J. Korean Math. Soc. **53** (2016), No. 2, pp. 447–459
http://dx.doi.org/10.4134/JKMS.2016.53.2.447

POWER SERIES RINGS OVER PRÜFER v-MULTIPLICATION DOMAINS

Gyu Whan Chang

Abstract. Let D be an integral domain, $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and $D[\![\{X_\alpha\}]\!]_1$ be the first type power series ring over D. We show that if D is a t-SFT Prüfer v-multiplication domain, then $D[\![\{X_\alpha\}]\!]_{1\,D-\{0\}}$ is a Krull domain, and $D[\![\{X_\alpha\}]\!]_1$ is a Prüfer v-multiplication domain if and only if D is a Krull domain.

1. Introduction

1.1. Motivation and results

Let D be an integral domain. An ideal I of D is called an *SFT-ideal* (an ideal of strong finite type) if there exist a finitely generated ideal $J \subseteq I$ and an integer $k \geq 1$ such that $a^k \in J$ for all $a \in I$. The ring D is called an *SFT-ring* if each ideal of D is an SFT-ideal. The t-operation analogue of the notions of SFT-ideals and SFT-rings, in [17], Kang-Park defined a nonzero ideal A of D to be a *t-SFT-ideal* if there exist a nonzero finitely generated ideal $B \subseteq A$ and a positive integer k such that $a^k \in B_v$ for all $a \in A_t$, and D to be a *t-SFT-ring* if each nonzero ideal of D is a t-SFT-ideal. (Definitions related to the t-operation will be reviewed in Section 1.2.) It is known that D is an SFT-ring (resp., a t-SFT-ring) if and only if each prime ideal (resp., prime t-ideal) of D is an SFT-ideal (resp., a t-SFT-ideal) [3, Proposition 2.2] (resp., [17, Proposition 2.1]). Hence, a t-SFT-ring contains an integral domain whose prime t-ideals are of finite type (see [5, Section 5] for such an integral domain). A *Mori domain* is an integral domain that satisfies the ascending chain condition on integral v-ideals. Clearly, a Noetherian domain is a Mori domain, and a Mori domain is a t-SFT-ring. It is well known that D is a Krull domain if and only if D is a completely integrally closed Mori domain, if and only if D is a Mori Prüfer v-multiplication domain (PvMD) (cf. [19, Theorem 2.5]). Hence, a Krull domain is a t-SFT PvMD. For more on basic properties of Krull domains, the reader can be referred to [13, Sections 43 and 44].

Received March 5, 2015.
2010 *Mathematics Subject Classification.* 13A15, 13F05, 13F25.
Key words and phrases. t-operation, t-SFT PvMD, power series ring, Krull domain.

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, $D[\{X_\alpha\}]$ be the polynomial ring over D, and $D[\![\{X_\alpha\}]\!]_1$ be the first type power series ring over D, i.e., $D[\![\{X_\alpha\}]\!]_1 = \bigcup D[\![X_1, \ldots, X_n]\!]$, where $\{X_1, \ldots, X_n\}$ runs over all finite subsets of $\{X_\alpha\}$; so if $|\{X_\alpha\}| < \infty$, then $D[\![\{X_\alpha\}]\!]_1 = D[\![\{X_\alpha\}]\!]$ (cf. [13, Section 1] for the power series ring). It was shown in [1, Theorem 3.7] that if D is an SFT Prüfer domain, then $D[\![\{X_\alpha\}]\!]_{1D-\{0\}}$ is a Krull domain. The purpose of this paper is to generalize [1, Theorem 3.7] to t-SFT PvMDs. Let $X^1(D)$ be the set of height-one prime ideals of D, $R = \bigcap_{P \in X^1(D)} D_P$, and $qf(D[\![\{X_\alpha\}]\!]_1)$ be the quotient field of $D[\![\{X_\alpha\}]\!]_1$. In Section 2, we show that if D is a t-SFT PvMD in which each maximal t-ideal of D contains a height-one prime ideal, then R is a Krull domain and $R[\![\{X_\alpha\}]\!]_{1R-\{0\}} \cap qf(D[\![\{X_\alpha\}]\!]_1) = D[\![\{X_\alpha\}]\!]_{1D-\{0\}}$. We also prove that if D is a t-SFT PvMD, then $D[\![\{X_\alpha\}]\!]_{1D-\{0\}}$ is a Krull domain, and $D[\![\{X_\alpha\}]\!]_1$ is a PvMD if and only if D is a Krull domain. In Section 3, we show that D is a t-SFT PvMD if and only if $D[\{X_\alpha\}]$ is a t-SFT PvMD, if and only if $D[\{X_\alpha\}]_{N_v}$ is an SFT Prüfer domain, where $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. Hence, if D is an SFT Prüfer domain, then $D[\{X_\alpha\}]$ is a t-SFT PvMD. We finally prove that if K is the quotient field of D and X is an indeterminate over D, then $D + XK[X]$ is a t-SFT PvMD if and only if D is a t-SFT PvMD.

1.2. Definitions related to the *t*-operation

Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D; so $f(D) \subseteq F(D)$. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, then $I^{-1} \in F(D)$. The v-operation is defined by $I_v = (I^{-1})^{-1}$ and the t-operation is by $I_t = \bigcup \{F_v \mid F \in f(D) \text{ and } F \subseteq I\}$. Clearly, if $I \in F(D)$, then $I \subseteq I_t \subseteq I_v$, and if I is finitely generated, then $I_t = I_v$. The v- and t-operation are examples of the so-called star operations. For a review of star operations, the reader may look up [13, Sections 32 and 34]. If $* = v$ or t, then I is called a $*$*-ideal* if $I = I_*$ and a $*$*-ideal of finite type* if $I = B_*$ for some $B \in f(D)$. A $*$-ideal of D is called a *maximal* $*$*-ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) be the set of all maximal $*$-ideals of D. It is well known that each proper integral t-ideal is contained in a maximal t-ideal; each maximal t-ideal is a prime ideal; $D = \bigcap_{P \in t\text{-Max}(D)} D_P$; and t-Max$(D) \neq \emptyset$ when D is not a field even though v-Max(D) can be empty as in the case of a rank-one non-discrete valuation domain D. An *overring* of D means a ring between D and K. We say that an overring R of D is t*-linked* over D if $I_v = D$ implies $(IR)_v = R$ for all $I \in f(D)$. It is known that R is t-linked over D if and only if $(Q \cap D)_t \subsetneq D$ for each prime t-ideal Q of R [9, Proposition 2.1].

An $I \in F(D)$ is said to be t*-invertible* if $(II^{-1})_t = D$, while D is a *Prüfer* v*-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is well known that D is a PvMD if and only if D_P is a valuation domain for each maximal t-ideal P of D [16, Theorem 3.2]; hence D is a Prüfer domain if and only if D is a PvMD whose maximal ideals are t-ideals. Also, it is clear that an invertible ideal is a t-ideal, and hence every nonzero finitely generated ideal of a Prüfer domain is a t-ideal; so t-SFT Prüfer domains $\Leftrightarrow$ SFT Prüfer domains. Let X be an indeterminate over D and $D[X]$ be the polynomial ring over D. An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ such that $Q \cap D = (0)$. We say that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. It is well known that D is an integrally closed UMT-domain if and only if D is a PvMD [15, Proposition 3.2].

2. Power series rings over a t-SFT PvMD

Let D be an integral domain with quotient field K. In this section, we show that if D is a t-SFT PvMD, then $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain (Theorem 9). This is a generalization of Anderson-Kang-Park's result [1, Theorem 3.7] that if D is an SFT Prüfer domain, then $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain. Many of the techniques for the proofs of Theorem 9(3) and Lemma 8(2) are borrowed from [1] and [4, Lemma 3.3] respectively, and the proofs of Proposition 2 and the (2)-(3) of Proposition 6 are similar to those of the counterparts in [1].

For a polynomial $f \in D[\{X_\alpha\}]$, let $c(f)$ denote the ideal of D generated by the coefficients of f; for an ideal A of $D[\{X_\alpha\}]$, $c(A)$ denotes the ideal $\sum_{f\in A} c(f)$ of D; and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$.

Lemma 1. (1) $\{P[\{X_\alpha\}]_{N_v} \mid P \in t\text{-Max}(D)\}$ *is the set of maximal ideals of* $D[\{X_\alpha\}]_{N_v}$.

(2) *The following statements are equivalent.*
  (a) *D is a PvMD.*
  (b) *$D[\{X_\alpha\}]$ is a PvMD.*
  (c) *$D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain.*
  (d) *Every ideal A of $D[\{X_\alpha\}]_{N_v}$ is extended from D, i.e., $A = ID[\{X_\alpha\}]_{N_v}$ for some ideal I of D. In this case, I can be chosen so that I is finitely generated when A is finitely generated.*

(3) *D is a UMT-domain if and only if every prime ideal of $D[\{X_\alpha\}]_{N_v}$ is extended from D.*

Proof. (1) and (2) [16, Proposition 2.1, Theorems 3.1 and 3.7]. Also, note that if $0 \neq f \in D[\{X_\alpha\}]$, then $c(f)$ is t-invertible, and hence $fD[\{X_\alpha\}]_{N_v} = c(f)D[\{X_\alpha\}]_{N_v}$ [16, Theorem 2.12]. Thus, if $A = (f_1, \dots, f_n)D[\{X_\alpha\}]_{N_v}$, where $0 \neq f_i \in D[\{X_\alpha\}]$, then $I = \sum_{i=1}^n c(f_i)$ is finitely generated and $A = ID[\{X_\alpha\}]_{N_v}$.

(3) Note that D is a UMT-domain if and only if D_P is a quasi-Prüfer domain for each prime t-ideal P of D, i.e., if Q is a prime ideal of $D_P[\{X_\alpha\}]$ with $Q \subseteq PD_P[\{X_\alpha\}]$, then $Q = (Q \cap D_P)[\{X_\alpha\}]$ [7, Lemma 2.1 and Corollary 2.4]. Thus, D is a UMT-domain if and only if for each prime t-ideal P of D, if Q is

a prime ideal of $D[\{X_\alpha\}]$ with $Q \subseteq P[\{X_\alpha\}]$, then $Q = (Q \cap D)[\{X_\alpha\}]$, if and only if every prime ideal of $D[\{X_\alpha\}]_{N_v}$ is extended from D by (1). (See [15, Theorem 3.1] for one indeterminate.) $\square$

An element $d \in D$ is said to be *Archimedean* if $\bigcap_{n=1}^{\infty} d^n D = (0)$ and d is *non-Archimedean* or *bounded* if d is not Archimedean, i.e., $\bigcap_{n=1}^{\infty} d^n D \neq (0)$. We say that D is *Archimedean* (resp., *anti-Archimedean*) if each nonzero element of D is Archimedean (resp., bounded). Recall from [1, Proposition 2.1] that if D is anti-Archimedean, then every nonzero prime ideal of D has infinite height (or equivalently, D has no height-one prime ideal).

Proposition 2 (cf. [1, Theorem 2.15]). *$D[\{X_\alpha\}]_{N_v}$ is an anti-Archimedean domain if and only if D is an anti-Archimedean UMT-domain.*

Proof. ($\Rightarrow$) If D is not a UMT-domain, there is an upper to zero Q in $D[X]$ that is not a maximal t-ideal, where $X \in \{X_\alpha\}$; so $Q \subsetneq P[X]$ for some maximal t-ideal P of D [15, Theorem 1.4]. Hence, $QD[\{X_\alpha\}]_{N_v} \subseteq P[\{X_\alpha\}]_{N_v} \subsetneq D[\{X_\alpha\}]_{N_v}$ and $\text{ht}(QD[\{X_\alpha\}]_{N_v}) = \text{ht}(QD[\{X_\alpha\}]) = \text{ht}Q = 1$, a contradiction because an anti-Archimedean domain has no height-one prime ideals. Thus, D is a UMT-domain. Next, if $0 \neq a \in D$, then $\bigcap_{n=1}^{\infty} a^n D[X]_{N_v} \neq (0)$. Hence if $0 \neq f \in \bigcap_{n=1}^{\infty} a^n D[X]_{N_v}$, then, for each integer $n \geq 1$, $f = \frac{a^n h_n}{g_n}$ for some $g_n \in N_v$ and $h_n \in D[\{X_\alpha\}]$; so $c(f) \subseteq c(f)_v = (c(f)c(g_n))_v = c(fg_n)_v = a^n c(h_n)_v \subseteq a^n D$. Thus, $(0) \neq c(f) \subseteq \bigcap_{n=1}^{\infty} a^n D$.

($\Leftarrow$) Let Q be a prime ideal of $D[\{X_\alpha\}]_{N_v}$. Then $Q = P[\{X_\alpha\}]_{N_v}$ for some prime ideal P of D by Lemma 1(3). So if $0 \neq d \in P \subseteq Q$, then $(0) \neq \bigcap_{n=1}^{\infty} d^n D \subseteq \bigcap_{n=1}^{\infty} d^n D[\{X_\alpha\}]_{N_v}$, and hence Q contains a bounded element d. Thus, $D[\{X_\alpha\}]_{N_v}$ is an anti-Archimedean domain [1, Proposition 2.8]. $\square$

Let R be a commutative ring with identity, and let I be an ideal of R. It is known that if every prime ideal of R minimal over I is the radical of a finitely generated ideal, then there are only a finite number of prime ideals minimal over I [14, Theorem 1.6], which was generalized by Chang as follows.

Lemma 3 ([6, Lemma 2.1]). *Let I be an integral t-ideal of D. If every prime ideal of D minimal over I is the radical of a t-ideal of finite type, there are only finitely many prime ideals of D minimal over I.*

If D is a t-SFT-ring, then every prime t-ideal of D is the radical of a t-ideal of finite type, and hence by Lemma 3, each t-ideal of D has only finitely many minimal prime ideals.

Corollary 4 (cf. [1, Proposition 2.3]). *If D is a t-SFT PvMD, then the following statements are equivalent.*

(1) *D is an anti-Archimedean domain.*
(2) *$X^1(D) = \emptyset$.*
(3) *$D[\{X_\alpha\}]_{N_v}$ is an anti-Archimedean domain.*

Proof. (1) $\Rightarrow$ (2) [1, Proposition 2.1].

(2) $\Rightarrow$ (1) Let a be a nonzero nonunit of D. Then, by Lemma 3, aD has only finitely many minimal prime ideals $Q_1, \dots, Q_m$, and since aD is a t-ideal, each Q_i is a t-ideal. Since $X^1(D) = \emptyset$, each Q_i contains a nonzero prime ideal P_i; so $a \in Q_i - P_i$. Let $M \in t\text{-Max}(D)$, $n \geq 1$ be an integer, and $I = P_1 \cap \cdots \cap P_m$. If $a^n D_M = D_M$, then $ID_M \subseteq D_M = a^n D_M$. Next, if $a^n D_M \subsetneq D_M$, then $ID_M = P_i D_M \subsetneq a^n D_M \subseteq Q_i D_M \subseteq MD_M \subsetneq D_M$ for some i, where the first equality follows because $P_j D_M = D_M$ for $P_j \neq P_i$. Hence, $a^n D = \bigcap_{M \in t\text{-Max}(D)} a^n D_M \supseteq \bigcap_{M \in t\text{-Max}(D)} ID_M \supseteq I$, and therefore $\bigcap_{n=1}^{\infty} a^n D \supseteq I \neq (0)$.

(1) $\Leftrightarrow$ (3) This follows directly from Proposition 2 because a PvMD is an integrally closed UMT-domain. □

We next show that if D is a t-SFT PvMD, there are t-SFT PvMDs D_1 and D_2 such that $D = D_1 \cap D_2$, $X^1(D_1) = \emptyset$, and each maximal t-ideal of D_2 contains a height-one prime ideal. We begin with the following lemma.

Lemma 5. *Let D be a PvMD and $\{P\} \cup \{P_\lambda\}_\lambda$ be a family of prime t-ideals of D. Then $D_P \supseteq \bigcap_\lambda D_{P_\lambda}$ if and only if each finitely generated ideal contained in P is contained in some P_λ.*

Proof. Let X be an indeterminate over D and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then $D[X]_{N_v}$ is a Prüfer domain by Lemma 1(2) and $\{P[X]_{N_v}\} \cup \{P_\lambda[X]_{N_v}\}$ is a family of prime ideals of $D[X]_{N_v}$. Thus, $D[X]_{P[X]} \supseteq \bigcap_\lambda D[X]_{P_\lambda[X]}$ if and only if each finitely generated ideal contained in $P[X]_{N_v}$ is contained in some $P_\lambda[X]_{N_v}$ [13, Ex. 16 on p. 332]. Also, note that each ideal A of $D[X]_{N_v}$ is of the form $I[X]_{N_v}$ for some ideal I of D, and in this case, I can be chosen so that I is finitely generated when A is finitely generated by Lemma 1(2). Hence, each finitely generated ideal contained in P is contained in some P_λ if and only if each finitely generated ideal contained in $P[X]_{N_v}$ is contained in some $P_\lambda[X]_{N_v}$. Thus, it suffices to show that $D[X]_{P[X]} \supseteq \bigcap_\lambda D[X]_{P_\lambda[X]} \Leftrightarrow D_P \supseteq \bigcap_\lambda D_{P_\lambda}$.

Claim 1. If P_β is a prime t-ideal of D and $0 \neq f \in D[X]$, then $\frac{1}{f} D_{P_\beta}(X) = c(f)^{-1} D_{P_\beta}(X)$, where $D_{P_\beta}(X) = D_{P_\beta}[X]_{P_\beta D_{P_\beta}[X]} = D[X]_{P_\beta[X]}$.

Proof. $fD_{P_\beta}(X) = c_\beta(f) D_{P_\beta}(X) = c(f) D_{P_\beta}(X)$, where $c_\beta(f) = c(f) D_{P_\beta}$, because D_{P_β} is a valuation domain. Note that $c(f)$ is finitely generated; so $(c(f)D_{P_\beta})^{-1} = c(f)^{-1} D_{P_\beta}$. Hence, $(c(f) D_{P_\beta}(X))^{-1} = c_\beta(f)^{-1} D_{P_\beta}(X) = c(f)^{-1} D_{P_\beta}(X)$ [16, Proposition 2.2], and since $c(f)c(f)^{-1} \not\subseteq P_\beta$, $(c(f)D_{P_\beta}(X))(c(f)D_{P_\beta}(X))^{-1} = (c(f)c(f)^{-1}) D_{P_\beta}(X) = D_{P_\beta}(X)$. Thus, $fD_{P_\beta}(X) = c(f) D_{P_\beta}(X)$ implies $\frac{1}{f} D_{P_\beta}(X) = c(f)^{-1} D_{P_\beta}(X)$.

Claim 2. $D[X]_{P[X]} \supseteq \bigcap_\lambda D[X]_{P_\lambda[X]} \Leftrightarrow D_P \supseteq \bigcap_\lambda D_{P_\lambda}$.

Proof. ($\Rightarrow$) $\bigcap_\lambda D_{P_\lambda} = (\bigcap_\lambda D_{P_\lambda}(X)) \cap K \subseteq D_P(X) \cap K = D_P$. ($\Leftarrow$) Let $\frac{g}{f} \in \bigcap_\lambda D_{P_\lambda}(X) = \bigcap_\lambda D[X]_{P_\lambda[X]}$, where $0 \neq f, g \in D[X]$. Then $\frac{g}{f} D_{P_\lambda}(X) \subseteq$

$D_{P_\lambda}(X)$ for all λ, and hence $c(g)c(f)^{-1} \subseteq (c(g)c(f)^{-1})D_{P_\lambda}(X) = \frac{g}{f}D_{P_\lambda}(X) \subseteq D_{P_\lambda}(X)$ by Claim 1. Thus, $c(g)c(f)^{-1} \subseteq (\bigcap_\lambda D_{P_\lambda}(X)) \cap K = \bigcap_\lambda D_{P_\lambda} \subseteq D_P$. So $\frac{g}{f} \in \frac{g}{f}D_P(X) = (c(g)c(f)^{-1})D_P(X) \subseteq D_P(X)$ by Claim 1. Therefore, $\bigcap_\lambda D_{P_\lambda}(X) \subseteq D_P(X)$. □

An overring R of D is said to be *t-flat* over D if $R_M = D_{M\cap D}$ for each maximal t-ideal M of R. Clearly, a t-flat overring of D is t-linked over D. Moreover, if D is a PvMD, then each t-linked overring of D is t-flat over D [18, Proposition 2.10].

Proposition 6 (cf. [1, Lemma 3.5])**.** *Let D be a t-SFT PvMD, Λ be a nonempty set of prime t-ideals of D, and $R = \bigcap_{P\in\Lambda} D_P$.*

1. *R is a t-SFT PvMD.*
2. *If no $P \in \Lambda$ contains a height-one prime ideal, then no prime t-ideal of R contains a height-one prime ideal.*
3. *If each $P \in \Lambda$ contains a height-one prime ideal, then each prime t-ideal of R contains a height-one prime ideal.*

Proof. (1) Note that R is t-linked over D [16, Theorem 3.8]; so R is a PvMD [16, Corollary 3.9] that is t-flat over D [18, Proposition 2.10]. Thus, R is a t-SFT PvMD [17, Proposition 2.3].

For (2) and (3), let M be a prime t-ideal of R, and put $M \cap D = P$. Then R is a PvMD by (1), and since R is t-linked over D, P is a t-ideal of D. Thus, $R_M = D_P$ is a valuation domain and $D_P = R_M \supseteq \bigcap_{Q\in\Lambda} D_Q$. Since D is a t-SFT ring, there is a nonzero finitely generated ideal I of D such that $P = \sqrt{I}$. Hence, by Lemma 5, $I \subseteq P'$ for some $P' \in \Lambda$, and thus $P = \sqrt{I} \subseteq P'$.

(2) If M contains a height-one prime ideal Q_0, then $Q_0 \cap D \subseteq M \cap D = P \subseteq P'$, and since $D_P = R_M$, $\mathrm{ht}(Q_0 \cap D) = 1$. Hence, $P' \in \Lambda$ contains a height-one prime ideal $Q_0 \cap D$, a contradiction.

(3) Let P_0 be a height-one prime ideal of D contained in P'. Then, since $D_{P'}$ is a valuation domain and $P \subseteq P'$, we have $P_0 = P_0D_{P'} \cap D \subseteq PD_{P'} \cap D = P$. Thus, $D_P = R_M$ implies that M contains a height-one prime ideal. □

Let Λ be a set of prime ideals of D, and for convenience, we let $\bigcap_{P\in\Lambda} D_P = K$ when $\Lambda = \emptyset$. Then, by Corollary 4 and Proposition 6, we have:

Corollary 7. *Let D be a t-SFT PvMD, Λ_1 be the set of maximal t-ideals of D that contain no height-one prime ideal, Λ_2 be the set of maximal t-ideals of D that contain a height-one prime ideal, and put $D_i = \bigcap_{P\in\Lambda_i} D_P$ for $i = 1, 2$.*

1. *D_1 and D_2 are t-SFT PvMDs such that $D_1 \cap D_2 = D$,*
2. *$X^1(D_1) = \emptyset$; so D_1 is anti-Archimedean, and*
3. *each prime t-ideal of D_2 contains a height-one prime ideal.*

Clearly, $X^1(D) = \emptyset$ if and only if every prime ideal of D has infinite height, and if D is a Krull domain, then t-$\mathrm{Max}(D) = X^1(D)$. We recall that if D_1 and

D_2 are Krull domains that are subrings of a field L, then $D_1 \cap D_2$ is a Krull domain [13, Corollary 44.10].

Lemma 8. *Let D be a t-SFT PvMD in which each maximal t-ideal contains a height-one prime ideal, $R = \bigcap_{P \in X^1(D)} D_P$, and $qf(D[\![\{X_\alpha\}]\!]_1)$ be the quotient field of $D[\![\{X_\alpha\}]\!]_1$.*

(1) *R is a Krull domain.*
(2) *$R[\![\{X_\alpha\}]\!]_{1_{R-\{0\}}} \cap qf(D[\![\{X_\alpha\}]\!]_1) = D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$.*
(3) *$D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain.*

Proof. (1) If $P \in X^1(D)$, then P is a t-ideal, and hence $P^2 \subseteq A_v \subseteq P$ for some finitely generated ideal A of D [17, Proposition 2.6]. Hence, $(PD_P)^2 = P^2D_P \subseteq A_vD_P = (AD_P)_v = AD_P \subseteq PD_P$, where the third equality follows because A is t-invertible and the fourth equality is because D_P is a valuation domain. Thus, if $AD_P = PD_P$, then PD_P is principal, and hence D_P is a rank-one DVR. If $AD_P \subsetneq PD_P$, then $(PD_P)^2 \subsetneq PD_P$, and so PD_P is principal. Thus, D_P is a rank-one DVR.

Let $a \in D$ be a nonzero nonunit, and let Q be a prime ideal of D minimal over aD. Then Q is a t-ideal, and so $Q = \sqrt{A_t}$ for some finitely generated ideal A. Hence, there are only finitely many prime ideals minimal over aD by Lemma 3, and thus there are only finitely many prime ideals in $X^1(D)$ containing a. This means that the intersection $R = \bigcap_{P \in X^1(D)} D_P$ is locally finite. Thus, $R = \bigcap_{P \in X^1(D)} D_P$ is a Krull domain.

(2) The containment $(\supseteq)$ is clear. For the reverse containment, note that if $u \in R[\![\{X_\alpha\}]\!]_{1_{R-\{0\}}} \cap qf(D[\![\{X_\alpha\}]\!]_1)$, then

$$u \in R[\![X_1, \dots, X_n]\!]_{R-\{0\}} \cap qf(D[\![X_1, \dots, X_n]\!])$$

for some $X_1, \dots, X_n \in \{X_\alpha\}$; so it suffices to show that

$$R[\![X_1, \dots, X_n]\!]_{R-\{0\}} \cap qf(D[\![X_1, \dots, X_n]\!]) \subseteq D[\![X_1, \dots, X_n]\!]_{D-\{0\}}.$$

For convenience, let $T[\![X_1, \dots, X_k]\!] = T[\![X_k]\!]$ for an integral domain T and an integer $k \geq 1$, $\xi(X_1, \dots, X_k) = \xi(X_k)$ for any $\xi(X_1, \dots, X_k) \in T[\![X_k]\!]$, K_n be the quotient field of $D[\![X_n]\!]$, and $X^1(D) = \Lambda$.

Let $\mathcal{F}(\Lambda)$ be the family of finite subsets of Λ. For $\lambda = \{P_{\alpha_1}, \dots, P_{\alpha_r}\} \in \mathcal{F}(\Lambda)$, let $\mathfrak{S}_\lambda$ denote the set of t-invertible ideals A of D such that $(\prod_{i=1}^r P_{\alpha_i})_t \subsetneq A_t \subseteq D$ but $A \nsubseteq P_{\alpha_i}$ for $i = 1, \dots, r$ (hence, $A \nsubseteq P$ for all $P \in X^1(D)$ because $\prod_{i=1}^r P_{\alpha_i} \subseteq A_t$). If $A \in \mathfrak{S}_\lambda$, then

$$P_{\alpha_i} \supseteq (\prod_{i=1}^r P_{\alpha_i})_t = (((\prod_{i=1}^r P_{\alpha_i})A^{-1})A)_t \text{ and } (\prod_{i=1}^r P_{\alpha_i})A^{-1} \subseteq D.$$

But, since $A \nsubseteq P_{\alpha_i}$ for $i = 1, \dots, r$, we have $(\prod_{i=1}^r P_{\alpha_i})A^{-1} \subseteq \bigcap_{i=1}^r P_{\alpha_i}$. Note that $(P_{\alpha_i} + P_{\alpha_j})_t = D$ for $i \neq j$; so $\bigcap_{i=1}^r P_{\alpha_i} = (\prod_{i=1}^r P_{\alpha_i})_t$, and therefore $(\prod_{i=1}^r P_{\alpha_i})_t = ((\prod_{i=1}^r P_{\alpha_i})A^{-1})_t$. In particular, if $A_1, A_2 \in \mathfrak{S}_\lambda$, then A_1A_2 is

t-invertible,

$$(A_1A_2)_t \supsetneq ((\prod_{i=1}^{r} P_{\alpha_i})A_1A_2)_t = (((\prod_{i=1}^{r} P_{\alpha_i})A_2^{-1}A_1^{-1})A_1A_2)_t = (\prod_{i=1}^{r} P_{\alpha_i})_t,$$

and $A_1A_2 \nsubseteq P_{\alpha_i}$ for $i = 1, \dots, r$; so $A_1A_2 \in \mathfrak{S}_\lambda$. Hence, $\mathfrak{S}_\lambda$ is a multiplicatively closed set of ideals of D. Thus, if we let $D_\lambda = D_{\mathfrak{S}_\lambda}(:= \{\xi \in K \mid \xi A \subseteq D$ for some $A \in \mathfrak{S}_\lambda\})$, then D_λ is t-linked over D [16, Lemma 3.10], D_λ is a t-SFT PvMD by the proof of Proposition 6(1), and $(D : D_\lambda) = \{x \in K \mid xD_\lambda \subseteq D\}$ contains $\prod_{i=1}^{r} P_{\alpha_i}$ (for if $x \in D_\lambda$, then $xA \subseteq D$ for some $A \in \mathfrak{S}_\lambda$, and since $\prod_{i=1}^{r} P_{\alpha_i} \subseteq A_t$, we have $x(\prod_{i=1}^{r} P_{\alpha_i}) \subseteq xA_t = (xA)_t \subseteq D$). Thus, $D[\![X_n]\!]_{D-\{0\}} = D_\lambda[\![X_n]\!]_{D_\lambda-\{0\}} = D_\lambda[\![X_n]\!]_{D-\{0\}}$.

Let $\mathfrak{S} = \bigcup_{\lambda\in\mathcal{F}(\Lambda)} \mathfrak{S}_\lambda$. If $A_1, A_2 \in \mathfrak{S}$, then $A_i \in \mathfrak{S}_{\lambda_i}$ for some $\lambda_i \in \mathcal{F}(\Lambda)$. Note that $\lambda_1 \cup \lambda_2 \in \mathcal{F}(\Lambda)$ and $A_i \in \mathfrak{S}_{\lambda_1\cup\lambda_2}$; so $A_1A_2 \in \mathfrak{S}_{\lambda_1\cup\lambda_2} \subseteq \mathfrak{S}$. Thus, $\mathfrak{S}$ is a multiplicatively closed set of ideals of D and $D_{\mathfrak{S}} = \bigcup_{\lambda\in\mathcal{F}(\Lambda)} D_\lambda$.

Claim 1. $R = D_{\mathfrak{S}}$.

Proof. ($\supseteq$) If $x \in D_{\mathfrak{S}}$, then $x \in D_\lambda$ for some $\lambda \in \mathcal{F}(\Lambda)$, and so $xA \subseteq D$ for some $A \in \mathfrak{S}_\lambda$. Note that $A \nsubseteq P$ for all $P \in X^1(D)$; so $x \in xD_P = xAD_P \subseteq D_P$. Thus, $x \in \bigcap_{P\in X^1(D)} D_P = R$. ($\subseteq$) Let $y \in R$. Since $D \subseteq D_{\mathfrak{S}}$, we assume that $y \notin D$. Hence, if we let $A_y = \{r \in D \mid ry \in D\}$, then $A_y \nsubseteq P$ for all $P \in X^1(D)$, $A_y \subsetneq D$, and A_y is a t-invertible t-ideal of D because D is a PvMD. Since D is a t-SFT-ring, by Lemma 3, there are only a finite number of prime ideals of D minimal over A_y, say, $Q_1, \dots, Q_k$. By assumption and D_{Q_i} being a valuation domain, each Q_i contains a unique prime ideal of $X^1(D)$, and hence there are finitely many (distinct) prime ideals $P_1, \dots, P_m$ in $X^1(D)$ that are contained in some Q_i. Let $I = \prod_{i=1}^{m} P_i$ and $M \in t$-Max(D). If $Q_j \subseteq M$ for some j, then $ID_M \subsetneq A_yD_M \subseteq Q_jD_M \subseteq D_M$ because $A_y \nsubseteq P_i$ for $i = 1, \dots, m$. Next, if $Q_i \nsubseteq M$ for $i = 1, \dots, k$, then $ID_M \subseteq D_M = A_yD_M$. Hence, $I_t = \bigcap_{M\in t\text{-Max}(D)} ID_M \subseteq \bigcap_{M\in t\text{-Max}(D)} A_yD_M = (A_y)_t = A_y$ [16, Theorem 3.5], and since $A_y \nsubseteq P$ for all $P \in X^1(D)$, we have $I_t \subsetneq A_y$. Thus, $\lambda = \{P_1, \dots, P_m\} \in \mathcal{F}(\Lambda)$, $A_y \in \mathfrak{S}_\lambda$, and $yA_y \subseteq D$. Thus, $y \in D_\lambda \subseteq D_{\mathfrak{S}}$.

Claim 2. $R[\![X_n]\!] \cap K_n = \bigcup_{\lambda\in\mathcal{F}(\Lambda)} D_\lambda[\![X_n]\!]$.

Proof. ($\supseteq$) This follows because $R = \bigcup_{\lambda\in\mathcal{F}(\Lambda)} D_\lambda$ by Claim 1 and $D_\lambda[\![X_n]\!] \subseteq D[\![X_n]\!]_{D-\{0\}} \subseteq K_n$ for each $\lambda \in \mathcal{F}(\Lambda)$. ($\subseteq$) Let $\{\xi_i\}_{i=1}^{\infty}$ be a subset of R, and suppose that there exist $0 \neq d \in D$ and positive integers $\{m_i\}_{i=1}^{\infty}$ such that $d^{m_i}\xi_i \in D$. If $dD = D$, then $\xi_i \in D$, so we assume $dD \subsetneq D$. Hence, by Lemma 3, there are only finitely many prime ideals $P_{\alpha_1}, \dots, P_{\alpha_r}$ in $X^1(D)$ that are contained in some minimal prime ideals of dD (cf. the proof of Claim 1). Let $\lambda = \{P_{\alpha_1}, \dots, P_{\alpha_r}\}$ and $A_{\xi_i} = \{a \in D \mid a\xi_i \in D\}$. Clearly, A_{ξ_i} is a t-invertible t-ideal and $A_{\xi_i} \nsubseteq P_{\alpha_j}$ for $j = 1, \dots, r$. Let $p \in \prod_{j=1}^{r} P_{\alpha_j}$ and $M \in t$-Max(D). If $d \notin M$, then $p\xi_i \in D_M$. If $d \in M$, then $P_{\alpha_j} \subseteq M$ for some j, whence $p\xi_i \in pR \subseteq P_{\alpha_j}D_{P_{\alpha_j}} = P_{\alpha_j}D_M \subsetneq D_M$. Hence, $p\xi_i \in \bigcap_{M\in t\text{-Max}(D)} D_M = D$. Thus,

$(\prod_{j=1}^{r} P_{\alpha_j})_t \subsetneq (A_{\xi_i})_t = A_{\xi_i}$, and so $\xi_i \in D_\lambda$. By induction, we can easily show that if $k \geq 0$ is an integer, $\{\xi_i(X_k)\}_{i=1}^{\infty}$ is a subset of $R[\![X_k]\!]$, $\{m_i\}_{i=1}^{\infty}$ is a set of positive integers, and $0 \neq d(X_k) \in D[\![X_k]\!]$ such that $d(X_k)^{m_i}\xi_i(X_k) \in D[\![X_k]\!]$, then $\{\xi_i(X_k)\}_{i=1}^{\infty} \subseteq D_\lambda[\![X_k]\!]$ for some $\lambda \in \mathcal{F}(\Lambda)$ (see the proof of [4, Lemma 3.3]).

Let $\xi(X_n) = \frac{f(X_n)}{g(X_n)} \in R[\![X_n]\!] \cap K_n$, where $0 \neq f(X_n), g(X_n) \in D[\![X_n]\!]$, and write $\xi(X_n) = \sum_{i=0}^{\infty} \xi_i(X_{n-1})X_n^i$ and $g(X_n) = \sum_{i=0}^{\infty} d_i(X_{n-1})X_n^i$. We may assume that $d_0(X_{n-1}) \neq 0$, then

$$\xi(X_n)g(X_n) = \sum_{k=0}^{\infty}(\sum_{i+j=k} \xi_i(X_{n-1})d_j(X_{n-1}))X_n^k \in D[\![X_n]\!].$$

Hence, $d_0(X_{n-1})^{i+1} \cdot \xi_i(X_{n-1}) \in D[\![X_{n-1}]\!]$ for all $i \geq 0$, and thus $\{\xi_i(X_{n-1})\} \subseteq D_\lambda[\![X_{n-1}]\!]$ for some $\lambda \in \mathcal{F}(\Lambda)$ by the above paragraph. Thus, $\xi(X_n) \in D_\lambda[\![X_n]\!]$.

Finally, note that $R[\![X_n]\!]_{R-\{0\}} = R[\![X_n]\!]_{D-\{0\}}$; so if $u(X_n) \in R[\![X_n]\!]_{R-\{0\}} \cap K_n$, then there is $0 \neq d \in D$ such that $d \cdot u(X_n) \in R[\![X_n]\!] \cap K_n$, and hence, by Claim 2, $d \cdot u(X_n) \in D_\lambda[\![X_n]\!]$ for some $\lambda \in \mathcal{F}(\Lambda)$. Therefore, $u(X_n) \in D[\![X_n]\!]_{D-\{0\}}$ since $D_\lambda[\![X_n]\!] \subseteq D_\lambda[\![X_n]\!]_{D-\{0\}} = D[\![X_n]\!]_{D-\{0\}}$.

(3) Since R is a Krull domain, $R[\![\{X_\alpha\}]\!]_1$ is a Krull domain [12, Theorem 2.1] and $R[\![\{X_\alpha\}]\!]_{1_{R-\{0\}}}$ is a Krull domain [13, Corollary 43.6]. Clearly, $qf(D[\![\{X_\alpha\}]\!]_1)$ is a Krull domain, and thus $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain by (2) and [13, Corollary 44.10]. □

We are now ready to prove the main result of this paper for which we let $\bigcap_{P\in X^1(D)} D_P = K$ when $X^1(D) = \emptyset$.

Theorem 9. *If D is a t-SFT PvMD, then*

(1) *$R = \bigcap_{P\in X^1(D)} D_P$ is a Krull domain,*
(2) *D is a Krull domain if and only if $X^1(D) = t\text{-Max}(D)$, and*
(3) *$D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain.*

Proof. (1) If $X^1(D) = \emptyset$, then $R = K$, and hence R is a Krull domain, whence we assume that $X^1(D) \neq \emptyset$. However, this can be proved by an argument similar to the proof of Lemma 8(1).

(2) It is well known that if D is a Krull domain, then $X^1(D) = t\text{-Max}(D)$. For the converse, note that if $X^1(D) = t\text{-Max}(D)$, then $D = \bigcap_{P\in X^1(D)} D_P = R$. Thus, by (1), D is a Krull domain.

(3) Let Λ_i and D_i for $i = 1, 2$ be as in Corollary 7. Note that if $\Lambda_i = \emptyset$, then $D_i[\![\{X_\alpha\}]\!]_1 = K[\![\{X_\alpha\}]\!]_1$ is a Krull domain; so we assume that $\Lambda_i \neq \emptyset$ for $i = 1, 2$. Then D_1 is anti-Archimedean by Corollary 7, and thus $D_1[\![\{X_\alpha\}]\!]_{1_{D_1-\{0\}}}$ is a Krull domain [1, Corollary 3.4]. Next, note that $D_2[\![\{X_\alpha\}]\!]_{1_{D_2-\{0\}}}$ is a Krull domain by Corollary 7(3) and Lemma 8(3), and

$$D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}} = D_1[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}} \cap D_2[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$$

$$= D_1[\![\{X_\alpha\}]\!]_{1_{D_1-\{0\}}} \cap D_2[\![\{X_\alpha\}]\!]_{1_{D_2-\{0\}}},$$

where the second equality follows because D_1 and D_2 are overrings of D. Thus, $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain [13, Corollary 44.10]. □

The next theorem shows that $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain but $D[\![\{X_\alpha\}]\!]_1$ is not a Krull domain when D is a t-SFT PvMD but not a Krull domain.

Theorem 10. *If D is a t-SFT PvMD, then $D[\![\{X_\alpha\}]\!]_1$ is a PvMD if and only if D is a Krull domain.*

Proof. Assume that D is a t-SFT PvMD. Then each prime t-ideal of D is a v-ideal [17, Proposition 2.10]; so if P is a prime t-ideal of D, then

$$(PD[\![\{X_\alpha\}]\!]_1)_v = P_v[\![\{X_\alpha\}]\!]_1 = P[\![\{X_\alpha\}]\!]_1,$$

and hence $P[\![\{X_\alpha\}]\!]_1$ is a t-ideal. Hence, $D[\![\{X_\alpha\}]\!]_{1_{P[\![\{X_\alpha\}]\!]_1}}$ is a valuation domain, and therefore, D is a Krull domain [8, Theorem 3.3]. Conversely, if D is a Krull domain, then $D[\![\{X_\alpha\}]\!]_1$ is a Krull domain, and thus a PvMD. □

3. Examples of t-SFT PvMDs

Let D be an integral domain with quotient field K, $D[\{X_\alpha\}]$ be the polynomial ring over D, and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$.

Theorem 11. *The following statements are equivalent for D.*

(1) *D is a t-SFT PvMD.*
(2) *$D[\{X_\alpha\}]$ is a t-SFT PvMD.*
(3) *$D[\{X_\alpha\}]_{N_v}$ is an SFT Prüfer domain.*

Proof. (1) ⇒ (2) By Lemma 1(2), $D[\{X_\alpha\}]$ is a PvMD; so it suffices to show that every prime t-ideal of $D[\{X_\alpha\}]$ is a t-SFT ideal [17, Proposition 2.1]. For this, let Q be a prime t-ideal of $D[\{X_\alpha\}]$.

If $c(Q)_t \subsetneq D$, then $Q \cap N_v = \emptyset$, and so $Q = (Q \cap D)[\{X_\alpha\}]$ by Lemma 1(2) because D is a PvMD. Let $I \subseteq P(:= Q \cap D)$ be a nonzero finitely generated ideal and $k \geq 1$ be an integer such that $a^k \in I_t$ for all $a \in P$. If $0 \neq f \in P[\{X_\alpha\}]$ with $c(f) = (a_1, \ldots, a_n)$, then $f^k \in c(f^k)[\{X_\alpha\}] \subseteq c(f^k)_v[\{X_\alpha\}] = (c(f)^k)_v[\{X_\alpha\}] = (a_1^k, \ldots, a_n^k)_v[\{X_\alpha\}] \subseteq I_t[\{X_\alpha\}] = (I[\{X_\alpha\}])_t$, where the second and third equalities are from [13, Corollary 28.3] and [2, Lemma 3.3] respectively because $c(f)$ is t-invertible. Thus, Q is a t-SFT ideal.

Next, assume $c(Q)_t = D$. Then Q is a maximal t-ideal of $D[\{X_\alpha\}]$ and $Q \cap D = (0)$ (cf. [11, Proposition 2.2]); so ht$Q = 1$ (cf. [11, Lemma 2.3]). Since $K[\{X_\alpha\}]$ is a UFD, there is an $f \in Q$ such that $QK[\{X_\alpha\}] = fK[\{X_\alpha\}]$. Then $Q = QK[\{X_\alpha\}] \cap D[\{X_\alpha\}] = fK[\{X_\alpha\}] \cap D[\{X_\alpha\}] = fc(f)^{-1}[\{X_\alpha\}]$, and so if $0 \neq d \in c(f)$, then $dQ \subseteq fD[\{X_\alpha\}]$. Clearly, $\frac{d}{f}Q \subseteq D[\{X_\alpha\}]$, but $\frac{d}{f} \cdot f = d \in Q^{-1}Q - Q$. Hence $Q \subsetneq QQ^{-1}$, and since Q is a maximal t-ideal, $(QQ^{-1})_t = D[\{X_\alpha\}]$, and so $Q = A_t$ for some finitely generated ideal $A \subseteq Q$. Thus, Q is a t-SFT ideal.

(2) ⇒ (3) $D[\{X_\alpha\}]_{N_v}$ is flat over $D[\{X_\alpha\}]$, and thus $D[\{X_\alpha\}]_{N_v}$ is a t-SFT PvMD. Note that $D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain by Lemma 1(2); so every ideal of $D[\{X_\alpha\}]_{N_v}$ is a t-ideal. Thus, $D[\{X_\alpha\}]_{N_v}$ is an SFT Prüfer domain.

(3) ⇒ (1) Let P be a prime t-ideal of D. Then $P[\{X_\alpha\}]_{N_v}$ is a proper prime ideal of $D[\{X_\alpha\}]_{N_v}$, and hence by (3) and Lemma 1(2), there is a finitely generated ideal

$I \subseteq P$ and an integer $k \geq 1$ such that $f^k \in I[\{X_\alpha\}]_{N_v}$ for all $f \in P[\{X_\alpha\}]_{N_v}$. In particular, if $a \in P$, then $a^k \in I[\{X_\alpha\}]_{N_v} \cap K = I_t$ (cf. [16, Propositions 2.2(3) and 2.8(1)] for the equality). □

If $|\{X_\alpha\}| = \infty$, then $D[\{X_\alpha\}]$ is not an SFT-ring because $(\{X_\alpha\})$ is not an SFT-ideal. However, since an SFT Prüfer domain is a t-SFT PvMD, by Theorem 11, we have:

Corollary 12. *If D is an SFT Prüfer domain, then $D[\{X_\alpha\}]$ is a t-SFT PvMD.*

Remark 13. It is well known that D is a PvMD if and only if $D[\{X_\alpha\}]$ is a PvMD, and a PvMD is integrally closed. Hence, the (1) $\Leftrightarrow$ (2) of Theorem 11 also follows from [17, Corollary 2.14] that if D is integrally closed, D is a t-SFT-ring if and only if $D[\{X_\alpha\}]$ is a t-SFT-ring. Also, we use Theorem 11 to give other proofs of Corollary 4 and Theorem 9.

(1) *Proof of Corollary 4.* It suffices to show the implication (2) $\Rightarrow$ (3). By Lemma 1(3), $X^1(D[\{X_\alpha\}]_{N_v}) = X^1(D) = \emptyset$. Also, $D[\{X_\alpha\}]_{N_v}$ is an SFT Prüfer domain by Theorem 11, and therefore $D[\{X_\alpha\}]_{N_v}$ is an anti-Archimedean domain [1, Proposition 2.3].

(2) *Proof of Theorem 9.* If D is a t-SFT PvMD, then $D[X]_{N_v}$ is an SFT Prüfer domain by Theorem 11, and hence $(D[X]_{N_v})[\![\{X_\alpha\}]\!]_{1_{D[X]_{N_v}-\{0\}}}$ is a Krull domain [1, Theorem 3.7]. Note that

$$(D[X]_{N_v})[\![\{X_\alpha\}]\!]_{1_{D[X]_{N_v}-\{0\}}} \cap K[\![\{X_\alpha\}]\!]_1 = D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}.$$

(For if $\xi \in (D[X]_{N_v})[\![\{X_\alpha\}]\!]_{1_{D[X]_{N_v}-\{0\}}} \cap K[\![\{X_\alpha\}]\!]_1$, then $f\xi \in (D[X]_{N_v})[\![\{X_\alpha\}]\!]_1 \cap K[\![\{X_\alpha\}]\!]_1$ for some $0 \neq f \in D[X]_{N_v}$. Hence, if ω is one of the nonzero coefficients of ξ, then $f\omega \in K \cap D[X]_{N_v} = D$, and thus $f \in D$ and $f\xi \in D[\![\{X_\alpha\}]\!]_1$. Therefore, $\xi \in D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$.) Clearly, $K[\![\{X_\alpha\}]\!]_1$ is a Krull domain. Thus, $D[\![\{X_\alpha\}]\!]_{1_{D-\{0\}}}$ is a Krull domain.

We end this paper with a theorem by which one can construct new t-SFT PvMDs from old ones (e.g., Krull domains).

Theorem 14. *Let T be an integral domain, M be a nonzero maximal ideal of T, $\varphi : T \to T/M$ be the canonical homomorphism, D be a subring of T/M, and $R = \varphi^{-1}(D)$. Then R is a t-SFT PvMD if and only if T/M is the quotient field of D, D and T are t-SFT PvMDs, and T_M is a valuation domain such that $P^2 \subsetneq P$ for all nonzero prime ideals P of T_M.*

Proof. The result follows from the facts that (i) R is a PvMD if and only if T/M is the quotient field of D, D and T are PvMDs, and T_M is a valuation domain [10, Theorem 4.1]; (ii) R is a t-SFT ring if and only if D and T are t-SFT-rings [17, Theorem 2.8]; (iii) if T is a t-SFT-ring, then T_M is a t-SFT-ring [17, Proposition 2.3]; and (iv) a valuation domain V is a t-SFT-ring if and only if V is an SFT-ring, if and only if $P^2 \subsetneq P$ for all nonzero prime ideals P of V (by the definitions). □

Corollary 15. *Let X be an indeterminate over D, and let $R = D + XK[X]$. Then R is a t-SFT PvMD if and only if D is a t-SFT PvMD.*

Proof. Let $T = K[X]$ and $M = XK[X]$. Then T is a t-SFT PvMD, $T/M \cong K$ is the quotient field D, and T_M is a rank-one DVR. Thus, the result follows directly from Theorem 14. □

Example 16. Let D be a Krull domain with quotient field K, $V = K[\![X]\!]$ be the power series ring over K, and $R = D + XK[\![X]\!]$.

(1) R is a t-SFT PvMD with a unique nonzero minimal prime ideal $XK[\![X]\!]$.

(2) $R[\![\{X_\alpha\}]\!]_{1\,R-\{0\}}$ is a Krull domain, but $R[\![\{X_\alpha\}]\!]_1$ is not a PvMD.

(3) D is a Dedekind domain if and only if R is a Prüfer domain.

Proof. (1) Note that $V = K[\![X]\!]$ is a rank-one DVR; so V is a t-SFT PvMD. Thus, by Theorem 14, R is a t-SFT PvMD. Also, $XK[\![X]\!]$ is contained in every nonzero prime ideal of R, and hence $XK[\![X]\!]$ is a unique nonzero minimal prime ideal of R.

(2) By Theorem 9, $R[\![\{X_\alpha\}]\!]_{1\,R-\{0\}}$ is a Krull domain. Clearly, R is not a Krull domain, and hence by Theorem 10, $R[\![\{X_\alpha\}]\!]_1$ is not a Krull domain.

(3) It is obvious that a Krull domain is a Prüfer domain if and only if it is a Dedekind domain. Thus, R is a Prüfer domain if and only if D is a Prüfer domain [13, Exercise 13 on page 286], if and only if D is a Dedekind domain. □

Acknowledgements. This work was supported by the Incheon National University Research Fund in 2013 (Grant No. 20130396).

References

[1] D. D. Anderson, B. G. Kang, and M. H. Park, *Anti-Archimedean rings and power series rings*, Comm. Algebra **26** (1998), 3223–3238.

[2] D. D. Anderson and M. Zafrullah, *Almost Bezout domains*, J. Algebra **142** (1991), 285–309.

[3] J. Arnold, *Power series rings over Prüfer domains*, Pacific J. Math. **44** (1973), 1–11.

[4] ———, *Power series rings with finite Krull dimension*, Indiana Univ. Math. J. **31** (1982), 897–911.

[5] G. W. Chang, *A pinched-Krull domain at a prime ideal*, Comm. Algebra **30** (2002), 3669–3686.

[6] ———, *Spectral localizing systems that are t-splitting multiplicative sets of ideals*, J. Korean Math. Soc. **44** (2007), 863–872.

[7] G. W. Chang and M. Fontana, *Upper to zero in polynomial rings and Prüfer-like domains*, Comm. Algebra **37** (2009), 164–192.

[8] G. W. Chang and D. Y. Oh, *The rings $D(\!(\mathcal{X})\!)_i$ and $D\{\!\{\mathcal{X}\}\!\}_i$*, J. Algebra Appl. **12** (2013), 1250147 (11 pages).

[9] D. Dobbs, E. Houston, T. Lucas, and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains*, Comm. Algebra **17** (1989), 2835–2852.

[10] M. Fontana and S. Gabelli, *On the class group and the local class group of a pullback*, J. Algebra **181** (1996), 803–835.

[11] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra **26** (1998), 1017–1039.

[12] R. Gilmer, *Power series rings over a Krull domain*, Pacific J. Math. **29** (1969), 543–549.

[13] ———, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[14] R. Gilmer and W. Heinzer, *Primary ideals with finitely generated radical in a commutative ring*, Manuscripta Math. **78** (1993), 201–221.

[15] E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra **17** (1989), 1955–1969.

[16] B. G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), 151–170.

[17] B. G. Kang and M. H. Park, *A note on t-SFT-rings*, Comm. Algebra **34** (2006), 3153–3165.

[18] D. J. Kwak and Y. S. Park, *On t-flat overrings*, Chinese J. Math. **23** (1995), 17–24.

[19] J. Mott and M. Zafrullah, *On Krull domains*, Arch. Math. **56** (1991), 559–568.

Department of Mathematics Education
Incheon National University
Incheon 22012, Korea
E-mail address: whan@inu.ac.kr