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RAD-SUPPLEMENTING MODULES

Salahattin Özdemir

Abstract. Let R be a ring, and let M be a left R-module. If M is Rad-supplementing, then every direct summand of M is Rad-supplementing, but not each factor module of M. Any finite direct sum of Rad-supplementing modules is Rad-supplementing. Every module with composition series is (Rad-)supplementing. M has a Rad-supplement in its injective envelope if and only if M has a Rad-supplement in every essential extension. R is left perfect if and only if R is semilocal, reduced and the free left R-module $({}_RR)^{(\mathbb{N})}$ is Rad-supplementing if and only if R is reduced and the free left R-module $({}_RR)^{(\mathbb{N})}$ is ample Rad-supplementing. M is ample Rad-supplementing if and only if every submodule of M is Rad-supplementing. Every left R-module is (ample) Rad-supplementing if and only if $R/P(R)$ is left perfect, where $P(R)$ is the sum of all left ideals I of R such that $\operatorname{Rad} I = I$.

1. Introduction

All rings consider in this paper will be associative with an identity element. Unless otherwise stated, R denotes an arbitrary ring and all modules will be *left* unitary R-modules. For a module M, by $X \subseteq M$, we mean X is a submodule of M or M is an extension of X. As usual, $\operatorname{Rad} M$ denotes the radical of M and J denotes the Jacobson radical of the ring R. $E(M)$ will be the injective envelope of M. For an index set I, $M^{(I)}$ denotes the direct sum $\oplus_I M$. By $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ we denote as usual the set of natural numbers, the ring of integers and the field of rational numbers, respectively. A submodule $K \subseteq M$ is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. Dually, a submodule $L \subseteq M$ is called *essential* in M (denoted by $L \trianglelefteq M$) if $L \cap X \neq 0$ for every nonzero submodule X of M.

The notion of a supplement submodule was introduced in [12] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules U and V of a module M, V is said to be a *supplement* of U in M or U is said to *have a supplement* V in M if $U + V = M$ and $U \cap V \ll V$. The module M is called *supplemented* if every

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submodule of M has a supplement in M. See [19, §41] and [9] for results and the definitions related to supplements and supplemented modules. Recently, several authors have studied different generalizations of supplemented modules. In [1], τ-supplemented modules were defined for an arbitrary preradical τ for the category of left R-modules. For submodules U and V of a module M, V is said to be a *τ-supplement* of U in M or U is said to *have a τ-supplement* V in M if $U + V = M$ and $U \cap V \subseteq \tau(V)$. M is called a *τ-supplemented* module if every submodule of M has a τ-supplement in M. For the particular case $\tau = \text{Rad}$, Rad-supplemented modules have been studied in [6]; rings over which all modules are Rad-supplemented were characterized. Also, in the recent paper [7], the relation between Rad-supplemented modules and local modules have been investigated. See [18]; these modules are called *generalized supplemented* modules. Note that Rad-supplements V of a module M are also called *coneat* submodules which can be characterized by the fact that each module with zero radical is injective with respect to the inclusion $V \subseteq M$; see [1], [9, §10] and [15]. On the other hand, modules that have supplements in every module in which it is contained as a submodule have been studied in [22]; the structure of these modules, which are called *modules with the property* (E), has been completely determined over Dedekind domains. Such modules are also called *Moduln mit Ergänzungseigenschaft* in [3] and *supplementing* modules in [9, p. 255]. We follow the terminology and notation as in [9]. We call a module M *supplementing* if it has a supplement in each module in which it is contained as a submodule. By considering these modules we define and study (ample) Rad-supplementing modules as a proper generalization of supplementing modules. A module M is called (*ample*) *Rad-supplementing* if it has a (an ample) Rad-supplement in each module in which it is contained as a submodule, where a submodule $U \subseteq M$ has *ample Rad-supplements* in M if for every $L \subseteq M$ with $U + L = M$, there is a Rad-supplement L' of U with $L' \subseteq L$.

In Section 2, we investigate some properties of Rad-supplementing modules. It is clear that every supplementing module is Rad-supplementing, but the converse implication fails to be true; Example 2.3. If a module M has a Rad-supplement in its injective envelope, M need not be Rad-supplementing. However, we prove that M has a Rad-supplement in its injective envelope if and only if M has a Rad-supplement in every essential extension; Proposition 2.5. We prove that for modules $A \subseteq B$, if A and B/A are Rad-supplementing, then so is B. Using this fact we also prove that every module with composition series is Rad-supplementing; Theorem 2.12. A factor module of a Rad-supplementing module need not be Rad-supplementing; Example 2.15. For modules $A \subseteq B \subseteq C$ with C/A injective, we prove that if B is Rad-supplementing, then so is B/A. As one of the main results, we prove that R is left perfect if and only if R is semilocal, ${}_RR$ is reduced and $({}_RR)^{(\mathbb{N})}$ is Rad-supplementing; Theorem 2.20. Finally, using a result of [22], we show that over a commutative ring R, a semisimple R-module M is Rad-supplementing if and only if it is supplementing and that is equivalent the fact that M is pure-injective; Theorem 2.21.

Section 3 contains some properties of ample Rad-supplementing modules. It starts by proving a useful property that a module M is ample Rad-supplementing if and only if every submodule of M is Rad-supplementing; Proposition 3.1. One of the main results of this part is that R is left perfect if and only if ${}_RR$ is reduced and the free left R-module $({}_RR)^{(\mathbb{N})}$ is ample Rad-supplementing; Theorem 3.3. In the proof of this result, Rad-supplemented modules plays an important role as, of course, every ample Rad-supplementing module is Rad-supplemented. Finally, using the characterization of Rad-supplemented modules given in [6], we characterize the rings over which every module is (ample) Rad-supplementing. We prove that every left R-module is (ample) Rad-supplementing if and only if every reduced left R-module is Rad-supplementing if and only if $R/P(R)$ is left perfect; Theorem 3.4.

2. Rad-supplementing modules

A module M is called *radical* if $\operatorname{Rad} M = M$, and M is called *reduced* if it has no nonzero radical submodule. See [21, p. 47] for details for the notion of reduced and radical modules.

Proposition 2.1. *Supplementing modules and radical modules are Rad-supplementing.*

Proof. Let M be a module and N be any extension of M. If M is supplementing, then it has a supplement, and so a Rad-supplement in N. Thus M is Rad-supplementing. Now, if $\operatorname{Rad} M = M$, then N is a Rad-supplement of M in N. □

By $P(M)$ we denote the sum of all *radical* submodules of the module M, that is,

$$P(M) = \sum \{U \subseteq M \mid \operatorname{Rad} U = U\}.$$

Clearly M is reduced if $P(M) = 0$.

Since $P(M)$ is a radical submodule of M we have the following corollary.

Corollary 2.2. *For a module M, $P(M)$ is Rad-supplementing.*

A subset I of a ring R is said to be *left T-nilpotent* in case, for every sequence $\{a_k\}_{k=1}^{\infty}$ in I, there is a positive integer n such that $a_1 \cdots a_n = 0$.

In general, Rad-supplementing modules need not be supplementing as the following example shows.

Example 2.3. Let k be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates x_n, $n \in \mathbb{N}$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{N}$. Then the quotient ring $R = k[x_1, x_2, \ldots]/I$ is a local ring with the unique maximal ideal

$J = J^2$ (see [6, Example 6.2] for details). Now let $M = J^{(\mathbb{N})}$. Then we have $\operatorname{Rad} M = M$, and so M is Rad-supplementing by Proposition 2.1. However, M does not have a supplement in $R^{(\mathbb{N})}$. Because, otherwise, by [5, Theorem 1], J would be a left T-nilpotent as R is semilocal, but this is impossible. Thus M is not supplementing.

For instance, over a left max ring, supplementing modules and Rad-supplementing modules coincide, where R is called a *left max ring* if every left R-module has a maximal submodule or equivalently, $\operatorname{Rad} M \ll M$ for every left R-module M.

Proposition 2.4. *Every direct summand of a Rad-supplementing module is Rad-supplementing.*

Proof. Let U be a direct summand of a Rad-supplementing module M, and let N be any extension of U. Then $M = A \oplus U$ for some submodule $A \subseteq M$. By hypothesis M has a Rad-supplement in the module $A \oplus N$ containing M, that is, there exists a submodule V of $A \oplus N$ such that

$$(A \oplus U) + V = A \oplus N \quad \text{and} \quad (A \oplus U) \cap V \subseteq \operatorname{Rad} V.$$

Now, let $g : A \oplus N \to N$ be the projection onto N. Then

$$U + g(V) = g(A \oplus U) + g(V) = g((A \oplus U) + V) = g(A \oplus N) = N, \text{ and}$$

$$U \cap g(V) = g((A \oplus U) \cap V) \subseteq g(\operatorname{Rad} V) \subseteq \operatorname{Rad}(g(V)).$$

Hence $g(V)$ is a Rad-supplement of U in N. □

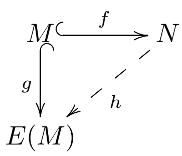
If a module M has a Rad-supplement in its injective envelope $E(M)$, M need not be Rad-supplementing. For example, for $R = \mathbb{Z}$, the R-module $M = 2\mathbb{Z}$ has a Rad-supplement in $E(M) = \mathbb{Q}$ since $\operatorname{Rad} \mathbb{Q} = \mathbb{Q}$ (and so $\mathbb{Q}$ is Rad-supplemented). But, M does not have a Rad-supplement in $\mathbb{Z}$, and thus M is not Rad-supplementing. However, we have the following result.

Proposition 2.5. *Let M be a module. Then the following are equivalent.*

(i) *M has a Rad-supplement in every essential extension;*
(ii) *M has a Rad-supplement in its injective envelope $E(M)$.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(i) Let $M \subseteq N$ with $M \trianglelefteq N$, and let $f : M \to N$ and $g : M \to E(M)$ be inclusion maps. Then we have the following commutative diagram with h necessarily monic:



By hypothesis, M has a Rad-supplement in $E(M)$, say K. That is, $M + K = E(M)$ and $M \cap K \subseteq \operatorname{Rad} K$. Since $M \subseteq h(N)$, we obtain that $h(N) =$

$h(N) \cap E(M) = h(N) \cap (M + K) = M + h(N) \cap K$. Now, taking any $n \in N$, we have $h(n) = m + h(n_1) = h(m + n_1)$ where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since h is monic, and so $M + h^{-1}(K) = N$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K) \subseteq h^{-1}(\operatorname{Rad} K) \subseteq \operatorname{Rad}(h^{-1}(K))$ since $h^{-1}(M) = M$ as h is monic. Hence $h^{-1}(K)$ is a Rad-supplement of M in N. □

Proposition 2.6. *Let B be a module, and let A be a submodule of B. If A and B/A are Rad-supplementing, then so is B.*

Proof. Let $B \subseteq N$ be any extension of B. By hypothesis, there is a Rad-supplement V/A of B/A in N/A and a Rad-supplement W of A in V. We claim that W is a Rad-supplement of B in N. We have epimorphisms $f : W \to V/A$ and $g : V/A \to N/B$ such that $\operatorname{Ker} f = W \cap A \subseteq \operatorname{Rad} W$ and $\operatorname{Ker} g = V/A \cap B/A \subseteq \operatorname{Rad}(V/A)$. Then $g \circ f : W \to N/B$ is an epimorphism such that $W \cap B = \operatorname{Ker}(g \circ f) \subseteq \operatorname{Rad} W$ by [20, Lemma 1.1]. Finally, $N = V + B = (W + A) + B = W + B$. □

Remark 2.7. The previous result holds for supplementing modules; see [22, Lemma 1.3-(c)].

Corollary 2.8. *If M_1 and M_2 are Rad-supplementing modules, then so is $M_1 \oplus M_2$.*

Proof. Consider the short exact sequence

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0.$$

Thus the result follows by Proposition 2.6. □

R is said to be a *left hereditary* ring if every left ideal of R is projective.

Corollary 2.9. *If $M/P(M)$ is Rad-supplementing, then M is Rad-supplementing. For left hereditary rings, the converse is also true.*

Proof. Since $P(M)$ is Rad-supplementing by Corollary 2.2, the result follows by Proposition 2.6. Over left hereditary rings, any factor module of a Rad-supplementing module is Rad-supplementing (see Corollary 2.18). □

We give the proof of the following known fact for completeness.

Lemma 2.10. *Every simple submodule S of a module M is either a direct summand of M or small in M.*

Proof. Suppose that S is not small in M, then there exists a proper submodule K of M such that $S + K = M$. Since S is simple and $K \neq M$, $S \cap K = 0$. Thus $M = S \oplus K$. □

Proposition 2.11. *Every simple module is* (*Rad-*)*supplementing.*

Proof. Let S be a simple module and N any extension of S. Then by Lemma 2.10, $S \ll N$ or $S \oplus S' = N$ for a submodule $S' \subseteq N$. In the first case, N is a (Rad-)supplement of S in N, and in the second case, S' is a (Rad-)supplement of S in N. So, in each case S has a (Rad-)supplement in N, that is, S is (Rad-)supplementing. □

Theorem 2.12. *Every module with composition series is* (*Rad-*)*supplementing.*

Proof. Let $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ be a composition series of a module M. The proof is by induction on $n \in \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple, and so M is (Rad-)supplementing by Proposition 2.11. Suppose that this is true for each $k \leq n - 1$. Then M_{n-1} is (Rad-)supplementing. Since M_n/M_{n-1} is also (Rad-)supplementing as a simple module, we obtain by Proposition 2.6 that $M = M_n$ is (Rad-)supplementing. □

Corollary 2.13. *A finitely generated semisimple module is* (*Rad-*)*supplementing.*

In general, a factor module of a Rad-supplementing module need not be Rad-supplementing. To give such a counterexample we need the following result.

R is called *Von Neumann regular* if every element $a \in R$ can be written in the form axa, for some $x \in R$.

Proposition 2.14. *Let R be a commutative Von Neumann regular ring. Then an R-module M is Rad-supplementing if and only if M is injective.*

Proof. Suppose that M is a Rad-supplementing module. Let $M \subseteq N$ be any extension of M. Then there is a Rad-supplement V of M in N, that is, $V+M = N$ and $V \cap M \subseteq \operatorname{Rad} V$. Since all R-modules have zero radical by [13, 3.73 and 3.75], we have $\operatorname{Rad} V = 0$, and so $N = V \oplus M$. Conversely, if M is injective and $M \subseteq N$ is any extension of M, then $N = M \oplus K$ for some submodule $K \subseteq N$. Thus K is a Rad-supplement of M in N. □

It is known that a ring R is lefty hereditary if and only if every quotient of an injective R-module is injective (see [8, Ch.I, Theorem 5.4]).

Example 2.15. Let $R = \prod_{i \in I} F_i$ be a ring, where each F_i is a field for an infinite index set I. Then R is a commutative Von Neumann regular ring. Indeed, let $a = (a_i)_{i \in I} \in R$ where $a_i \in F_i$ for all $i \in I$. Taking $b = (b_i)_{i \in I} \in R$ where $b_i \in F_i$ such that

$$b_i = \begin{cases} a_i^{-1} & \text{if } a_i \neq 0, \\ 0 & \text{if } a_i = 0. \end{cases}$$

Then we obtain that

$$aba = (a_i)_I (b_i)_I (a_i)_I = (a_i b_i a_i)_{i \in I} = (a_i)_{i \in I} = a.$$

Now, by Proposition 2.14, R is a Rad-supplementing module over itself since it is injective (see [13, Corollary 3.11B]). Since R is not noetherian, it cannot be semisimple (by [14, Corollary 2.6]). Thus R is not hereditary by [16, Corollary]. Hence, there is a factor module of R which is not injective.

The following technical lemma will be useful to show that Rad-supplementing modules are closed under factor modules, under a special condition.

Lemma 2.16. *Let $A \subseteq B \subseteq C$ be modules with C/A injective. Let N be a module containing B/A. Then there exists a commutative diagram with exact rows*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Proof. By pushout we have the following commutative diagram, where φ exists since C/A is injective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B/A & \hookrightarrow & N & \longrightarrow & N/(B/A) & \longrightarrow & 0 \\ & & \big\downarrow & {\scriptstyle (1)}\ \swarrow{\scriptstyle \varphi} & \big\downarrow{\scriptstyle g} & {\scriptstyle \alpha}\ \swarrow\ {\scriptstyle (2)} & \big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & C/A & \xrightarrow{\beta} & N' & \longrightarrow & N/(B/A) & \longrightarrow & 0 \end{array}$$

In the diagram, since the triangle-(1) is commutative, there exists a homomorphism $\alpha : N/(B/A) \longrightarrow N'$ making the triangle-(2) is commutative by [11, Lemma I.8.4]. So, the second row splits. Then we can take $N' = (C/A) \oplus (N/(B/A))$, and so we may assume that $\beta : C/A \longrightarrow N'$ is an inclusion. Therefore, we have the following commutative diagram since $B/A = \beta(B/A) = g(B/A) \subseteq N'$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\gamma} & C \oplus (N/(B/A)) & \xrightarrow{\sigma} & N' & \longrightarrow & 0 \end{array}$$

where $\gamma(a) = (a, 0)$ for every $a \in A$, $\phi(b) = (b, 0)$ for every $b \in B$, and $\sigma(c, \overline{x}) = (c + A, \overline{x})$ for every $c \in C$ and $\overline{x} \in N/(B/A)$. Finally, taking $P = \sigma^{-1}(g(N))$ and defining a homomorphism $\widetilde{\sigma} : P \longrightarrow g(N)$ by $\widetilde{\sigma}(x) = \sigma(x)$ for every $x \in P$ (in fact, $\widetilde{\sigma}$ is an epimorphism as so is σ), we obtain the following desired commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{\widetilde{\sigma}} & g(N) \cong N & \longrightarrow & 0 \end{array}$$

□

Proposition 2.17. *Let $A \subseteq B \subseteq C$ with C/A injective. If B is Rad-supplementing, then so is B/A.*

Proof. Let $B/A \subseteq N$ be any extension of B/A. By Lemma 2.16, we have the following commutative diagram with exact rows since C/A is injective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{\sigma} & B/A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

Since h is monic and B is Rad-supplementing, $B \cong \operatorname{Im} h$ has a Rad-supplement in P, say V. That is, $\operatorname{Im} h + V = P$ and $\operatorname{Im} h \cap V \subseteq \operatorname{Rad} V$. We claim that $g(V)$ is a Rad-supplement of B/A in N.

$$N = g(P) = g(h(B)) + g(V) = (f\sigma)(B) + g(V) = (B/A) + g(V), \text{ and}$$

$$(B/A) \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap V] \subseteq g(\operatorname{Rad} V) \subseteq \operatorname{Rad}(g(V)). \square$$

Corollary 2.18. *If R is a left hereditary ring, then every factor module of Rad-supplementing module is Rad-supplementing.*

Proposition 2.19. *If M is a reduced, projective and Rad-supplementing module, then* $\operatorname{Rad} M \ll M$.

Proof. Suppose $X + \operatorname{Rad} M = M$ for a submodule X of M. Then since M is projective, there exists $f \in \operatorname{End}(M)$ such that $\operatorname{Im} f \subseteq X$ and $\operatorname{Im}(1 - f) \subseteq \operatorname{Rad} M = JM$ where J is a Jacobson radical of R. Therefore f is a monomorphism by [4, Theorem 3]. Since M is Rad-supplementing and $\operatorname{Im} f \cong M$, $\operatorname{Im} f$ has a Rad-supplement V in M, that is, $\operatorname{Im} f + V = M$ and $\operatorname{Im} f \cap V \subseteq \operatorname{Rad} V$. Now we have an epimorphism $g : V \to M/\operatorname{Im} f$ such that $\operatorname{Ker} g = V \cap \operatorname{Im} f \subseteq \operatorname{Rad} V$. Moreover, since $M = \operatorname{Im} f + \operatorname{Im}(1 - f) = \operatorname{Im} f + \operatorname{Rad} M$ we have $\operatorname{Rad}(M/\operatorname{Im} f) = M/\operatorname{Im} f$. Thus $\operatorname{Rad} V = V$, and so $V = 0$ since M is reduced. Hence $M = \operatorname{Im} f \subseteq X$ implies that $X = M$ as required. □

R is said to be a *semilocal* ring if R/J is a semisimple ring, that is a left (and right) semisimple R-module (see [14, §20]).

Theorem 2.20. *A ring R is left perfect if and only if R is semilocal, ${}_RR$ is reduced and the free left R-module $F = ({}_RR)^{(\mathbb{N})}$ is Rad-supplementing.*

Proof. If R is left perfect, then R is semilocal by [2, 28.4], and clearly ${}_RR$ is reduced. Since all left R-modules are supplemented and so Rad-supplemented, F is Rad-supplementing. Conversely, since $P({}_RR) = 0$ we have $P(F) = (P({}_RR))^{(\mathbb{N})} = 0$, that is, F is reduced. Thus by Proposition 2.19, $JF = \operatorname{Rad} F \ll F$, that is, J is left T-nilpotent by, for example, [2, 28.3]. Hence R is left perfect by [2, 28.4] since it is moreover semilocal. □

Supplementing modules over commutative noetherian rings have been studied in [3]; the author showed that if a module M is supplementing, then it is *cotorsion*, that is, $\operatorname{Ext}^1_R(F, M) = 0$ for every flat module F (see [10] for cotorsion modules). So the question was raised When Rad-supplementing modules are cotorsion? Since any pure-injective module is cotorsion, the following result gives an answer of the question for a semisimple module over a commutative ring. The relation between (Rad-)supplementing modules and cotorsion modules needs to be further investigated.

The part (iii)⇒(i) of the proof of the following theorem follows from [22, Theorem 1.6-(ii)⇒(i)], but we give it by explanation for completeness.

Theorem 2.21. *Let R be a commutative ring. Then the following are equivalent for a semisimple R-module M.*

(i) *M is supplementing;*
(ii) *M is Rad-supplementing;*
(iii) *M is pure-injective.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii) Let $M \subseteq N$ be a pure extension of M. By hypothesis M has a Rad-supplement V in N, that is, $M + V = N$ and $M \cap V \subseteq \operatorname{Rad} V$. Since M is pure in N, we have $\operatorname{Rad} M = M \cap \operatorname{Rad} N$ (as R is commutative). Thus $M \cap V \subseteq M \cap \operatorname{Rad} N = \operatorname{Rad} M = 0$ as M is semisimple. Hence $N = M \oplus V$ as required.

(iii)⇒(i) Let $M \subseteq N$ be any extension of M. Then the factor module $X = (M + \operatorname{Rad} N)/\operatorname{Rad} N$ of M is again semisimple and pure-injective. Since semisimple submodules are pure in every module with zero radical and $\operatorname{Rad}(N/\operatorname{Rad} N) = 0$, it follows that X is a direct summand of $N/\operatorname{Rad} N$. Now let

$$(V/\operatorname{Rad} N) \oplus X = N/\operatorname{Rad} N$$

for a submodule $V \subseteq N$ such that $\operatorname{Rad} N \subseteq V$. So we have $V + M = N$ with V minimal, and thus V is a supplement of M in N. This is because, if $T + M = N$ for a submodule T of N with $T \subseteq V$, then from

$$\operatorname{Rad}(N/T) = \operatorname{Rad}((M+T)/T) = \operatorname{Rad}(M/M \cap T) = 0$$

as $M/M \cap T$ is semisimple, we obtain that $\operatorname{Rad} N \subseteq T$. Moreover, since

$$\operatorname{Rad} N = V \cap (M + \operatorname{Rad} N) = V \cap M + \operatorname{Rad} N,$$

we have $V \cap M \subseteq \operatorname{Rad} N$ and $V = T + V \cap M \subseteq T + \operatorname{Rad} N = T$, thus $T = V$. □

3. Ample Rad-supplementing modules

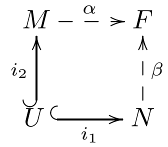
The following useful result gives a relation between Rad-supplementing modules and ample Rad-supplementing modules.

Proposition 3.1. *A module M is ample Rad-supplementing if and only if every submodule of M is Rad-supplementing.*

Proof. (⇐) Let M be a module and N be any extension of M. Suppose that for a submodule $X \subseteq N$, $X + M = N$. By hypothesis the submodule $X \cap M$ of M has a Rad-supplement V in X containing $X \cap M$, that is, $(X \cap M) + V = X$ and

$(X \cap M) \cap V \subseteq \operatorname{Rad} V$. Then $N = M + X = M + (X \cap M) + V = M + V$ and, $M \cap V = M \cap (V \cap X) = (X \cap M) \cap V \subseteq \operatorname{Rad} V$. Hence V is a Rad-supplement of M in N such that $V \subseteq X$.

($\Rightarrow$) Let U be a submodule of M and N be any module containing U. Thus we can draw the pushout for the inclusion homomorphisms $i_1 : U \hookrightarrow N$ and $i_2 : U \hookrightarrow M$:



In the diagram, α and β are also monomorphisms by the properties of pushout (see, for example, [17, Exercise 5.10]). Let $M' = \operatorname{Im} \alpha$ and $N' = \operatorname{Im} \beta$. Then $F = M' + N'$ by the properties of pushout. So by hypothesis, $M' \cong M$ has a Rad-supplement V in F such that $V \subseteq N'$, that is, $M' + V = F$ and $M' \cap V \subseteq \operatorname{Rad} V$. Therefore V is a Rad-supplement of $M' \cap N'$ in N', because $N' = N' \cap F = N' \cap (M' + V) = (M' \cap N') + V$ and $(M' \cap N') \cap V = M' \cap V \subseteq \operatorname{Rad} V$. Now, we claim that $\beta^{-1}(V)$ is a Rad-supplement of U in N. Since $\beta : N \to F$ is a monomorphism with $N' = \operatorname{Im} \beta$, we have an isomorphism $\widetilde{\beta} : N \to N'$ defined as $\widetilde{\beta}(x) = \beta(x)$ for all $x \in N$. By this isomorphism, since V is a Rad-supplement of $M' \cap N'$ in N', we obtain $\widetilde{\beta}^{-1}(V)$ is a Rad-supplement of $\widetilde{\beta}^{-1}(M' \cap N')$ in $\widetilde{\beta}^{-1}(N')$. Since it can be easily shown that $\widetilde{\beta}^{-1}(V) = \beta^{-1}(V)$, $\widetilde{\beta}^{-1}(N') = N$, and $\widetilde{\beta}^{-1}(M' \cap N') = U$ the result follows. □

Corollary 3.2. *Every ample Rad-supplementing module is both Rad-supplementing and Rad-supplemented.*

Theorem 3.3. *A ring R is left perfect if and only if ${}_RR$ is reduced and the free left R-module $F = ({}_RR)^{(\mathbb{N})}$ is ample Rad-supplementing.*

Proof. If R is left perfect, then ${}_RR$ is reduced and all left R-modules are supplemented, and so Rad-supplemented. Thus every submodule of F is Rad-supplementing. Hence F is ample Rad-supplementing by Proposition 3.1. Conversely, if F is ample Rad-supplementing, then it is Rad-supplemented by Corollary 3.2, and so R is left perfect by [6, Theorem 5.3]. □

Finally, we give the characterization of the rings over which every module is (ample) Rad-supplementing.

Theorem 3.4. *For a ring R, the following are equivalent*:

(i) *Every left R-module is Rad-supplementing*;
(ii) *Every reduced left R-module is Rad-supplementing*;
(iii) *Every left R-module is ample Rad-supplementing*;
(iv) *Every left R-module is Rad-supplemented*;
(v) *$R/P(R)$ is left perfect.*

Proof. Let M be a module. (i)⇒(ii) is clear.

(ii)⇒(i) Since $M/P(M)$ is reduced, it is Rad-supplementing by hypothesis. So M is Rad-supplementing by Corollary 2.9.

(i)⇒(iii) Since every submodule of M is Rad-supplementing, M is ample Rad-supplementing by Proposition 3.1.

(iii)⇒(iv) by Corollary 3.2.

(iv)⇒(i) Let $M \subseteq N$ be any extension of M. By hypothesis, N is Rad-supplemented, and so M has a Rad-supplement in N.

(iv)⇔(v) by [6, Theorem 6.1]. □

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References

[1] K. Al-Takhman, C. Lomp, and R. Wisbauer, *τ-complemented and τ-supplemented modules*, Algebra Discrete Math. (2006), no. 3, 1–16.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, New-York, Springer, 1992.

[3] J. Averdunk, *Moduln mit Ergänzungseigenschaft*, Dissertation, Ludwig-Maximilians-Universität München, Fakultät für Mathematik, 1996.

[4] I. Beck, *Projective and free modules*, Math. Z. **129** (1972), 231–234.

[5] E. Büyükaşik and C. Lomp, *Rings whose modules are weakly supplemented are perfect: Applications to certain ring extensions*, Math. Scand. **105** (2009), no. 1, 25–30.

[6] E. Büyükaşik, E. Mermut, and S. Özdemir, *Rad-supplemented modules*, Rend. Semin. Mat. Univ. Padova **124** (2010), 157–177.

[7] E. Büyükaşik, R. Tribak, *On w-local modules and Rad-supplemented modules*, J. Korean Math. Soc. **51** (2014), no. 5, 971–985.

[8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Landmarks in Mathematics and Physics series, New Jersey: Princeton Univesity, 1956.

[9] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules*, Frontiers in Mathematics, Basel: Birkhäuser Verlag, Supplements and projectivity in module theory, 2006.

[10] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, vol. 30 of de Gruyter Expositions in Mathematics, Berlin: Walter de Gruyter & Co., 2000.

[11] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, vol. 84 of Mathematical Surveys and Monographs, Providence, RI: American Mathematical Society, 2001.

[12] F. Kasch and E. A. Mares, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J. **27** (1966), 525–529.

[13] T. Y. Lam, *Lectures on modules and rings*, vol. 189 of Graduate Texts in Mathematics, New York: Springer-Verlag, 1999.

[14] ———, *A first course in noncommutative rings*, vol. 131 of Graduate Texts in Mathematics, New York: Springer-Verlag, 2001.

[15] E. Mermut, *Homological Approach to Complements and Supplements*, Ph.D. thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences, İzmir-Turkey, 2004.

[16] B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.

[17] J. J. Rotman, *An Introduction to Homological Algebra*, Universitext, New York: Springer, 2009.

[18] Y. Wang and N. Ding, *Generalized supplemented modules*, Taiwanese J. Math. **10** (2006), no. 6, 1589–1601.

[19] R. Wisbauer, *Foundations of Module and Ring Theory*, Reading: Gordon and Breach, 1991.

[20] W. Xue, *Characterization of semiperfect and perfect rings*, Publ. Mat. **40** (1996), no. 1, 115–125.

[21] H. Zöschinger, *Komplementierte Moduln über Dedekindringen*, J. Algebra **29** (1974), 42–56.

[22] ———, *Moduln, die in jeder Erweiterung ein Komplement haben*, Math. Scand. **35** (1974), 267–287.

Department of Mathematics
Faculty of Sciences
Dokuz Eylül University
Buca, İzmir, 35390, Turkey
E-mail address: salahattin.ozdemir@deu.edu.tr