of profiles such that the materials presented for profiles are reduced and corresponded to scalar variables by limiting profiles to random but constant profiles on the studied domain. Since statistical tests are performed by sampling, the concepts required for sampling are presented in Section 3, and an estimator is made for them considering the family of random profiles. Using these instruments, two tests are designed for mean of profiles in Section 4 and their similarity to similar tests is shown in scalar random variables. In Section 5, performance of the presented tests is compared with that of a classic test by simulating and obtaining numerical results. Section 6 concludes the paper.

2. THEORY OF RANDOM PROFILES

In this section, theory of classic statistics and probability is developed to space of random profiles with a functional approach. While presenting the definitions, both fundamental dimensions of the nature of these objects, i.e. functionality and randomness, are considered. At the same time, properties of profiles are expressed and proved as some theorems and lemmas. Like other mathematical fields, this development is done such that the presented concepts could be consistent with the corresponding cases in space of scalar random variables in the case of using constant but random profiles. In fact, some needed parts of functional analysis are presented and proven with the understandable words for an engineer and not a mathematician.

Definition 1: (Profile random test) It is an experiment whose outcome is not specified in advance but its result is a profile.

Definition 2: (Random profile) It refers to the outcome of a profile random test. Through this paper, random profiles are shown in capital letters.

Definition 3: (Observed profile) After performing the test, each of the test results is called an observed profile. Observed profiles are shown in lower case letters.

Definition 4: (Property of a random profile) If each of the achievable profiles in a profile random test has a property in common (for example, all are continuous), the random profile is said to have that property.

Random profiles can be highly varied in terms of specifications (such as being discrete or continuous, single variable or multivariate types, etc.). In this paper, only continuous single variable random profiles are considered. More explicitly, assume a profile random test, the result of which is a single variable continuous random profile belongs to $[a, b]$:

$$Y = F(x),\ a \le x \le b \tag{5}$$

In the rest of this paper, random profile means such profiles.

Definition 5: (Algebra of random profiles) If F_1 and F_2 are two random profiles on $[a, b]$, F_1+F_2, F_1-F_2, $F_1\times F_2$, $F_1 \div F_2$ are random profiles which are defined as follows for each $x \in [a, b]$:

$$(F_1 + F_2)(x) = F_1(x) + F_2(x)$$
$$(F_1 - F_2)(x) = F_1(x) - F_2(x)$$
$$(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$$
$$(F_1 \div F_2)(x) = F_1(x) \div F_2(x),\ F_2(x) \ne 0.$$

Definition 6: (Induced random variable) Consider random profile (5). For each point like $\xi \in [a, b]$, $F(\xi)$ will be a scalar random variable. This random variable is called induced random variable of F at point ξ.

Theorem 1. Induced random variables obtained from points along a smooth random profile (with continuous derivative) cannot be assumed to be independent.

Proof: Assume that F has continuous derivative on (a, b). Suppose $f'^{-}(c) > 0$ while performing a test for recording an observed profile, f, at a point like $c \in (a, b)$. Since f has continuance derivative in c, there exist $\delta > 0$ such that $f'(x) > 0$ if $x \in (c, c+\delta)$, or equivalently $f(x) > f(c)$. Therefore, $f(x)$ has a value which is necessarily greater than $f(c)$; i.e. $f(x)$ is not independent from $f(c)$. Namely,

$$P\big(F(x) \big| F(c)\big) \neq P\big(F(x)\big). \quad \square$$

Definition 7: (Independent random profiles) Some random profiles defined on interval [a, b] are called independent if their resulting induced variables are independent at each point of this interval.

It is necessary to note that independence of random profiles means independence of behavior of each profile from other profiles, not independent behavior of each point from other points along a profile (which is violated in Theorem 1).

Definition 8: (Expected value (mean) for a random profile) Expected value (mean) of random profile (5) is function μ_F (or μ in case that there is no fear of ambiguity) which is defined as follows:

$$\mu(x) = E\big(F(x)\big), \quad a \le x \le b \tag{6}$$

μ is a nonrandom function which shows the center of profile and random profile fluctuates around it. It is necessary to note that μ may not be defined on [a, b]. For example, let F be a random profile from a test, the result of which is $f_i(x) = 4ix(1-x),\ x \in [0, 1]$ per natural number i with probability of $\frac{1}{i^2}$. In this case, induced random variable at point x = 0.5 has the probability density function as $P(F(0.5) = i) = \frac{1}{i^2},\ i \in \mathbf{N}$, for which it can be easily observed that there is no mean. Therefore, μ has not been defined in x = 0.5. Henceforth, it is assumed that the means of discussed random profiles are available at all points of [a, b]. In other words, μ has been defined on [a, b].

Lemma 1: μ -if available- is continuous on [a, b].

Proof: Assume that μ is defined on [a, b] and t is an arbitrary point of this interval. From continuity of linear operator E and random profile F, it would follow that $\lim_{x\to t} \boldsymbol{\mu}(x) = \lim_{x\to t}\big(E\big(F(x)\big)\big) = E\Big(\lim_{x\to t}\big(F(x)\big)\Big) = E\big(F(x)\big)$. This means $\lim_{x\to t} \boldsymbol{\mu}(x) = \boldsymbol{\mu}(t)$; therefore, $\boldsymbol{\mu}$ is continuous on (a, b). Similarly, it can be shown that there is right and left continuity in a and b, respectively. □

Theorem 2: (Generalizing linear property of expected value in space of profiles)

If F_1 and F_2 are two random profiles in [a, b] with means of $\boldsymbol{\mu}_{F_1}$ and $\boldsymbol{\mu}_{F_2}$ respectively, and g_1 and g_2 are arbitrary nonrandom functions defined in this interval, then, $\boldsymbol{\mu}_{g_1F_1+g_2F_2} = g_1\boldsymbol{\mu}_{F_1} + g_2\boldsymbol{\mu}_{F_2}$.

Proof: For each $x \in [a, b]$, $F_1(x)$ and $F_2(x)$ (induced random variables of F_1 and F_2 at point x) are random variables and $g_1(x)$ and $g_2(x)$ are constant values; therefore,

$$\begin{aligned} &E\big(g_1(x)F_1(x) + g_2(x)F_2(x)\big) \\ &\quad = g_1(x)E\big(F_1(x)\big) + g_2(x)E\big(F_2(x)\big),\ \forall x \in [a, b] \end{aligned}$$

can be written based on linear property of expected value in space of scalar random variables; hence, $\boldsymbol{\mu}_{g_1F_1+g_2F_2} = g_1 \boldsymbol{\mu}_{F_1} + g_2\boldsymbol{\mu}_{F_2}$. □

The above theorem can be easily generalized to $\boldsymbol{\mu}_{g_1F_1+\cdots+g_nF_n} = g_1\boldsymbol{\mu}_{F_1} + \cdots + g_n\boldsymbol{\mu}_{F_n}$. The following lemma is an immediate result of Theorem 2.

Lemma 2: If f and g are arbitrary nonrandom functions defined in interval [a, b] and F is a random profile in this interval, then $E(f \cdot F + g) = f \cdot E(F) + g$.

Definition 9: (Error profile) For random profile F, error profile is defined as follows:

$$\boldsymbol{e}_F(x) = F(x) - \boldsymbol{\mu}(x),\ \forall x \in [a, b].$$

It is evident that error profile is a random profile itself, the value of which at each point of the domain is deviation of random profile from the mean. If there is no fear of ambiguity, error profile is shown by $\boldsymbol{e}$. Considering the above definition, $F = \boldsymbol{\mu} + \boldsymbol{e}_F$, and considering that $\boldsymbol{\mu}$ is a nonrandom function, F will be definite in case $\boldsymbol{e}_F$ is available. Therefore, as will be seen, its corresponding error profile will be given while discussing a random profile with a definite mean. The following theorem is a straightforward consequence of lemma 2.

Theorem 3: $\boldsymbol{\mu}_{\boldsymbol{e}}(x) = \mathbf{0}$.

This theorem shows that error profile is a random profile which fluctuates around x-axis. Induced variable of $\boldsymbol{e}_F$ is a scalar random variable which shows measure of error (distance of profile from its mean) at each point. As shown in Theorem 1, assumption of errors' independence (independence of changes at consecutive points along a profile) is violated. As shown in Figure 1, manner of depicting error of points by a continuous profile states the doubt that error distribution of points can be a bimodal distribution; therefore, assumption of errors'

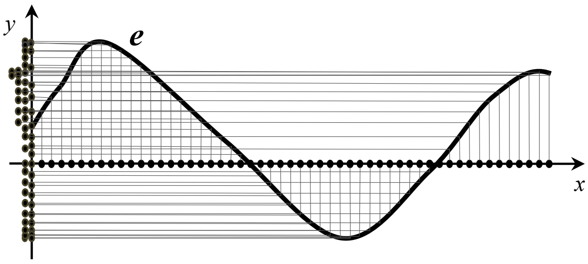


Figure 1. Depiction manner of points' error by a continuous profile.

normality does not seem to be axiomatic. This point is observed in results obtained from numerical simulation given in Section 5. Therefore, assumption of error distribution normality which has been widely used in the conducted researches on profile monitoring (as referred to in the first section) may not have the expected generality.

Therefore, none of the two assumptions of errors' independence and normality are used in this research. However, it is only assumed that they have variance of σ_e^2 without considering independence of errors and any assumption about their distribution; it is known from Theorem 3 that they have mean of zero.

As μ_F shows center of profile, an index which shows dispersion of profiles around this mean is required. In the following definitions we introduce this index.

Definition 10: Let g be an integrable function in an interval like *I*. Norm of *g* is defined as follows:

$$\|g\|_I = \int_I g(x)dx$$

Definition 11: (Random dispersion and deviation) For random profile (5), the random variables defined by $\mathbf{E}_F = \frac{1}{b-a}\|\boldsymbol{e}_F\|_{[a,b]}$ and $\mathbf{D}_F^2 = \frac{1}{b-a}\|\boldsymbol{e}_F^2\|_{[a,b]}$ are called random deviation and dispersion, respectively.

The above definition shows that random deviation, measures resultant random profile deviation from its mean along its domain. It is intuitively expected from positive and negative deviations to neutralize each other and have zero resultant. This will be proved in Theorem 4. Therefore, random deviation cannot be used as a suitable criterion for measuring distance of a random profile from its mean. Definition of random dispersion, which is a scalar random variable, reveals that this quantity can be considered a suitable index for measuring distance of random profile from its mean. Because by definition,

$$\mathbf{D}_F^2 = \frac{1}{b-a}\|e_F^2\|_{[a,b]} = \frac{1}{b-a}\int_a^b \left(F(x)-\mu(x)\right)^2 dx$$

and by applying Riemann sum of recent integral, it can be written:

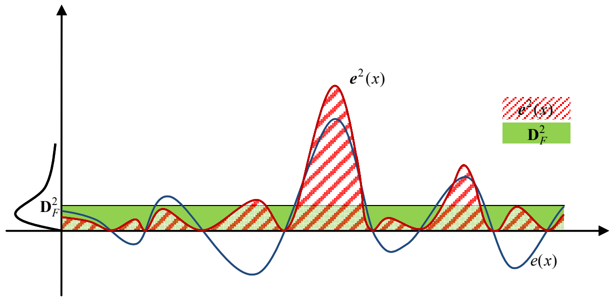


Figure 2. $\mathbf{D}_F^2$ is an index which shows distance of profile from its mean.

$$\mathbf{D}_F^2 = lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left(F\left(a+i\frac{b-a}{n}\right)-\mu\left(a+i\frac{b-a}{n}\right)\right)^2.$$

It means that random dispersion is means quared distance of profile points from their means along the random profile. For this reason, it is used as an index which indicates distance of random profile from its mean (Figure 2).

According to the recent term (writing integral as a limit sum), it can be seen that deviation and random dispersion of random profiles will be also independent in the case of their independent. In what follows, some properties of deviation and random dispersion and their relationship to the norm which is later needed are proved.

Theorem 4: $E(\mathbf{E}_F)=0$

Proof: By Definitions 9 and 10, we have:

$$E(\mathbf{E}_F) = E\left(\frac{1}{b-a}\|\boldsymbol{e}_F\|_{[a,b]}\right) = E\left(\frac{1}{b-a}\int_a^b (e_F(x))dx\right)$$

According to continuity of *F* on [a, b] and by Lemma 1, μ is continuous on [a, b]. Therefore, $\boldsymbol{e}_F = \mathrm{F}-\boldsymbol{\mu}$ is continuous on [a, b]. Hence Riemann sum of recent integral can be written as

$$E\left(\frac{1}{b-a}\int_a^b (e_F(x))dx\right) = E\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left[e_F\left(a+i\frac{b-a}{n}\right)\right]\right)$$
$$= lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left[E\left(\boldsymbol{e}_F\left(a+i\frac{b-a}{n}\right)\right)\right]=0. \quad \square$$

Lemma 3: $E(\|F\|) = \|E(F)\|$.

Proof:
$$E(\|F\|) = E(\|\boldsymbol{\mu}+(F-\boldsymbol{\mu})\|) = E(\|\boldsymbol{\mu}\|+\|F-\boldsymbol{\mu}\|)$$
$$= E(\|\boldsymbol{\mu}\|)+E(\|F-\boldsymbol{\mu}\|) = \|\boldsymbol{\mu}\|+E(\mathbf{E}_F) = \|\boldsymbol{\mu}\|+0 = \|E(F)\|. \quad \square$$

The following lemma is an immediate application of Lemmas 2 and 3.

Lemma 4: If f is an arbitrary nonrandom function on $[a, b]$ and F is a random profile on this interval, then $E(\|f \cdot F\|) = \|f \cdot E(F)\|$.

Theorem 5: If F and G are two independent random profiles defined on [a, b], then $E(FG) = E(F)E(G)$.

Proof: Because F and G are independent, then $F(x)$ and $G(x)$ are independent for each $x \in [\mathrm{a}, \mathrm{b}]$ by Definition 7. Thus, according to the property of expected value for independent scalar random variables, we have

$$E\big(F(x)G(x)\big) = E\big(F(x)\big)E\big(G(x)\big),\ \forall x \in [\mathrm{a}, \mathrm{b}]$$

Thus, $E(FG) = E(F)E(G)$. □

Definition 12: (Variance and standard deviation of random profile) For random profile (5), variance is defined as follows:

$$\boldsymbol{\sigma}_F^2 = Var(F) = E\left(\mathbf{D}_F^2\right)$$

and square root of variance is called random profile's standard deviation; that is, $\boldsymbol{\sigma}_F = \sqrt{\sigma_F^2}$.

Theorem 6: If all induced variables along random profile have common variance of σ_e^2, then $\boldsymbol{\sigma}_F^2 = \sigma_e^2$.

Proof.

$$\boldsymbol{\sigma}_F^2 = E\left(\mathbf{D}_F^2\right) = E\left(\frac{1}{b-a}\left\|e_F^2\right\|_{[\mathrm{a,b}]}\right)$$
$$= E\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[\boldsymbol{e}_F^2\left(a + i\frac{b-a}{n}\right)\right]\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum\nolimits_{i=1}^{n} E\left[e_F^2\left(a + i\frac{b-a}{n}\right)\right]$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum\nolimits_{i=1}^{n}\sigma_e^2 = \lim_{n\to\infty}\sigma_e^2 = \sigma_e^2. \quad \square$$

Theorem 7: (Properties of random profile variance)

a) For arbitrary scalar a and nonrandom function g and random profile F which are defined in an interval: $Var(aF + g) = a^2 Var(F)$.

b) $Var(F) = E\left(\left\|F^2\right\|\right) - \left\|\boldsymbol{\mu}^2\right\|$.

Proof:

By Definition 12, we can write:

a) $Var(aF+g) = E\left(\left\|\big(aF + g - E(aF+g)\big)^2\right\|\right)$

$$= E\left(\left\|\big(aF - aE(F)\big)^2\right\|\right) = a^2 E\left(\left\|\big(F - E(F)\big)^2\right\|\right) = a^2 Var(F).$$

b) $Var(F) = E\left(\mathbf{D}_F^2\right) = E\left(\left\|(F-\mu)^2\right\|\right)$

$$= E\left(\left\|F^2 - 2F\boldsymbol{\mu}^2 + \boldsymbol{\mu}^2\right\|\right)$$
$$= \left\|E\left(F^2 - 2F\boldsymbol{\mu} + \boldsymbol{\mu}^2\right)\right\| = \left\|E(F^2) - 2\boldsymbol{\mu}E(F) + \mu^2\right\|$$
$$= \left\|E\left(F^2\right) - 2\boldsymbol{\mu}^2 + \boldsymbol{\mu}^2\right\|$$
$$= E\left(\left\|F^2\right\|\right) - \left\|F^2\right\|. \quad \square$$

Definition 13: (Covariance of two random profiles) Covariance of two random profiles F and G are defined as follows:

$$Cov(F, G) = E\left(\left\|(F - \boldsymbol{\mu}_F\right\| - \left\|(G - \boldsymbol{\mu}_G\right\|\right).$$

Theorem 8. $Cov(F, G) = \|F(FG)\| - \|\boldsymbol{\mu}_F\boldsymbol{\mu}_G\|$.

Proof:

$$Cov(F, G) = E\left(\left\|\big(F - \boldsymbol{\mu}_F\big)\big(G - \boldsymbol{\mu}_G\big)\right\|\right)$$
$$= E\left(\left\|FG - \boldsymbol{\mu}_F G - \boldsymbol{\mu}_G F + \boldsymbol{\mu}_F\boldsymbol{\mu}_G\right\|\right)$$
$$= \left\|E(FG) - \boldsymbol{\mu}_F E(G) - \boldsymbol{\mu}_F E(F) + \boldsymbol{\mu}_F\boldsymbol{\mu}_G\right\|$$
$$= \left\|E(FG) - \boldsymbol{\mu}_F\boldsymbol{\mu}_G\right\|$$
$$= \left\|E(FC)\right\| - \left\|\boldsymbol{\mu}_F\boldsymbol{\mu}_G\right\|. \quad \square$$

Theorem 9: If F and G are two independent random profiles, $Cov(F, G) = 0$.

Proof: It is clear from Theorem 8 that $Cov(F, G) = \|E(FG)\| - \|\boldsymbol{\mu}_F\boldsymbol{\mu}_G\|$; since F and G are assumed to be independent, then Theorem 5 shows that $E(FG) = \boldsymbol{\mu}_F\boldsymbol{\mu}_G$ and the theorem is proved. □

Theorem 10: For random profiles $F_1, \cdots, F_n$ and constant scalars $a_1, \cdots, a_n$,

$$Var\left(\sum_{i=1}^{n} a_i F_i\right) = \sum_{i=1}^{n} a_i^2 Var(F_i) + \sum_{i<j}\sum a_i a_j Cov(F_i, F_j)$$

therefore,

$$Var\left(\sum_{i=1}^{n} a_i F_i\right) = \sum_{i=1}^{n} a_i^2 Var(F_i)$$

can be given in case of independent random profiles.

Proof: By Definition 12 and 13, we can write:

$$Var\left(\sum_{i=1}^{n} a_i F_i\right) = E\left(\left\|\left(\sum_{i=1}^{n} a_i F_i - \sum_{i=1}^{n} a_i \boldsymbol{\mu}_{F_i}\right)^2\right\|\right.$$

$$=\left\|\left(\sum_{i=1}^{n} a_i\left(F_i-\mu_{F_i}\right)\right)^2\right\|$$

$$=E\left(\left\|\sum_{i=1}^{n} a_i^2\left(F_i-\mu_{F_i}\right)^2+2\sum\sum_{i<j}^{n} a_i a_j\left(F_i-\mu_{F_i}\right)\left(F_j-\mu_{F_j}\right)\right\|\right)$$

$$=E\left(\sum_{i=1}^{n} a_i^2\left\|\left(F_i-\mu_{F_i}\right)^2\right\|+2\sum\sum_{i<j}^{n} a_i a_j\left\|\left(F_i-\mu_{F_i}\right)\left(F_j-\mu_{F_j}\right)\right\|\right)$$

$$=\sum_{i=1}^{n} a_i^2 E\left(\left\|\left(F_i-\mu_{F_i}\right)^2\right\|\right)$$

$$+2\sum\sum_{i<j}^{n} a_i a_j E\left(\left\|\left(F_i-\mu_{F_i}\right)\left(F_j-\mu_{F_j}\right)\right\|\right)$$

$$=\sum_{i=1}^{n} a_i^2 Var(F_i)+2\sum\sum_{i<j} a_i a_j Cov\left(F_i, F_j\right) \quad \square$$

3. SAMPLING AND ESTIMATION IN PROFILES

To perform statistical tests, random profiles should be sampled. In this section, the related definitions and concepts are presented. Then considering a family of common profiles, some estimators of parameters are presented.

Definition 14: (Identically distributed random profiles) Random profiles are called identically distributed if they have equal mean and their random dispersion is identically distributed.

Definition 15: (Profile random sample) If $F_1, F_2, \cdots, F_n$ are independent and identically distributed random profiles, they form a profile random sample of infinite population, determined by their means and common distribution of random dispersion.

Definition 16: (Profile statistic) A random profile which is a function of profile random sample is called a profile statistic.

Definition 17: (Mean and variance of profile sample) If $F_1, F_2, \cdots, F_n$ is a profile random sample, then statistic:

a) $\bar{F}=\frac{1}{n}\sum_{i=1}^{n} F_i$ is called profile sample mean.

b) $S^2=\frac{1}{n-1}\sum_{i=1}^{n}\frac{1}{b-a}\left\|\left(F_i-\bar{F}\right)^2\right\|$ is called profile sample variance.

$\bar{F}$ estimates center of random profile, while S^2 is an estimation for dispersion amount of profiles around this center; and it can be seen intuitively that the center of profile can be transferred without changing dispersion; also dispersion amount of profile can be changed without changing its center; i.e. $\mathbf{D}_{\bar{F}}^2$ and S^2 are independent.

Theorem 11: If $\bar{F}$ is the mean of profile random sample with size of n from infinite profile population with a mean μ and a variance σ^2, then

a) $E\left(\bar{F}\right)=\mu.$

b) $Var\left(\bar{F}\right)=\frac{1}{n}\sigma^2.$

Proof:

a) $E\left(\bar{F}\right)=E\left(\frac{1}{n}\sum_{i=1}^{n} F_i\right)=\frac{1}{n}\sum_{i=1}^{n} E(F_i)=\frac{1}{n}\sum_{i=1}^{n}\mu=\mu$

b) $Var\left(\bar{F}\right)=Var\left(\frac{1}{n}\sum_{i=1}^{n} F_i\right)=\frac{1}{n^2}\sum_{i=1}^{n} Var\left(F_i\right)$

$=\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2=\frac{1}{n}\sigma^2. \quad \square$

Considering definition of random profile variance, $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ always has mean of 1. The conducted study shows that variance of $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ depends on shrinkage of random profile along its domain. It means that, the higher the fluctuation of profile along the domain, the smaller the variance for $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ (Figure 3).

Then, a family of random profiles is considered, for which division of random deviation by random profile variance $\left(\frac{\mathbf{D}_F^2}{\sigma_F^2}\right)$ has chi-squared distribution with one degree of freedom. Such profiles have conventional fluctuations in practice. This point will be more analyzed in simulation and its numerical results in Section 5. It is evident that, if this assumption does not hold in an application, the related distribution should be used in the future analysis. Considering $\sigma_F^2=E\left(\mathbf{D}_F^2\right)=\sigma_e^2$ according to Theorem 6, then $\frac{\mathbf{D}_F^2}{\sigma_e^2}\sim\chi_1^2$. On the other hand, according to Theorem 11: $\sigma_{\bar{F}}^2=\frac{1}{n}\sigma_F^2=\frac{1}{n}\sigma_e^2$; then $\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_{\bar{F}}^2}=\frac{\mathbf{D}_{\bar{F}}^2}{\frac{1}{n}\sigma_e^2}$ has chi-squared distribution with one degree of freedom.

Theorem 12: $\frac{(n-1)S^2}{\sigma_e^2}$ statistic has chi-squared distribution with n-1 degree of freedom.

Proof: With a brief Manipulation, we can write

$$\sum_{i=1}^{n}\left\|\left(F_i-\mu\right)^2\right\|=\sum_{i=1}^{n}\left\|\left(\left(F_i-\bar{F}\right)+\left(\bar{F}-\mu\right)\right)^2\right\|$$

$$= \sum_{i=1}^{n} \left\| \left(F_i - \bar{F} \right) \right\| + n \left\| \left(\bar{F} - \boldsymbol{\mu} \right)^2 \right\|$$

or equivalently

$$\sum_{i=1}^{n} D_{F_i}^2 = (n-1)S^2 + n\mathbf{D}_{\bar{F}}^2$$

which implies that

$$\sum_{i=1}^{n} \frac{\mathbf{D}_{F_i}^2}{\sigma_e^2} = \frac{(n-1)S^2}{\sigma_e^2} + \frac{\mathbf{D}_{\bar{F}}^2}{\frac{1}{n}\sigma_e^2}$$

In the recent relation, the right hand side statement is the sum of n independent random variable, each of which has chi-squared distribution with one degree of freedom; therefore, the result has chi-squared distribution with n degree of freedom. The second statement on the left has chi-squared distribution with one degree of freedom. Therefore, the first statement on the left has chi-squared distribution with n-1 degree of freedom according to Cochran's theorem. □

Theorem 13: $\bar{F}$ is an unbiased estimator for $\boldsymbol{\mu}$ and S^2 is an unbiased estimator for $Var(F)$.

Proof: By part (a) of Theorem 11, it can be seen obviously that $\bar{F}$ is an unbiased estimator for $\boldsymbol{\mu}$. To prove the second part, it can be written that

$$E\left(S^2\right) = \frac{\sigma_e^2}{(n-1)} E\left(\frac{(n-1)S^2}{\sigma_e^2} \right)$$

which $\frac{(n-1)S^2}{\sigma_e^2}$ has chi-squared distribution with n-1 degree of freedom on the right side of the recent relation, according to Theorem 12. Then, its expected value is equal to n-1; therefore, $E\left(S^2\right) = \frac{\sigma_e^2}{(n-1)}(n-1) = \sigma_e^2 = \sigma_F^2$, which proves the theorem. □

3.1 Error Estimation

As is known about scalar random variables, $P\left(|z| < z_{\frac{\alpha}{2}}\right) = 1-\alpha$ holds in the case of using $\bar{X}$ statistic (mean of random sample with size of n from normal population with a definite variance of σ^2) as mean estimator, considering that $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ statistic has standard normal distribution.However, considering $\left(z_{\frac{\alpha}{2}}\right)^2 = \chi_{\alpha,1}^2$ and that Z^2 has chi-squared distribution with one degree of freedom, the above term can be written as $P\left(Z^2 < \chi_{\alpha,1}^2\right) = 1-\alpha$ or $P\left(\left| \left(\bar{X} - \mu_0 \right)^2 \right| < \chi_{\alpha,1}^2,\ \sigma/\sqrt{n} \right) = 1-\alpha$. It means that the maximum of square of estimation error with probability of $1-\alpha$ is equal to $\chi_{\alpha,1}^2 \cdot \sigma/\sqrt{n}$. Now, considering the similarity between left side statement of the last inequality, i.e. $\left| \left(\bar{X} - \mu \right)^2 \right|$ and $(b-a)\mathbf{D}_{\bar{F}}^2 = \left\| \left(\bar{F} - \boldsymbol{\mu} \right)^2 \right\|$, and considering equal nature of these quantities, both of which have sizes for showing distance magnitude of mean estimator from real mean, $\mathbf{D}_{\bar{F}}^2$ is considered estimation error resulting from use of $\bar{F}$ as an estimation of $\boldsymbol{\mu}_F$ value. The following theorem is obtained for the maximum error of this estimation.

Theorem 14: If $\bar{F}$, mean of a profile random sample of size n from a profile population (with a known variance σ_e), is used as $\boldsymbol{\mu}_F$ estimator, the probability that estimation error is less than $\chi_{\alpha,1}^2, \frac{\sigma_e^2}{n}$ is equal to $1-\alpha$

Proof: Because $\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_e^2/n}$ has chi-squared distribution with one degree of freedom, then, $P\left(\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_e^2/n} < \chi_{\alpha,1}^2 \right) = 1-\alpha.$ or $P\left(\mathbf{D}_{\bar{F}}^2 < \chi_{\alpha,1}^2, \frac{\sigma_e^2}{n} \right) = 1-\alpha.$ □

4. DESIGNING STATISTICAL TEST FOR MEAN OF RANDOM PROFILES

By developing classic statistical test techniques to the space of random profiles, in this section the method of performing statistical test for mean of profile population is presented and their existing similarities are shown. As is known, while testing $\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$ hypothesis at significance level of $1-\alpha$ for a scalar population with normal distribution and known variance, σ^2, critical region of maximum likelihood test is obtained as $|z| > z_{\frac{\alpha}{2}}$ considering that $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ statistic (where $\bar{X}$ is mean of random sample of size n from the assumed population) has standard normal distribution. Therefore, considering $\left(z_{\frac{\alpha}{2}}\right)^2 = \chi_{\alpha,1}^2$ and that Z^2 has chi-squared distribution with one degree of freedom, rejection region can be written as $Z^2 > \chi_{\alpha,1}^2$ or $\left| \left(\bar{X} - \mu_0 \right)^2 \right| > \chi_{\alpha,1}^2, \sigma^2/n$. To generalize this test to a similar test in space of random profiles, in case variance of profile population is known σ_e^2, $1-\alpha = P\ (Type\ I\ error) = P\ (H_0\ is\ rejected \mid H_0\ is\ true) =$

$P\left(\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_e^2/n} > \chi_{\alpha,1}^2\right)$ can be written considering the point that $X^2 = \frac{D_{\bar{F}}^2}{\sigma_e^2/n}$ statistic has chi-squared distribution with one degree of freedom. Therefore, the rejection region can be written as $D_{\bar{F}}^2 > \chi_{\alpha,1}^2, \sigma_e^2/n$ for performing this test. In the case profile population variance is unknown, its estimation, i.e. S^2, should be used. Because $F = \frac{\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_e^2}/n}{\frac{(n-1)S^2}{\sigma_e^2}/(n-1)} = \frac{\mathbf{D}_{\bar{F}}^2}{S^2/n}$ statistic is the result of dividing two independent chi-squared variables, each of which is divided by its degree of freedom, it has Fisher distribution with 1 and n-1 degrees of freedom. Therefore, critical region of the test is obtained as $F > f_{\alpha,1,n-1}$ which is converted into $\mathbf{D}_{\bar{F}}^2 > f_{\alpha,1,n-1} \cdot S^2/n$ after slight algebraic manipulation. It can be seen again that the resulting critical region is similar to the corresponding state of its scalar variable because critical region is as $|T| > t_{\frac{\alpha}{2},n-1}$ where $T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$. Since $(t_{n-1})^2 = f_{1,n-1}$ and $\left(t_{\frac{\alpha}{2},n-1}\right)^2 = f_{\alpha,1,n-1}$, critical region can be written as $T^2 > f_{\alpha,1,n-1}$ or equivalently $\left|(\bar{X}-\mu)^2\right| > f_{\alpha,1,n-1} \cdot S^2/n$, and the similarity of these two corresponding states is observable.

5. SIMULATION

In this section, the results obtained from simulation are given, which are used to evaluate ability of the presented tests by drawing operating characteristic (OC) curves. In the first step, the random profile with the characteristics based on the applied assumption is presented and those properties which theoretically proved are numerically investigated. Here, through some examples, some functions are selected as mean of random profile and by applying deviation for them, ability of the test for recognizing these deviations is studied. According to Definition 9, $F = \boldsymbol{\mu} + e_F$, and $\boldsymbol{\mu}$ is a nonrandom function; therefore, F is also obtained in case of having error profile of $\boldsymbol{e}_F$. As the error profile, the following function is considered in interval [2, 4].

$$\boldsymbol{e}_F(x) = 0.2|r_1|^{0.2} e^{cos(vx+\pi r_2)}(cos(vx+10\pi r_3)+\pi r_4) \\ \times cos(vx \cdot \cos(vx+10\pi r_s)+\pi r \quad (7)$$

where r_1 to r_6 are random numbers with standard normal distribution and v (fluctuation coefficient) is a con-

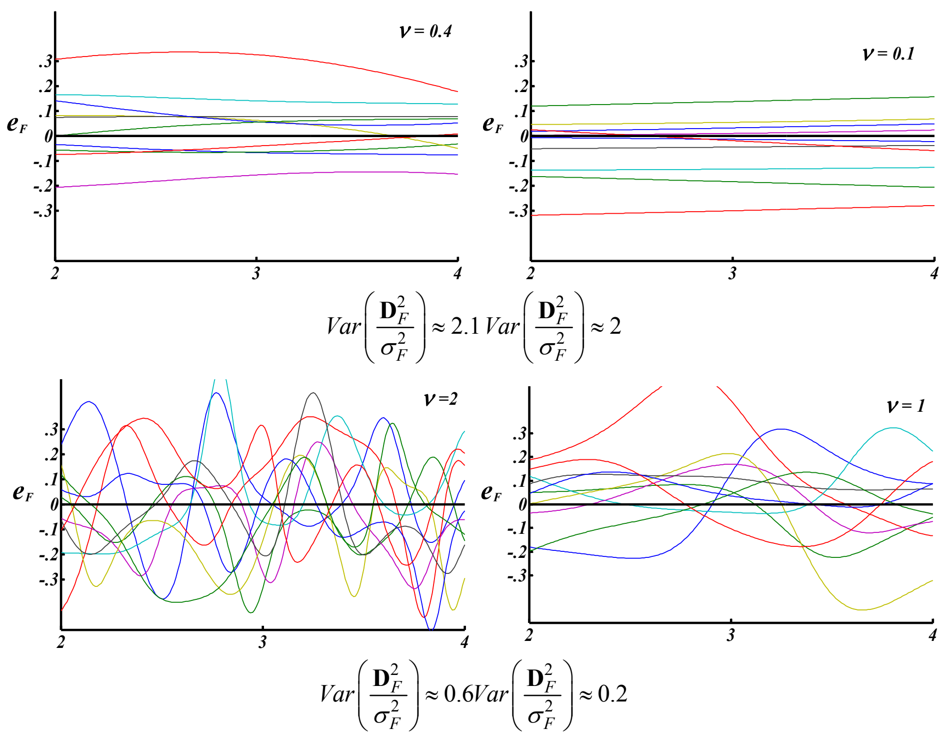


Figure 3. Variance of $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ decreases with increase of profile fluctuation.

stant, with increase of which value of profile fluctuation increases as well. In Figure 3, diagram of some of these profiles is given for different values of *v*. Variance of $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ is also mentioned for each of these values; as mentioned previously, it is observed that with increase of profile fluctuation rate, variance of $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ decreases. Simulation has been done long enough, 10,000 replication, such that standard deviation of the estimated values is less than 5 percent, and model results are reasonable. For the simulation, family of the profiles in which $v = 0.4$ is considered. Fluctuation rate of this group of profiles is observed to be at the conventional level in many applications. As shown by the numerical results, the assumed properties in Section 3 are also satisfied for them.

Investigating distribution of induced variables of this family of random profiles (Figure 4) at different error profile points shows that, first, all induced variables have mean of zero and equal standard deviation of $\sigma_e = 0.1639$ at the points located in the profile domain; therefore, assumption of error variance equality is established

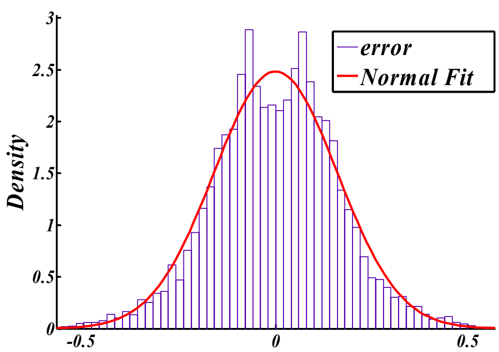
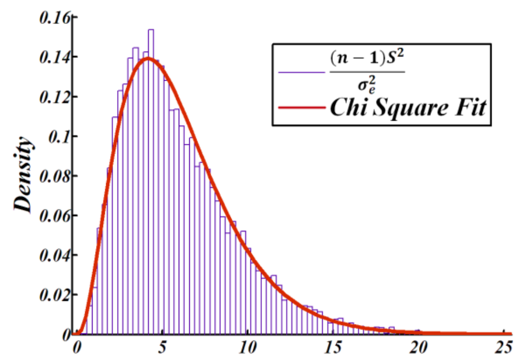


Figure 4. Goodness of fit test rejects normality of error distribution (*p-value* = 1.1e-16).

Investigating distribution of induced variables of this family of random profiles (Figure 4) at different error profile points shows that, first, all induced variables have mean of zero and equal standard deviation of $\sigma_e = 0.1639$ at the points located in the profile domain; therefore, assumption of error variance equality is established along the profile and, second, this distribution is bimodal and their distribution cannot be considered as random variables with normal distribution (small value of *p-value* obtained from chi-square goodness of fit test for hypothesis of error distribution normality also confirmed this issue).

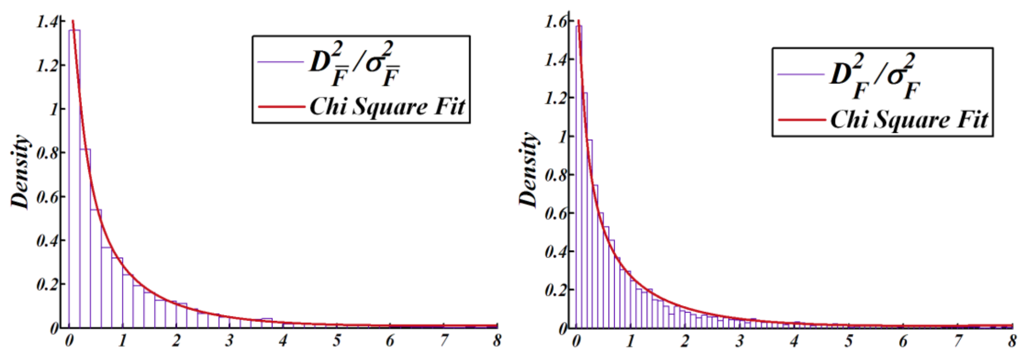
Considering that the designed tests are performed based on this assumption that division of random dispersion by random profile variance in the family of considered profiles has chi-squared distribution with one degree of freedom, this hypothesis is tested for profiles and their profile sample means. In Figure 5, distribution of these variables is given and the *p-value* obtained from goodness of fit test shows that these variables acceptably followed chi-squared distribution with one degree of freedom.

Figure 6 shows distribution of $\frac{(n-1)s^2}{\sigma_e^2}$. The p_



P_value ≈ 0.6

Figure 6. Goodness of fit test confirms chi-square distribution with *n*-1 degree of freedom for $\frac{(n-1)S^2}{\sigma_e^2}$.



P_value ≈ 0.3P_value ≈ 0.2

Figure 5. Goodness of fit test confirms chi-squared distribution with one degree of freedom for $\frac{\mathbf{D}_F^2}{\sigma_F^2}$ and $\frac{\mathbf{D}_{\bar{F}}^2}{\sigma_{\bar{F}}^2}$.

value obtained from goodness of fit test confirmed the proved fact in Theorem 12 that $\frac{(n-1)s^2}{\sigma_e^2}$ has chi-squared distribution with n-1 degree of freedom.

Now, statistical test can be done for profiles using some examples while ensuring that the assumption mentioned in this research are satisfied for the error profile introduced in (7). In the first example, null hypothesis $H_0 : \boldsymbol{\mu}_F(x) = 3 + 4x$ versus alternative hypothesis $H_1 : \boldsymbol{\mu}_F(x) \neq 3 + 4x$ in interval [2, 4] is tested when shape of this random profile changed with deviation from slope or intercept. In case null hypothesis is established, the random profile would be as $F(x) = 3 + 4x + \boldsymbol{e}_F(x)$, where $\boldsymbol{e}_F(x)$ is the error profile given in (7). Because so far no similar method has been presented for testing random profiles, its ability is compared with a classic test for mean of a scalar variable with standard normal distribution as control test when relative errors (ratio of null hypothesis error to variance) are equal. It means that, in the case of violating the null hypothesis, $H_0 : \mu = \mu_0$, and if $\mu = \mu_1$, relative errors of the null hypothesis of both tests are considered equal when:

$$\frac{1}{\sigma_e^2}\frac{\left\|(\mu_1-\mu_0)^2\right\|}{(b-a)} = \frac{1}{\sigma^2}\left|\mu_1-\mu_0\right|^2 \tag{8}$$

In the first study, intercept increased as much as $\lambda\sigma_e$ (mean of random profile is converted into $\mu_F(x) = \left(3+\lambda\sigma_e\right)+4x$) and, in the control test, mean increased to $\mu=\lambda$ such that relative error of null hypothesis of both tests became equal with concept (8) and comparison is possible. Figure 8 shows experimental OC curve for two profile tests using $X^2 = \frac{\mathbf{D}_F^2}{\sigma_e^2/n}$ and $F = \frac{\mathbf{D}_F^2}{s^2/n}$ statistic along with theoretical OC curve of the control test for some sample sizes at significance level of $\alpha = 0.05$. For more accurate experimental OC curves, length of simulation is considered to have 1,000 replication.

Performance of profile tests is very close to that of control test and both showed desirable performance.

The second study is related to increase of slope as much as $\beta\sigma_e$ (mean of random profile is converted into $\boldsymbol{\mu}_F(x) = 3 + (4+\beta\sigma_e)x$). To equalize relative error of null

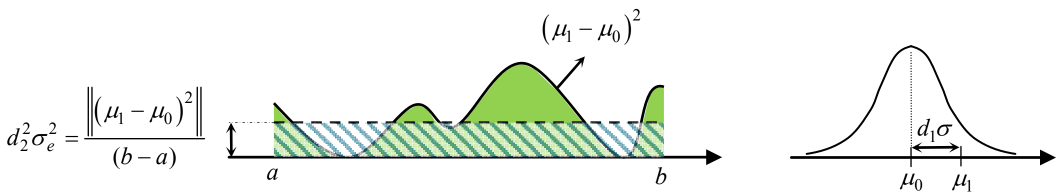


Figure 7. In case d_1 and d_2 are equal, relative errors of the two null hypotheses are considered equal.

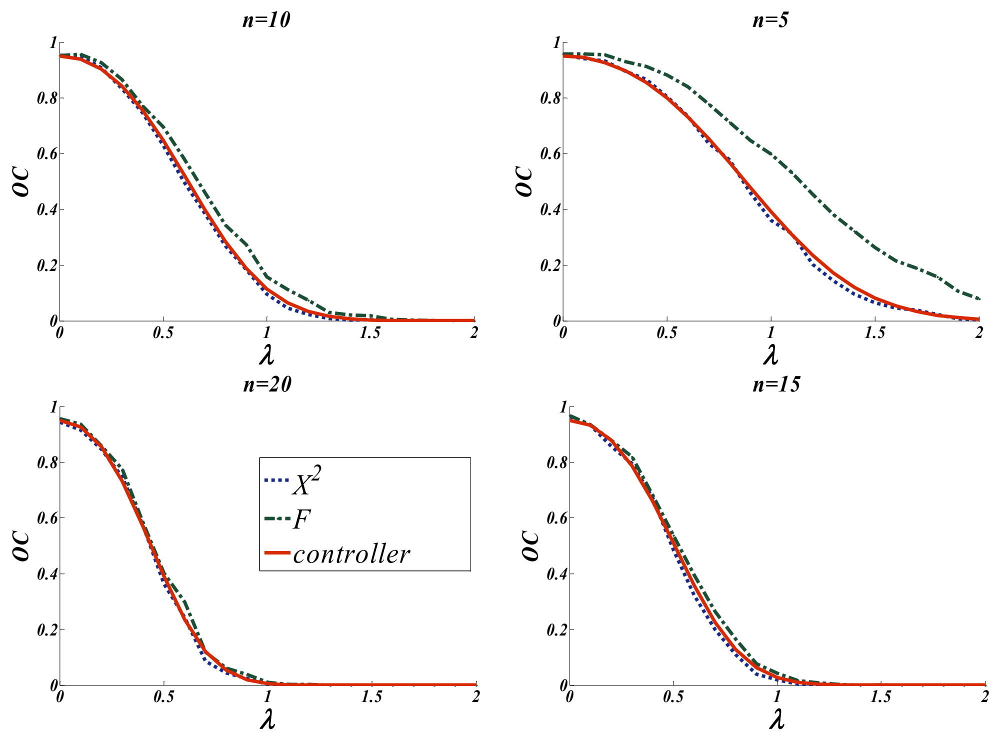


Figure 8. Comparing OC curves of the tests when intercept increases as much as $\lambda\sigma_e$.

hypothesis with concept (8), $\mu = \sqrt{\frac{28}{3}}\beta$ is considered in the control test. Figure 9 shows OC curves at significance level of $\alpha = 0.05$ in this case.

In this state, both profile tests have desirable performance compared with the control test, and their ability for recognizing deviation in null hypothesis is very close to each other's and also to that of the control test, especially for sample sizes of more than 10.

If the middle point of mean, $\mu_F(x) = 3 + 4x$, is kept fixed in the center of interval [2, 4] and the profile is

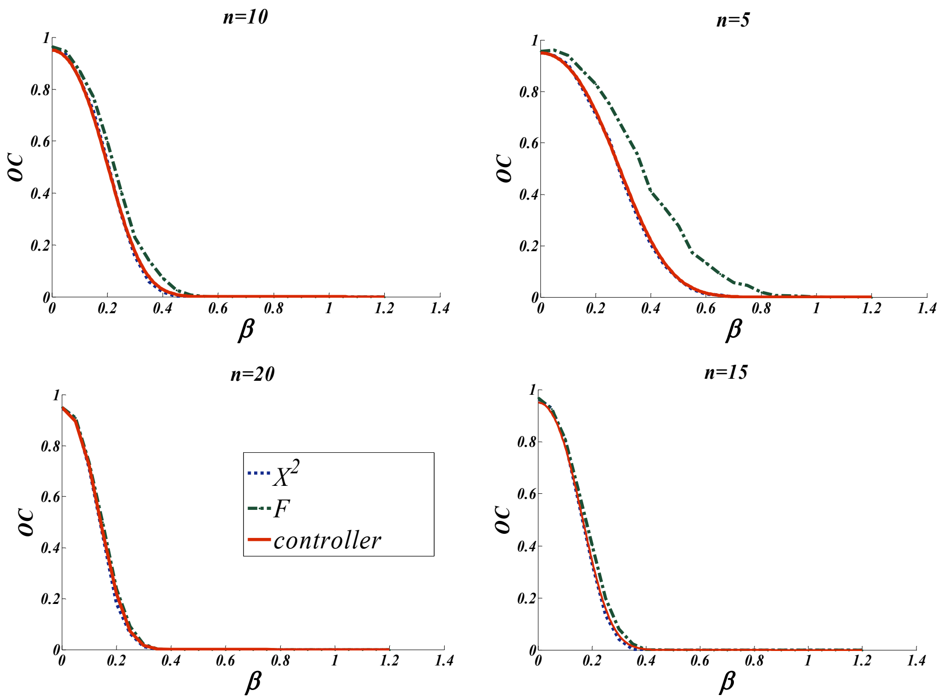


Figure 9. Comparing OC curves of the tests when slope increases as much as $\beta\sigma_e$.

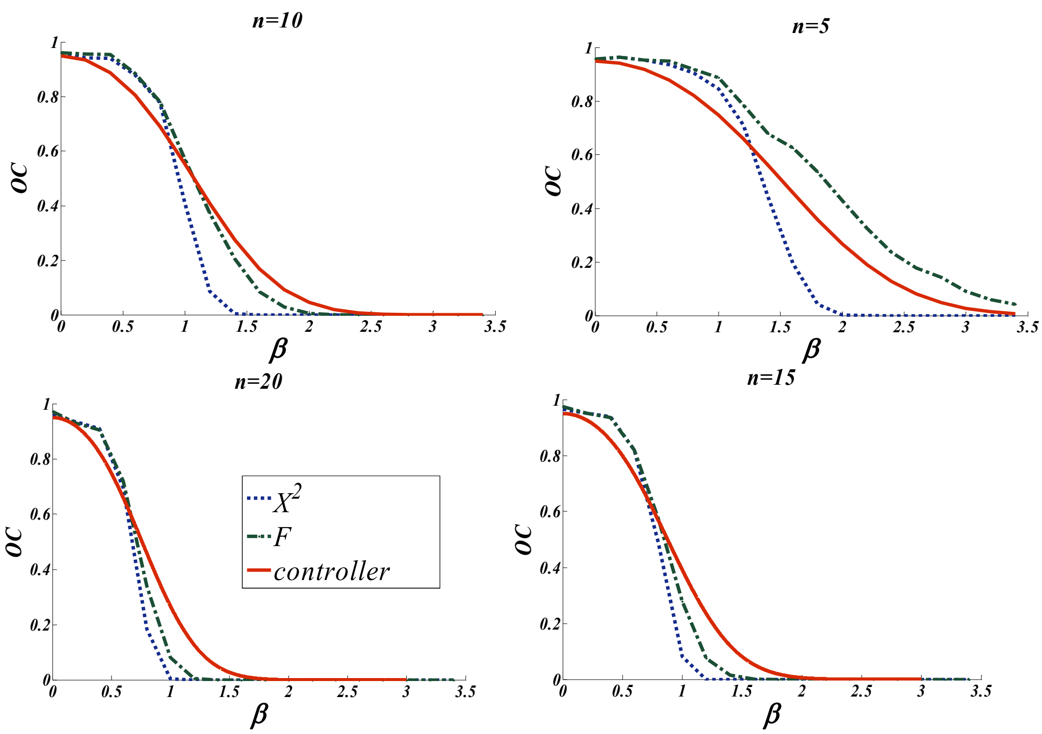


Figure 10. Comparing OC curves of the tests when mean line rotates around its center point as much as $\beta\sigma_e$.

rotated around it, the mean will change to $\mu_F(x)=(3-3\beta\sigma_e)+(4+\beta\sigma_e)x$. Investigating the ability of the presented profile test in recognizing this change is the third study subject in this section. Using the definition presented in (8), relative error of null hypothesis of two profile and control tests became equal by placing $\mu=\beta/\sqrt{3}$, and comparison of their ability is possible by OC curves. In Figure 10, some of these curves are given for different sizes at significance level of $\alpha=0.05$.

In this case, profile tests showed weaker performance than the control test in recognizing small deviations; but, when deviation from null hypothesis increased, their recognition ability remarkably improved by the profile tests and became better than the performance of the control test.

The last problem which is presented here considers a process, the output of which is always a sine function and a profile test is applied in it for recognizing phase difference. It means that, when $\mu_F(x)=sin\left(x+\frac{\pi}{2}\gamma\sigma_e\right)$, null hypothesis $H_0:\mu_F(x)=sin(x)$ is tested against alternative hypothesis $H_1:\mu_F(x)\neq sin(x)$. In order to obtain μ value in the control test so that the resulting OC curves which are given in Figure 11 can be comparable, definition of equality of relative error of null hypothesis in (8) isre-applied as follows:

$$\mu^2=\frac{1}{2\sigma_e^2}\int_2^4\left(sin\left(x+\frac{\pi}{2}\gamma\sigma_e\right)-sin(x)\right)^2dx$$

$$=\frac{1}{8\sigma_e^2}\left(8-8cos\left(\frac{\pi}{2}\gamma\sigma_e\right)-sin(8+\pi\gamma\sigma_e)-sin(8)\right.$$
$$\left.-sin\left(\frac{\pi}{2}\gamma\sigma_e\right)+sin(4+\pi\gamma\sigma_e)+sin(4)-2sin\left(4+\frac{\pi}{2}\gamma\sigma_e\right)\right)$$

6. CONCLUSION

Considering the presence of two elements of functionality and randomness in the nature of profiles, expanding classic statistics and probability concepts to the space of random profiles is scientifically valuable. Also this expansion makes it possible to perform other statistical techniques for random profiles, which have been left free thus far. In this paper, this development is done such that the presented definitions and theorems became consistent with the corresponding cases in space of scalar random variables by limiting profiles with constant but ran-dom functions. Then, two tests were presented for mean of random profiles using $X^2=\frac{\mathbf{D}_F^2}{\sigma_e^2/n}$ and $F=\frac{\mathbf{D}_F^2}{\sigma^2/n}$ sta-tistics as an example of applications resulting from this theory. The first statistic can be only used when variance of the profile population is known. Because no test has been presented for profiles so far,

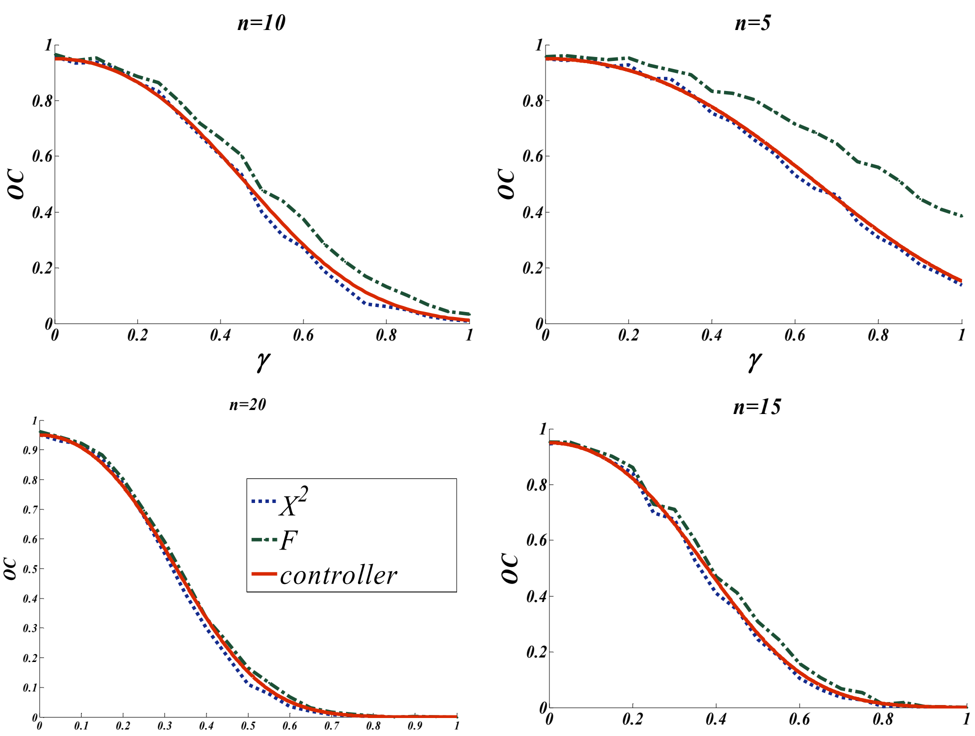


Figure 11. Comparing OC curves of the tests when phase difference of $\frac{\pi}{2}\gamma\sigma_e$ is created in profile's sinusoidal mean.

their performance is compared with that of a classic test related to scalar random variables in order to study the ability of these tests in recognizing violation of the null hypothesis. Results of the performed simulation showed that performance of profile test with X^2 statistic is very close to that of the control test in small sample sizes and it also had even better performance than the control test for lar-ger sample sizes. Its performance is always better than the test performed by F statistic. Generally, both tests showed proper performance.

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