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Zeros of Polynomials in East Asian Mathematics

東洋 數學에서 多項方程式의 解

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Since Jiuzhang Suanshu, mathematical structures in the traditional East Asian mathematics have been revealed by practical problems. Since then, polynomial equations are mostly the type of $p(x) = a_0$ where $p(x)$ has no constant term and a_0 is a positive number. This restriction for the polynomial equations hinders the systematic development of theory of equations. Since tianyuanshu (天元術) was introduced in the 11th century, the polynomial equations took the form of $p(x) = 0$, but it was not universally adopted. In the mean time, East Asian mathematicians were occupied by kaifangfa so that the concept of zeros of polynomials was not materialized. We also show that Suanxue Qimeng inflicted distinct developments of the theory of equations in three countries of East Asia.

Keywords: East Asian mathematics, zeros of polynomials, polynomial equations, tianyuanshu, kaifangfa, Suanxue Qimeng

MSC: 01A13, 01A25, 01A27, 11D09, 12-03, 26-03

1 Introduction

As is well known, the theory of equations is one of the most important subjects throughout the history of mathematics. In East Asian mathematics, the theory of linear equations together with systems of linear equations was established already in Jiuzhang Suanshu (九章筭術). It is also well known that equations of higher degrees were dealt in Jiuzhang Suanshu as extractions of square and cube roots and that a general quadratic equation was introduced in its last chapter gougu (勾股). The extractions were introduced by finding sides of squares or cubes with given areas or volumes, respectively so that the corresponding equations are of the type

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$x^n = a_0$, where a_0 is a positive number. Since Jiuzhang Suanshu set a standard format for mathematical literatures in East Asia and they reveal mathematical structures through practical problems, for East Asian mathematicians, polynomial equations are of the form $p(x) = a_0$, where $p(x)$ is a polynomial without a constant term and a_0 is a positive number. We call them *the first type* in the sequel. Thus the traditional mathematicians did not pay any attention to zeros of the polynomial $p(x)-a_0$.

Tianyuanshu (天元術) was introduced in the early Song dynasty (960–1279) and the equations constructed by tianyuanshu took the form of $q(x) = 0$ for a polynomial $q(x)$, called *the second type*. But tianyuanshu was not universally adopted by Song–Yuan mathematicians so that the two forms of $p(x) = a_0$ and $q(x) = 0$ coexisted in Song–Yuan mathematics. Zengchengfa (增乘法) was introduced in the 11th century and also applied to solve polynomial equations of the two types. Further tianyuanshu was lost in the Ming dynasty (1368–1643) until the late 18th century. These led Chinese mathematicians to miss the concept of zeros of polynomial equations.

Suanxue Qimeng (算學啓蒙, 1299) of Zhu Shijie (朱世傑) has been the most basic reference for the development of the mathematics throughout the Joseon dynasty (1392–1910). Thus their polynomial equations are mostly the second type and zengcheng kaifangfa was based on equations of the second type.

In the last decade of the 16th century, Suanxue Qimeng was introduced into Japan through Joseon and became the most influential reference to the development of Japanese mathematics as well. Thus their theory of equations was also based on equations of the second type.

Although the polynomial equations in Joseon and Japan are the second type, the concept of zeros of polynomials was not materialized because mathematicians in both countries were occupied to zengcheng kaifangfa as Chinese mathematicians. Thus they could not contribute to the development of modern theory of equations.

For the Chinese books included in [1, 8], they will not be numbered as an individual reference.

2 Polynomial equations in East Asia

As we mentioned in the previous section, the theory of equations in East Asia was originated in the fourth chapter, shaoguang (少廣) of Jiuzhang Suanshu for extracting square and cube roots which dealt with a side of square or cube given with its area or volume. The given numbers are called shi (實) as b for a linear equation $ax = b$. We note that the problem is not practical until the last chapter, gougu (勾股). Indeed, Liu Hui used the extraction of square roots to derive the formula for the area of a circle in the first chapter, fangtien (方田). Since extractions could be interpreted as solving equation $x^n = a_0$, they also give an important contribution to solving equations. The first general quadratic equation appears in Problem 20 of gougu chapter as follows:

> 術日 以出北門步數(20)乘西行步數(1,775), 倍之, 爲實
> 幷出南北門步數$(20 + 14)$, 爲從法 開方除之, 卽邑方

The above quote indicates the equation for the problem is $x^2 + 34x = 71,000$, i.e., the first type introduced in the previous section. Here shi is first set by a positive number and then the coefficient of linear term is decided. Further for a monic equation, the degree of the equation is indicated by solving equation without mentioning the highest term.

This pattern is retained in Jigu Suanjing (緝古筭經) of Wang Xiaotong (王孝通, fl. 7th C.) which deals with equations of degrees up to 4.

There has been a long blank of Chinese mathematical works until the 13th century. We could assume some development before the period by the following books among others. Yiguji (益古集) of an anonymous author is quoted in Yigu Yanduan (益古演段, 1259) of Li Ye (李冶, 1192–1279). Further, one can find some results of Huangdi Jiuzhang Suanjing Xichao (黃帝九章算經細草) of Jia Xian (賈憲, fl. 11th C.) in Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261) of Yang Hui, and those of Yigu Genyuan (議古根源) of Liu Yi (劉益, fl. 11th C.) in Yang Hui Suanfa (楊輝算法, 1274–1275).

Yigu Yanduan took examples in Yiguji to explain the advantage of tianyuanshu for constructing equations and compared the new algebraic method with the old geometrical one, called tiaoduan (條段) together with jiushu (舊術). Here jiushu should be the result in the original Yiguji. Equations given by the latter follow the first type originated by Jiuzhang Suanshu. The constant term by tiaoduan for Problem 18 is negative but it is positive by jiushu. We don't know why Li Ye took the equation between two negative terms and see [9] for the detailed proof. Unlike Jigu Suanjing, the equations by tiaoduan have all kinds of coefficients where the signs are exactly described except yiyu (益隅) of Problem 10 and only four cases among them are monic. Clearly the equations by tianyuanshu in Yigu Yanduan are the second type, namely $p(x) = 0$. Unfortunately, Yigu Yanduan was lost along with Ceyuan Haijing (測圓海鏡, 1248) in the Ming dynasty. We note that both books dealt with tianyuanshu and that Yigu Yanduan precisely showed the systematic advantage of tianyuanshu for constructing equations, in particular of the second type.

The equations of Liu Yi quoted by Yang Hui Suanfa are also denoted by the first type as those in Yiguji. Yang Hui Suanfa became a major reference for mathematicians in the Ming dynasty. We note that Wu Jing (吳敬) left his book Jiuzhang Suanfa Bilei Daquan (九章算法比類大全, 1450) and that Yang Hui used the term Bilei for mathematical structures [6]. Further Wu Jing included kaifang zuofa benyuan (開方作法本源), the triangle of Jia Xian from the missing chapter, shaoguang of Xiangjie Jiuzhang Suanfa. All the more he included that it is obtained by zengchengfa (增乘方求廉法). Cheng Dawei (程大位, 1533–1592) took Jiuzhang Suanfa Bilei Daquan as a main source for his Suanfa Tongzong (算法統宗, 1592). He quoted the Jia's triangle as kaifang qiulianlü zuofa benyuantu (開方求廉率作法本源圖) from Wu's book but deleted the sentence relating the triangle and zengchengfa. Because of the influence by Yang Hui's books, mathematicians of the Ming dynasty used the first type of equations. These were also retained by mathematicians of the Qing dynasty until the late 18th century when tianyuanshu was revived instead of jiegenfang (借根方) introduced in Shuli Jingyun (數理精蘊, 1723).

As is well known, Suanxue Qimeng is another book which deals with tianyuanshu. It has been one of the most important books for the development of Joseon mathematics since it was brought into Joseon in the middle of 15th century. Since the revival of Joseon mathematics in the 17th century, tianyuanshu became the main subject for Joseon mathematicians and hence their equations are mostly the second type [2]. Suanxue Qimeng was also one of the most important references for the development of wasan (和算) so that their equations were also mainly of the second type [10].

One of the most important works in the 13th century China is Shushu Jiuzhang (數書九章, 1247) of Qin Jiushao (秦九韶, 1202–1261). Although it contains the famous zengcheng kaifangfa, zhengfu kaisanchengfangtu (正負開三乘方圖) for an equation of the second type, Qin Jiushao used also the first type to represent equations. We will discuss how to solve equations based on the two types of equations in the next section.

In all, because of the Chinese notions of equations, mainly the first type, the concept of zeros of polynomials has not been materialized in China.

3 Zeros of polynomials and solving equations

In this section, we will discuss that solving polynomial equations in East Asia also interfere to lay the foundation of zeros of polynomials in East Asian mathematics.

As extractions of roots were introduced to solve geometrical problems in Jiuzhang Suanshu, they were solved by geometrical method in the book. The method is based on the expansion $(y+\alpha)^n$ to find the digits of the solution from the highest position. Further, an inductive device was formulated such that the remaining area (volume, resp.) (餘積) resulted by the previous digits becomes shi (實), or subtrahend and the minuend for the next digit, area (volume, resp.) is calculated exactly same as the previous ones. In the 11th century, algebraic approach to solve the extractions was introduced by Jia Xian, and they are shisuo pingfangfa (釋鎖平方法) and zengcheng kaipingfangfa (增乘開平方法) and corresponding ones for extractions of cube roots. They are quoted in Xiangjie Jiuzhang Suanfa Zuanlei (纂類). The former is exactly the same with the one in Jiuzhang Suanshu. One can easily discern that the constant terms in both cases are positive, i.e., the equation is the first type.

For solving general equations before the 13th century, we can find some information for constructing and solving equations by Liu Yi in Yang Hui Saunfa. Indeed, they were quoted in its second book Tianmu Bilei Chengchu Jiefa (田畝比類乘除捷法, 1275). They were dealt in Problem 6 to 16. We discuss Problem 7 in detail because its equation has a negative linear term and the solving methods follow basically Jia's two kaifangfa, respectively.

The equation for the problem is $x^2 - 12x = 864$, where the linear term is called fucong (負從) 12, i.e., -12. We quote the process of solving equation

> 草曰置積八百六十四於第二級 置差十二步於第四級爲負從 置隅一筭於五級
> 於第一級上商置長三十步 以乘隅於第三級置方法三十 以上商三十乘負從十二
> 添積三百六十 御除積九百餘積三百二十四步 二因方法共六十改名廉法一退
> 負從一退 隅二退 又於實上商置長六步 以乘隅一置六於廉次名 以上商六命負從
> 添積七十二 共積三百九十六 以廉隅之數命上商除實適盡得三十六步合問

We first note that we delete fu in fuyu (負隅) in the book. The above first solving method is called yiji kaifang (益積開方) resulting from the negative linear term and it follows basically shisuo kaifangfa. We also note that there are two linear terms, the given one cong and fangfa (方法) the latter of which is derived from the expansion $(y + 30)^2$ as in the extraction of square root. In the end, subtracting the remaining shi (餘積) by terms fucong and fangfa originated by $(y + 30)^2$, shi becomes 0 (除實適盡). Thus the author claims that the solution is 36.

We now quote the second method of solving the equation, called jiancong kaifang (減從開方)

> 草曰依五級資次布置 商積 方法 負從 隅筭 置積爲實 於實上商置長三十
> 以乘隅筭 置三十於實數之下 名曰方法 以負從十二減三十餘一十八命上商除實
> 五百四十餘積三百二十四 復以上商三十乘隅得三十併入方法共四十八退位爲廉
> 其隅筭再退 又於實上商長六步 以乘隅筭得六併入廉法共五十四
> 命上商六步除實盡得長三十六步合問

The difference between the above two methods is exactly the same with that of shisuo kaifangfa and zengcheng kaifangfa of Jia Xian. Zengchengfa reduces the calculations through adding cong and fangfa and then $2\times$(fangfa) is obtained by one more step of zengcheng. We refer to [3] for the structure of zengchengfa obtained by Joseon mathematician, Hong Jeong-ha (洪正夏, 1684–1727) in his Gu-il jib (九一集, 1713–1724) [2].

Problem 23 of Tianmu Bilei Chengchu Jiefa deals with an equation $p(x) = 4,096$ where $p(x) = -5x^4 + 52x^3 + 128x^2$ and $p(4) = 4,096$ is calculated by zengcheng as follows:

上商 $(= 4)$ 命二廉增乘至此爲法除,

which says that applying zengcheng to $p(x)$ at 4, one has the linear term and then $p(4)$. We suggest readers to compare ordinary calculation of $p(4)$ with the above zengchengfa.

We recall that all the examples of extracting roots in shaoguang of Jiuzhang Suanshu have *essentially* natural number solutions but cases of fujin (不盡), namely solutions being decimal, are mentioned and Liu Hui added commentary on how to evaluate their approximations. Sun Ji (孫子) dealt with examples of fujin in his Sun Ji Suanjing (孫子筭經). Presumably Liu Yi may deal with the cases of fujin but Yang Hui included the case of fujin in the third book, Xugu Zhaiqi Suanfa (續古摘奇算法, 1275) by quoting Biangu Tongyuan (辯古通源).

Although Qin Jiushao completed his book Shushu Jiuzhang in 1247, we discussed Yang Hui in the above for Yang quoted Liu Yi. Qin's methods of solving equations are basically the same with Liu's one based on zengcheng. Qin represented the process by calculating rods without indicating the negative numbers by the slashes but added the comments that indicate negative numbers and processes of zengcheng. For extracting roots, he followed Zia's zengcheng kaifangfa as shi is positive or the first type $x^n = a_0$ and for general equations, he used the second type, $p(x) = 0$ even though shi is given as positive when the equation is constructed. He also used the word chushi shijin (除實適盡) as the above example of Liu Yi, although the remaining shi is negative. We point out that the remaining shi for the cishang in the famous zhengfu kaisanchengfangtu, is positive because it is the case of fanji (飜積), also called huangu (換骨). Qin also had the case of yiji (益積), called toutai (投胎) in the process of solving equations. In zengcheng, the processes are simply done by multiplications and additions but Qin used the terminology xiangxiao (相消) for additions of the two terms with different signs (正負).

As mentioned in the previous section, Suanxue Qimeng also introduced tianyuanshu with which Zhu Shijie exemplified the advantage of tianyuanshu for constructing equations. For solving equations, Zhu included only extractions of roots via the first type $x^n = a_0$ and zengcheng kaifangfa and introduced the terminology qiajin (恰盡) instead of shijin in Liu Yi and Qin Jiushao.

Qin Jiushao extended the approximation of extraction of roots to those for general equations. We recall that the basic field for East Asian mathematics is the field

$\mathbb{Q}$ of rational numbers. Thus throughout the history of East Asian mathematics, approximations of solutions have been accepted as real solutions.

In the above discussions, an equation has an exact solution if and only if the remaining shi (餘積) for its last digit is zero. In other words, the last digit is a zero of the equation for the last digit in the zengcheng kaifangfa.

We recall that tianyuanshu is a perfect method to show the algebraic structure of polynomials but the tianyuan (天元) is always chosen as an unknown for a practical problem. Thus tianyuan has not played a role of a variable of polynomial functions even though its value at a given number is easily obtained by zengcheng. Thus the concept of polynomial functions through tianyuanshu has not been materialized in East Asian mathematics (see [5, 7] for the detail).

As mentioned above, jiegenfang in Shuli Jingyun replaced tianyuanshu and its equations are of the first type. In order to discuss solving equations in the book, we take Problem 5 in Chapter 33 whose equation is $x^3 + 13x^2 + 30x = 27,144$.

One can hardly conceive that zengchengfa is completely forgotten in Shuli Jingyun. First cushang 20 is chosen and $p(20) = 13,800$ is calculated by substitution, where $p(x) = x^3 + 13x^2 + 30x$ and then $27,144 - p(20) = 13,344$ called cishangji (次商積). The coefficient of the linear term of $p(y + 20)$, called cishang lianfa (次商廉法) is obtained by the expansions $a_k(y + 30)^k$ in $p(y + 30)$. They are easily obtained by zengchengfa. Dividing cishangji by cishang lianfa, cishang 6 is determined by substitution. Finally $p(26) = 27,144$ is calculated again by substition and concluded that 26 is a solution of the equation by adding qiajin (恰盡). The method is much more cumbersome than zengcheng kaifangfa but its only advantage is that the solution of an equation is precisely one which makes the equality at the solution, or in this case, 26 is a zero of $p(x) - 27,144$. Further, the guessing cishang by the tangent line at cushang as above does not work in general as shown in [4]. We also note that divisions of polynomials in jiegenfang are dealt in Shuli Jingyun but they can't connect solution α of $p(x) = 0$ with its divisibility by $x - \alpha$

Mathematicians in East Asia have been so occupied by the advantage of zengcheng together with inevitable approximations that they probably took it for granted that the exact solution obtained by shisuo or zengcheng kaifangfa, is really one that satisfies the original equation, i.e., zero of the given polynomial except authors of Shuli Jingyun.

4 Conclusions

In the traditional mathematics in East Asia, its mathematical structures have been illustrated by practical examples and its basic field is the field $\mathbb{Q}$ of rational numbers. These restrictions have formed a great impediment to the development of East

Asian mathematics, in particular that of theory of equations throughout its whole history. But they could build a perfect theory of equations in such restrictions by tianyuanshu up to siyuanshu and kaifangfa.

Investigating major sources dealing with theory of equations, we find that there are two types of representing or constructing equations and that these entail also some confusion for solving equations. Further, along the changes of dynasties, the achievements in the previous dynasty were disregarded or even completely lost in the next dynasties. This improbable incident happened most notably in the theory of equations. Thus the concept of zeros of polynomials has not materialized in East Asian mathematics so that the traditional East Asian mathematics fails to contribute to the development of modern theory of equations even though they did have a complete theory of equations in the field of rational numbers.

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