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ON QUASI-COMMUTATIVE RINGS

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Abstract. We study the structure of central elements in relation with polynomial rings and introduce *quasi-commutative* as a generalization of commutative rings. The Jacobson radical of the polynomial ring over a quasi-commutative ring is shown to coincide with the set of all nilpotent polynomials; and locally finite quasi-commutative rings are shown to be commutative. We also provide several sorts of examples by showing the relations between quasi-commutative rings and other ring properties which have roles in ring theory. We examine next various sorts of ring extensions of quasi-commutative rings.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let $J(R)$, $N_0(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the Wedderburn radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in a given ring R (possibly without identity), respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$. $C(R)$ means the center of R, i.e., the set of all central elements in R. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$) and for any polynomial (resp., power series) $f(x)$ in $R[x]$ (resp., $R[[x]]$), let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. $\mathbb{Z}_n$ denotes the ring of integers modulo n. Denote the n by n upper triangular matrix ring over R by $U_n(R)$. Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $|S|$ denotes the cardinality of a given set S.

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. Let R be a reduced ring and suppose that $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. In this situation, Armendariz [3, Lemma 1] proved that $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. Rege and Chhawchharia [12] called a ring (possibly without identity) *Armendariz* if it satisfies such property. So

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reduced rings are clearly Armendariz. This fact will be used freely in this note. A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [2, Theorem 6] (or [11, Lemma 7]).

We see in the following an equivalent condition to the commutativity of rings by applying the Armendariz property to power series rings on centers.

Lemma 1.1. *For a ring R the following conditions are equivalent*:

(1) *R is commutative*;

(2) *If $f(x)g(x) \in C(R)[[x]]$ for $f(x), g(x) \in R[[x]]$, then $ab \in C(R)$ for all $a \in C_{f(x)}, b \in C_{g(x)}$*;

(3) *If $ab \in C(R)$ for $a, b \in R$ then $aRb \subseteq C(R)$.*

Proof. It suffices to show (2) ⇒ (1) and (3) ⇒ (1).

(2) ⇒ (1). Suppose that the condition (2) holds. Let $a \in R$ and consider two power series $f(x) = 1 - ax, g(x) = 1 + ax + a^2x^2 + \cdots + a^nx^n + \cdots$ in $R[[x]]$. Then $f(x)g(x) = 1$, so we have $a \in C(R)$ by the condition (2).

(3) ⇒ (1). Suppose that the condition (3) holds. Then $1 \cdot 1 = 1$ implies that $r = 1r1 \in C(R)$ for all $r \in R$. □

We shall consider next a class of rings which is provided by Armendariz ring property over centers, applying the condition (2) in Lemma 1.1 to polynomials. This is given in the following.

Definition 1.2. A ring R is said to be *quasi-commutative* if $ab \in C(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in C(R)[x]$.

Every commutative ring is clearly quasi-commutative ring, but there exist quasi-commutative rings which are not commmutative as follows. It is easily checked that $C(R[x]) = C(R)[x]$, so we will use this fact freely.

Example 1.3. (1) Let K be a field and S be a set of noncommuting indeterminates over K. Suppose $|S| \geq 2$. Let R be the free algebra generated by S over K, and B be the set of all polynomials of zero constant in R. R is clearly a noncommutative ring.

We first claim $C(R) = K$. To see this, let $a = k + b \in C(R)$ with $k \in K$ and $b \in B$. Assume $b \neq 0$. Then there exists $s \in S$ such that $sb \neq bs$. But $ks + sb = sa = as = ks + bs$ implies $sb = bs$, contradicting $sb \neq bs$. Thus we have $b = 0$, entailing $a = k$. We obtain next $C(R[x]) = C(R)[x] = K[x]$ by claim.

Suppose that $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in C(R)[x]$. Here we can express $f(x), g(x)$ by

$$f(x) = f_0 + f_1 \text{ and } g(x) = g_0 + g_1$$

for $f_0, g_0 \in K[x], f_1, g_1 \in B[x]$. Then

$$f(x)g(x) = f_0g_0 + f_0g_1 + f_1g_1 + f_1g_0 \in C(R)[x],$$

entailing

$$f_0g_1 + f_1g_1 + f_1g_0 \in C(R)[x].$$

Here if $f_0 = 0$ (i.e., $f(x) = f_1$) or $g_0 = 0$ (i.e., $g(x) = g_1$), then $0 \neq f(x)g(x) = f_0g_1 + f_1g_1 + f_1g_0 \in C(R)[x]$. This induces a contradiction since $0 \neq f_0g_1 + f_1g_1 + f_1g_0 \in B[x]$ and this cannot be contained in $C(R)[x]$. Thus $f_0 \neq 0$ and $g_0 \neq 0$, entailing $f(x)g(x) = f_0g_0 + f_0g_1 + f_1g_1 + f_1g_0$ with $f_0g_0 \neq 0$.

Since $f_0g_1 + f_1g_1 + f_1g_0 \notin C(R[x])$ if nonzero, we must have $f_0g_1 + f_1g_0 + f_1g_1 = 0$. But $f_0g_1 + f_1g_0 + f_1g_1 = 0$ implies

$$f_0g_1 + f_1g_0 = 0 \text{ and } f_1g_1 = 0$$

since the degree of f_1g_1 is larger than one of $f_0g_1 + f_1g_0$. Thus we have $f_1 = 0$ or $g_1 = 0$.

If $f_1 = 0$ (i.e., $f(x) = f_0$), then $f(x)g(x) = f_0(g_0 + g_1) = f_0g_0 + f_0g_1 \in C(R)[x]$. This forces $f_0g_1 = 0$ (hence $g_1 = 0$) since $f_0g_1 \notin C(R[x])$ if nonzero. So $f(x) = f_0$ and $g(x) = g_0$.

Similarly if $g_1 = 0$ (i.e. $g(x) = g_0$), then $f_1g_0 = g_0f_1 = 0$ (hence $f_1 = 0$) and $f(x) = f_0$ and $g(x) = g_0$.

Consequently $\alpha\beta \in K = C(R)$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$, and therefore R is quasi-commutative.

(2) Let K be a field and $T = \{a_i, c \mid i \in I\}$ be a set of noncommuting indeterminates over K, where I is an index set. Set $S = \{a_i \mid i \in I\}$ and assume $|S| \geq 2$. Let A (resp., A_0) be the free algebra generated by T (resp., S) over K, and J be the ideal of A generated by

$$ac - ca, abc - bac, \text{ and } c^n \text{ for all } a, b \in S,$$

where $n \geq 2$. Let B (resp., B_0) be the set of all polynomials of zero constant in A (resp., A_0), and $R = A/J$. Identify a_i's and c with their images in R for simplicity.

Every element of R can be written by $r = k + s + t$ with $k \in K, s \in B_0$, and $t \in cA$. Assume $s \neq 0$. Then there exists $u \in B_0$ such that $su \neq us$. This yields $uk + us + ut = ur \neq ru = ku + su + tu$, entailing $r \notin C(R)$. So we get $C(R) = K + cR$.

Every polynomial $f(x)$ over R can be expressed by

$$f(x) = f_0 + f_1 + f_2 \text{ for } f_0 \in K[x],\ f_1 \in B_0[x],\ f_2 \in cA[x].$$

Now let $f(x)g(x) \in C(R)[x]$ for

$$0 \neq f(x) = f_0 + f_1 + f_2 \text{ and } 0 \neq g(x) = g_0 + g_1 + g_2$$

with $f_0, g_0 \in K[x]$, $f_1, g_1 \in B_0[x]$ and $f_2, g_2 \in cA[x]$. Then we have

$$f(x)g(x) = f_0g_0 + f_0g_1 + f_1g_1 + f_1g_0 + h \in C(R)[x]$$

for some $h \in cA[x]$. This yields

$$f_0g_1 + f_1g_1 + f_1g_0 \in C(R)[x]$$

since $f_0g_0, h \in C(R)[x]$. Here if $f_1 \neq 0$ and $g_1 \neq 0$, then $f_0g_1 + f_1g_1 + f_1g_0 \neq 0$ since the degree of f_1g_1 is larger than one of $f_0g_1 + f_1g_0$. This induces a contradiction because $0 \neq f_0g_1 + f_1g_1 + f_1g_0$ cannot be contained in $C(R)[x]$. Thus we have $f_1 = 0$ or $g_1 = 0$.

Suppose $f_1 = g_1 = 0$. Then $f(x) = f_0 + f_2, g(x) = g_0 + g_2 \in C(R)[x]$, so $\alpha\beta \in C(R)$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$.

Suppose $f_1 = 0$ (i.e., $f(x) = f_0 + f_2$). Then $f(x)g(x) = f_0g_1 + h_1 \in C(R)[x]$ for some $h_1 \in C(R)[x]$, so $f_0g_1 = 0$. Thus $f_0 = 0$ or $g_1 = 0$. Consequently we have that "$f(x) = f_0 + f_2$ (with $f_0 \neq 0$) and $g(x) = g_0 + g_2$" or "$f(x) = f_2$ and $g(x) = g_0 + g_1 + g_2$". So $\alpha\beta \in C(R)$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$.

Suppose $g_1 = 0$. Then we have similarly that $\alpha\beta \in C(R)$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$.

Consequently, in any case, $\alpha\beta \in C(R)$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$. Thus R is quasi-commutative.

The ring in Example 1.3(2) is a noncommutative non-reduced quasi-commutative ring. But there can also exists many noncommutative non-reduced quasi-commutative rings by help of Corollary 2.7 to follow.

One may hope division rings to be quasi-commutative, considering the quasi-commutative domain R in Example 1.3. However the Hamilton quaternions $\mathbb{H}$ over the real number field $\mathbb{R}$ is not quasi-commutative as can be seen by the fact that $(1 - ix)(1 + ix) = 1 + x^2 \in C(\mathbb{H})[x]$ and $i \notin C(\mathbb{H}) = \mathbb{R}$. Division rings are clearly Armendariz, and there exists a commutative ring which is not Armendariz by [12, Example 3.2]. Therefore the concepts of Armendariz and quasi-commutativity are independent of each other.

Proposition 1.4. *Let R be a quasi-commutative ring and suppose that $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are polynomials in $R[x]$. If $f_1(x)f_2(x)\cdots f_n(x) \in C(R)[x]$, then $a_1a_2\cdots a_n \in C(R)$ for all $a_i \in C_{f_i(x)}$.*

Proof. We apply the proof of [2, Proposition 1]. Suppose that

$$f_1(x)f_2(x)\cdots f_n(x) \in C(R)[x]$$

for $f_1(x), f_2(x), \ldots, f_n(x) \in R[x]$. We have $f_1(x)(f_2(x)\cdots f_n(x)) \in C(R)[x]$, so the quasi-commutativity of R implies $a_1b \in C(R)$ for any $b \in C_{f_2(x)\cdots f_n(x)}$. This entails $(a_1f_2(x))(f_3(x)\cdots f_n(x)) \in C(R)[x]$. Since $a_1a_2 \in C_{a_1f_2(x)}$, the quasi-commutativity of R implies $a_1a_2c \in C(R)$ for any $c \in C_{f_3(x)\cdots f_n(x)}$. This entails $(a_1a_2f_3(x))(f_4(x)\cdots f_n(x)) \in C(R)[x]$. Continuing, we obtain inductively $a_1a_2\cdots a_n \in C(R)$. □

We next introduce a lemma which will make our approach to the nature of quasi-commutative rings much easier.

Lemma 1.5. *Let R be a quasi-commutative ring. Then we have the following.*

(1) *Let $a \in R$. If $a^n \in C(R)$ for some $n \geq 1$ then $a \in C(R)$.*

(2) *$N(R) \subseteq C(R)$.*

(3) *$N(R) = N_*(R) = N^*(R) = N_0(R)$.*

(4) *Let $a, b, c \in R$ and suppose $ac \in C(R)$. If $ab^n c \in C(R)$ for some $n \geq 1$, then $abc \in C(R)$.*

Proof. (1) For $a \in R$, suppose that $a^n \in C(R)$ for some $n \geq 1$. Then

$$(1 - ax)(1 + ax + a^2x^2 + \cdots + a^{n-1}x^{n-1}) = 1 - a^n x^n \in C(R)[x].$$

Since R is quasi-commutative, we have $a \in C(R)$.

(2) is an immediate consequences of (1).

(3) Let $a \in N(R)$, say $a^n = 0$ for some $n \geq 1$. Then $a \in C(R)$ by (2), and so we have $(RaR)^n = a^n R = 0$. This implies $a \in N_0(R)$. Thus $N(R) = N_*(R) = N^*(R) = N_0(R)$.

(4) Suppose that $ab^n c \in C(R)$ for some $n \geq 1$. Then $a(1 - bx)(1 + bx + \cdots + b^{n-1}x^{n-1})c = a(1 - b^n x^n)c = ac - ab^n cx^n \in C(R)[x]$. Since R is quasi-commutative, we have $abc \in C(R)$ by Proposition 1.4. □

The Hamilton quaternions over any subring of the real number field cannot be quasi-commutative by Lemma 1.5(1). If a ring R has a non-central self-invertible unit, then R is not quasi-commutative by Lemma 1.5(1). A group ring is not quasi-commutative by Lemma 1.5(1) if it is generated by a non-Abelian group which has a self-invertible non-central element.

Proposition 1.6. *Let R be a quasi-commutative ring. Then*

$$J(R[x]) = N_0(R[x]) = N_*(R[x]) = N^*(R[x]) = N_0(R)[x] = N(R)[x] = N(R[x]),$$

and $R[x]/J(R[x])$ is a reduced ring.

Proof. Let R be a quasi-commutative ring. Then $N_0(R) = N_*(R) = N^*(R) = N(R)$ by Lemma 1.5(3). This yields $J(R[x]) \subseteq N^*(R)[x]$ by help of [1, Theorem 1], entailing $J(R[x]) \subseteq N_*(R)[x]$. But $N_*(R)[x] = N_*(R[x])$ by [1, Theorem 3], and $N_*(R[x]) \subseteq J(R[x])$. Thus we have

$$J(R[x]) = N^*(R)[x] = N_*(R[x]) = N_*(R)[x] = N(R)[x] = N_0(R)[x],$$

combining these results. Moreover $N_0(R)[x] = N_0(R[x])$ by [6, Corollary 4], so we get the equality

$$J(R[x]) = N^*(R)[x] = N_*(R[x]) = N_*(R)[x] = N(R)[x] = N_0(R)[x] = N_0(R[x]).$$

Since $N_*(R) = N(R)$ as above, $N(R[x]) = N_*(R[x])$ by [5, Proposition 2.6]. This implies that

$$\frac{R[x]}{J(R[x])} = \frac{R[x]}{N_*(R[x])} = \frac{R[x]}{N(R[x])}$$

is a reduced ring. □

For a ring R and $n \geq 2$, we consider the subring

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\}$$

of the n by n upper triangular matrix ring $U_n(R)$. We next provide a basic relation which combines the commutativity and the quasi-commutativity, via the structure of $D_2(R)$.

Proposition 1.7. *For a ring R, the following conditions are all equivalent*:

(1) *R is a commutative ring*;

(2) *$D_2(R)$ is a commutative ring*;

(3) *$D_2(R)$ is a quasi-commutative ring.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): Let $E = D_2(R)$ be a quasi-commutative ring. Then $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right) \subseteq C(E)$ by Lemma 1.5(2) since $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right) \in N(E)$. This implies that for any $r \in R$,

$$\begin{pmatrix} 0 & rs \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & sr \\ 0 & 0 \end{pmatrix}$$

for all $s \in R$. Thus we get $rs = sr$, concluding that R is a commutative ring. □

Base on Proposition 1.7, one may ask whether $D_n(R)$ is also quasi-commutative for $n \geq 3$. However the following erases the possibility.

Remark. Let A be any ring. Then $R = D_n(A)$ is not a quasi-commutative ring for all $n \geq 3$. Note that $e_{12} \in N(R)$, but $e_{12} \notin C(R)$. Thus R is not quasi-commutative by Lemma 1.5(2). □

Following Bell [4], a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reduced rings are clearly IFP by simple computation. It is also easily checked that IFP rings are Abelian.

Recall that IFP rings need not be quasi-commutative. But we do not know whether quasi-commutative rings are IFP.

Proposition 1.8. (1) *Quasi-commutative rings are Abelian.*

(2) *Let R be a quasi-commutative ring. Then we have*

$$(RaRbR)^2 = 0 \text{ and } (RbRaR)^2 = 0$$

whenever $ab = 0$ for $a, b \in R$.

Proof. (1) Let R be a quasi-commutative ring. Assume on the contrary that there exist $e^2 = e, r \in R$ such that $er(1-e) \neq 0$. Let $a = er(1-e)$. Consider $f(x) = e + ax$, $g(x) = (1-e) - ax$, then $f(x)g(x) = 0 \in C(R)[x]$. Since R is quasi-commutative, $ea = a \in C(R)$, and so $0 \neq a = ea = ae = 0$. This induces a contradiction.

(2) Let R be a quasi-commutative ring and suppose that $ab = 0$ for $a, b \in R$. Then $bRa \subseteq N(R)$, and so $bRa \subseteq C(R)$ by Lemma 1.5(2). This yields that

$$r_1ar_2br_3ar_4br_5 = r_1a(br_3a)r_2r_4br_5 = 0,$$

where r_i's are any elements in R. This implies $(RaRbR)^2 = 0$. A similar computation gives us $(RbRaR)^2 = 0$. □

Proposition 1.8(1) may be proved in various ways as can be seen by the equality of $[e+(1-e)x][(1-e)+ex] = x$. There exist many Abelian rings which are not quasi-commutative by help of Lemma 1.5(2). For example, Hamilton quaternions over any subring of the real number field, and $D_n(R)$ over any Abelian ring R for $n \geq 3$ (refer to [8, Lemma 2]). So the converse of Proposition 1.8(1) is not true in general. The converse of Proposition 1.8(2) also need not hold by the following.

Example 1.9. We use the ring and argument in [10, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$, and B be the set of polynomials of zero constant term in A.

Let I be the ideal of A generated by $a_0rb_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$, and $r_1r_2r_3r_4$, where $r \in A$ and $r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \in I$. Let $R = A/I$. We identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity.

Then R is not quasi-commutative by Lemma 1.5(2) as can be seen by the fact that $a_0 \in N(R)$ but $a_0b_0 \neq b_0a_0$ (i.e., $a_0 \notin C(R)$).

Next let $f(x)g(x) = 0$ for $0 \neq f(x), g(x) \in R[x]$. Here we can express $f(x), g(x)$ by

$$f(x) = f_0 + f_1 \text{ and } g(x) = g_0 + g_1$$

for $f_0, g_0 \in \mathbb{Z}_2[x], f_1, g_1 \in B[x]$. Then

$$f(x)g(x) = f_0g_0 + f_0g_1 + f_1g_1 + f_1g_0 = 0,$$

entailing that

$$f_0g_0 = 0 \text{ and } f_0g_1 + f_1g_1 + f_1g_0 = 0.$$

So $f_0 = 0$ or $g_0 = 0$ from $f_0g_0 = 0$.

Assume $f_0 \neq 0$. Then $g_0 = 0$; hence we have $g_1 \neq 0$ and $f_0g_1 + f_1g_1 = 0$. Note that $f_1 \neq 0$ since $f_0g_1 \neq 0$. The degree of the term of smallest degree in f_1g_1 is larger than the degree of f_0g_1. This forces $f_0g_1 = 0$ and $f_1g_1 = 0$ since $f_0g_1 + f_1g_1 = 0$. But since $f_0 \neq 0$, we have $g_1 = 0$. This entails $g(x) = 0$, a contradiction to $g(x) \neq 0$. Consequently $f_0 = 0$. Similarly we get $g_0 = 0$. Therefore we now have $f(x) = f_1$ and $g(x) = g_1$. This also yields

$$(Rf(x)Rg(x)R)^2, (Rg(x)Rf(x)R)^2 \in (RBRBR)^2[x].$$

But $(RBRBR)^2 \subseteq B^4 = 0$, and so this implies $(Rf(x)Rg(x)R)^2 = 0 = (Rg(x)Rf(x)R)^2$.

The preceding example also shows that the converse of Proposition 1.8(1) need not hold. In fact, the ring R in Example 1.9 is IFP (hence Abelian) by [10, Example 2], but R is not quasi-commutative.

Note. Suppose that R be a quasi-commutative ring of characteristic 2, and let $I = N(R)$. Let $a + I \in C(R/I)$ and $r \in R$. Then $a^2 \in C(R)$ and $(ar)^k = (ra)^k$ for some $k \geq 2$.

Proof. Let $a + I \in C(R/I)$ and $r \in R$. Then $ar - ra \in I$. By Lemma 1.5(2), we get

$$a^2r - ara = a(ar - ra) = (ar - ra)a = ara - ra^2.$$

Since R is of characteristic 2, we have $a^2r = ra^2$. This implies $a^2 \in C(R)$. We obtain similarly $ar^2 = r^2a$, using $r(ar - ra) = (ar - ra)r$.

Next since $ar - ra \in N(R)$, $(ar - ra)^m = 0$ for some $m \geq 1$. We use freely $a^2 \in C(R)$ and $r^2a = ar^2$ in the following computation.

$$(ar-ra)^2 = arar-ar^2a-ra^2r+rara = arar-a^2r^2-a^2r^2+rara = (ar)^2+(ra)^2;$$

$$(ar - ra)^4 = ((ar)^2 + (ra)^2)^2 = (ar)^4 + a^4r^4 + a^4r^4 + (ra)^4 = (ar)^4 + (ra)^4;$$

$$\cdots\cdots\cdots$$

$$0 = (ar - ra)^{2m} = (ar)^{2m} + a^{2m}r^{2m} + a^{2m}r^{2m} + (ra)^{2m} = (ar)^{2m} + (ra)^{2m}.$$

Letting now $k = 2m$, we get $(ar)^k = (ra)^k$. □

Following [10], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. It is shown that a ring is locally finite if every finite subset generates a finite subring by [9, Theorem 2.2(1)]. It is obvious that the class of locally finite rings contains finite rings and algebraic closures of finite fields.

Corollary 1.10. (1) *Let R be a locally finite ring. Then R is quasi-commutative if and only if R is commutative.*

(2) *Finite quasi-commutative rings are commutative.*

Proof. It suffices to show the necessity. Let R be quasi-commutative and $a \in R$. Since R is locally finite, a^n is an idempotent for some $n \geq 1$ by the proof of [10, Propostion 16]. But $a^n \in C(R)$ by Proposition 1.8(1) since R is quasi-commutative. This yields $a \in C(R)$ by Lemma 1.5(1) also since R is quasi-commutative. Thus R is commutative.

(2) is an immediate consequence of (1). □

We can see in the following a little different proof of Corollary 1.10(1) by using Lemma 1.5(5). The idempotent a^n in the proof is central, we have $1a^n1 = a^n \in C(R)$; hence $a = 1a1 \in C(R)$ by Lemma 1.5(5).

We shall use the definition of quasi-commutative ring also for rings without identity. One may conjecture that a ring R may be quasi-commutative when both R/I and I are quasi-commutative, where I is a proper ideal of R and is quasi-commutative as a ring without identity. However the following erases the possibility.

Example 1.11. Let K be a field. Let $A = K\langle a, b, c\rangle$ be the free algebra generated by the noncommuting indeterminates a, b, c over K. Set I be the ideal of A generated by

$$ac - ca, bc - cb, c^2$$

and R be the factor ring A/I. We identify a, b, c with their images in R/I. It is easily checked that $N(R) = Rc = cR = RcR = N_0(R)$. Here we have

$$\frac{R}{N_0(R)} \cong K\langle a, b\rangle.$$

So $R/N_0(R)$ is quasi-commutative by Example 1.3. Moreover $N(R)^2 = (Rc)^2 = Rc^2 = 0$, and so $N(R)$ is a (quasi-)commutative ring.

But $ac \in N(R)$ and $bac \neq acb = abc$, entailing $ac \notin C(R)$. Thus R is not quasi-commutative by Lemma 1.5(2).

2. Examples of quasi-commutative rings

In this section we investigate the quasi-commutativity of various kinds of ring extensions which have roles in ring theory. We first examine the quasi-commutativity can pass to polynomial rings. We use $\deg(f(x))$ to denote the degree of a given polynomial $f(x)$.

Proposition 2.1. *A ring R is quasi-commutative if and only if so is $R[x]$.*

Proof. Let R be a quasi-commutative ring. Suppose that $f(t)g(t) \in C(R[x])[t]$ for $0 \neq f(t) = \sum_{i=0}^{m} f_i(x)t^i$, $g(t) = \sum_{j=0}^{n} g_j(x)t^j \in R[x][t]$, where $R[x][t]$ is the polynomial ring with an indeterminate t over $R[x]$. Next let $f_i(x) = a_{i_0} + a_{i_1}x + \cdots + a_{i_w}x^{i_w}$ and $g_j(x) = b_{j_0} + b_{j_1}x + \cdots + b_{j_v}x^{j_v}$ for all i and j, where $a_{i_0}, \ldots, a_{i_w}, b_{j_0}, \ldots, b_{j_v} \in R$.

Note $C(R[x][t]) = C(R[x])[t] = C(R)[x][t]$. So $f(t)g(t) \in C(R)[x][t]$.

We apply the proof of [2, Theorem 2] to show that $R[x]$ is quasi-commutative. Let $k = \sum_{i=0}^{m} \deg(f_i(x)) + \sum_{j=0}^{n} \deg(g_j(x))$, where the degree is considered as polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0. Set

$$F(x) = f(x^k) = \sum_{i=0}^{m} f_i(x)(x^k)^i \text{ and } G(x) = g(x^k) = \sum_{j=0}^{n} g_j(x)(x^k)^j \in R[x].$$

Then the set of all coefficients of f_i's (resp., g_j's) equals the set of all coefficients of $F(x)$ (resp., $G(x)$). Thus we have

$$F(x)G(x) \in C(R)[x]$$

from $f(t)g(t) \in C(R)[x][t]$.

Since R is quasi-commutative, $ab \in C(R)$ for all $a \in C_{F(x)}, b \in C_{G(x)}$. This yields $f_i(x)g_j(x) \in C(R[x])$, recalling $C(R[x]) = C(R)[x]$. Therefore $R[x]$ is quasi-commutative.

For the proof of the converse, let $f(x)g(x) \in C(R)[x]$ for $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Then we can write $f(t)g(t) \in C(R[x])[t]$ for $f(t) = \sum_{i=0}^{m} a_i t^i, g(t) = \sum_{j=0}^{n} b_j t^j \in R[t] \subset R[x][t]$, noting that $C(R[x]) = C(R)[x]$ and $C(R[x])[t] = C(R)[x][t]$.

Here if $R[x]$ is quasi-commutative, then $a_i b_j \in C(R[x])$ for all i, j. But since $C(R[x]) = C(R)[x]$, we have $a_i b_j \in C(R)$. Thus R is quasi-commutative. □

Let X denote a set of commuting indeterminates over a given ring R, and use $R[X]$ to denote the polynomial ring with indeterminates X over R. We extend Proposition 2.1 to the case of $|X| \geq 2$ as follows. It is also checked easily that $C(R[X]) = C(R)[X]$.

Corollary 2.2. *Let R be a ring and X be a set of commuting indeterminates over R. Suppose $|X| \geq 2$. Then R is quasi-commutative if and only if so is $R[X]$.*

Proof. Let R be a quasi-commutative ring, and suppose that

$$f(t)g(t) \in C(R[X])[t]$$

for $0 \neq f(t), g(t) \in R[X][t]$, where $R[X][t]$ is the polynomial ring with an indeterminate t over $R[X]$. Then there exists a finite subset X_0 of X such that $f(t), g(t) \in R[X_0][t]$. But $R[X_0]$ is quasi-commutative by Proposition 2.1, iterating this if necessary. This result induces that $R[X]$ is also quasi-commutative, noting that $C(R[X_0]) = C(R)[X_0] \subseteq C(R)[X] = C(R[X])$.

The converse can be proved by applying the proof of Proposition 2.1. □

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

Proposition 2.3. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is quasi-commutative if and only if so is $M^{-1}R$.*

Proof. Write $E = M^{-1}R$. First note $M^{-1}C(R) \subseteq C(E)$. Let $a^{-1}b \in C(E)$. Then $a^{-1}br = ra^{-1}b = a^{-1}rb$ for all $r \in R$, entailing $br = rb$. This implies $b \in C(R)$, and thus $C(E) \subseteq M^{-1}C(R)$. Consequently $C(E) = M^{-1}C(R)$. We will use this fact freely.

Suppose that R is quasi-commutative. Let $F(x) = \sum_{i=0}^{m} \alpha_i x^i$ and $G(x) = \sum_{j=0}^{n} \beta_j x^j$ be in $E[x]$ such that $F(x)G(x) \in C(E)[x]$, where $\alpha_i = u^{-1}a_i$, $\beta_j = v^{-1}b_j$ with $a_i, b_j \in R$ for all i, j and regular $u, v \in R$. But $F(x)G(x) = u^{-1}(a_0 + a_1x + \cdots + a_mx^m)v^{-1}(b_0 + b_1x + \cdots + b_nx^n) = (uv)^{-1}(a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n)$.

Here let $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$. Then $f(x)$ and $g(x)$ are in $R[x]$. Moreover $f(x)g(x) \in C(R)[x]$ since

$$F(x)G(x) \in C(E)[x] \text{ and } C(E)[x] = (M^{-1}C(R))[x].$$

Since R is quasi-commutative, $a_ib_j \in C(R)$ for all i, j. This entails $\alpha\beta = u^{-1}a_iv^{-1}b_j = u^{-1}v^{-1}a_ib_j \in M^{-1}C(R) = C(E)$. Thus E is quasi-commutative.

Suppose that E is quasi-commutative. Let $f(x)g(x) \in C(R)[x]$ for $f(x), g(x) \in R[x]$. Then $f(x)g(x) \in (M^{-1}C(R))[x] = C(E)[x]$. Since E is quasi-commutative, $ab \in C(E)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Thus $ab \in C(R)$ since $C(R) = R \cap C(E)$, and so R is quasi-commutative. □

Let R be a ring. Recall that the ring of *Laurent polynomials*, in an indeterminate x over R, consists of all formal sums $\sum_{i=k}^{n} a_ix^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. We denote this ring by $R[x; x^{-1}]$.

Corollary 2.4. *Let R be a ring. Then R is quasi-commutative if and only if $R[x]$ is quasi-commutative if and only if $R[x; x^{-1}]$ is quasi-commutative.*

Proof. The first equivalence is Proposition 2.1. The second one is an immediate consequence of Proposition 2.3, noting that $R[x; x^{-1}] = M^{-1}R[x]$ if $M = \{1, x, x^2, \ldots\}$. □

We use $\oplus$ to denote the direct sum. Let A be an algebra (with or without identity) over a commutative ring K. Due to Dorroh [7], the *Dorroh extension* of A by K, written by $A \oplus_D K$, is the Abelian group $A \oplus K$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in K$. Note that $(A \oplus_D K)[x]$ is isomorphic to $A[x] \oplus_D K[x]$ through $\sum_{i=0}^{m}(a_i, h_i)x^i \mapsto (\sum_{i=0}^{m} a_ix^i, \sum_{i=0}^{m} h_ix^i)$, so we treat here these two rings without discrimination.

Proposition 2.5. *Let A be a free algebra generated by a set X of noncommuting indeterminates over a commutative domain K, where $|X| \geq 2$. Let B be the subalgebra of all polynomials of zero constant in A. Then the Dorroh extension of B by K is quasi-commutative.*

Proof. Let R be the Dorroh extension of B by K. Clearly $0 \oplus K \subseteq C(R)$. Let $(b, k) \in C(R)$. Then $(ba + ka, 0) = (b, k)(a, 0) = (a, 0)(b, k) = (ab + ka, 0)$ for all $a \in B$, forcing $ab = ba$. This yields $b = 0$ since every nonzero element of B is non-central in B, entailing $C(R) = 0 \oplus K$. So

$$C(R[x]) = C(R)[x] = (0 \oplus K)[x] = 0 \oplus K[x].$$

We will use this fact freely.

Suppose that $0 \neq f(x) = \sum_{i=0}^{m}(a_i, h_i)x^i$ and $0 \neq g(x) = \sum_{j=0}^{n}(b_j, k_j)x^j \in R[x]$ satisfy $f(x)g(x) \in C(R)[x]$. Here $f(x), g(x)$ can be expressed by

$$f(x) = (f_0, f_1) \text{ and } g(x) = (g_0, g_1)$$

for $f_0 = \sum_{i=0}^{m} a_i x^i$, $g_0 = \sum_{j=0}^{n} b_j x^j \in B[x]$, $f_1 = \sum_{i=0}^{m} h_i x^i$, and $g_1 = \sum_{j=0}^{n} k_j x^j \in K[x]$. Then

$$f(x)g(x) = (f_0g_0 + g_1f_0 + f_1g_0, f_1g_1) \in C(R)[x],$$

entailing

$$f_0g_0 + g_1f_0 + f_1g_0 = 0.$$

Here if $f_0 \neq 0$ and $g_0 \neq 0$, then $f_0g_0 + g_1f_0 + f_1g_0 \neq 0$ since the degree of f_0g_0 is larger than one of $g_1f_0 + f_1g_0$. This induces a contradiction. Thus we have $f_0 = 0$ or $g_0 = 0$.

If $f_0 = 0$ (i.e., $0 \neq f(x) = f_1$), then $f(x)g(x) = (f_1g_0, f_1g_1) \in C(R)[x]$. This forces $f_1g_0 = 0$, and hence $g_0 = 0$ since $f_1 \neq 0$. So $f(x) = (0, f_1)$ and $g(x) = (0, g_1)$. The same result is obtained when $g_0 = 0$.

Consequently $(0, h_i)(0, k_j) = (0, h_ik_j) \in 0 \oplus K(= C(R))$ for all i and j, and therefore R is quasi-commutative. □

Use $\prod$ to denote the direct product of rings.

Proposition 2.6. *Let R_i be rings for $i \in I$ and $R = \prod_{i\in I} R_i$, where I is an index set. Then R_i is quasi-commutative for all $i \in I$ if and only if R is quasi-commutative.*

Proof. It is easily shown that $C(R) = \prod_{i\in I} C(R_i)$. We will use this freely in the proof. Suppose that R_i is quasi-commutative for all $i \in I$, and let $F(x)G(x) \in C(R)[x]$ for $F(x) = \sum_{s=0}^{m}(a(s)_i)x^s, G(x) = \sum_{t=0}^{n}(b(t)_i)x^t \in R[x]$.

$F(x)$ and $G(x)$ can be rewritten by $F(x) = (f(x)_i)_{i\in I}$ and $G(x) = (g(x)_i)_{i\in I}$, where $f(x)_i = \sum_{s=0}^{m} a(s)_i x^s$ and $g(x)_i = \sum_{t=0}^{n} b(t)_i x^t$. So we have

$$F(x)G(x) = (f(x)_i)(g(x)_i) = (h(x)_i),$$

where $h(x)_i = f(x)_i g(x)_i$ for all $i \in I$. But since $F(x)G(x) \in C(R)[x]$ and $C(R)[x] = (\prod_{i\in I} C(R_i))[x] = \prod_{i\in I}(C(R_i)[x]) = C(R[x])$, we get $f(x)_i g(x)_i \in C(R_i)[x]$ for all $i \in I$.

Now since every R_i is quasi-commutative, we have $a(s)_i b(t)_i \in C(R_i)$ for all $i \in I$. This implies $(a(s)_i)(b(t)_i) \in C(R)$, and so R is quasi-commutative.

Conversely assume that R is quasi-commutative, and let $f(x)g(x) \in C(R_i)[x]$ for $f(x) = \sum_{u=0}^{m} c_u x^u, g(x) = \sum_{v=0}^{n} d_v x^v \in R_i[x]$. Consider sequences $(a(u)_i)$, $(b(v)_i) \in R$ for all u, v such that $a(u)_i = c_u$, $b(v)_i = d_v$, and $a(u)_j = 0 = b(v)_j$ for all $j \neq i$. Next set

$$F(x) = (f(x)_i) = \sum_{u=0}^{m}(a(u)_i)x^u \text{ and } G(x) = (g(x)_i) = \sum_{v=0}^{n}(b(v)_i)x^v$$

such that

$$f(x)_i = \sum_{u=0}^{m} a(u)_i x^u \text{ and } g(x)_i = \sum_{v=0}^{n} b(v)_i x^v.$$

Note $f(x)_j = 0 = g(x)_j$ for all $j \neq i$. Now we have $F(x)G(x) \in C(R)[x]$. Since R is quasi-commutative, $(a(u)_i b(v)_i) = (a(u)_i)(b(v)_i) \in C(R)$ for all $i \in I$. This entails $c_u d_v \in C(R_i)$ for all u, v. Thus R_i is quasi-commutative. □

We see next an application of Proposition 2.6.

Corollary 2.7. (1) *There exist noncommutative non-reduced quasi-commutative rings.*

(2) *Let R be a ring and $e^2 = e \in C(R)$. Then R is quasi-commutative if and only if both eR and $(1-e)R$ are quasi-commutative.*

Proof. (1) Let R_1 be a noncommutative quasi-commutative ring as in Example 1.3, and R_2 be a non-reduced commutative ring. Then the direct product $\prod_{i=1}^{2} R_i$ is a noncommutative non-reduced quasi-commutative ring by Proposition 2.6.

(2) is an immediate consequence of Proposition 2.6, considering $R = eR \oplus (1-e)R$. □

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