

ON A GENERALIZATION OF HIRZEBRUCH'S THEOREM TO BOTT TOWERS

JIN HONG KIM

ABSTRACT. The primary aim of this paper is to generalize a theorem of Hirzebruch for the complex 2-dimensional Bott manifolds, usually called Hirzebruch surfaces, to more general Bott towers of height n . To do so, we first show that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes. As a consequence, in this paper we show that two Bott manifolds $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ and $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n)$ are isomorphic to each other, as Bott towers if and only if both $\alpha_n \equiv \alpha'_n \pmod{2}$ and $\alpha_n^2 = (\alpha'_n)^2$ hold in the cohomology ring of $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$ over integer coefficients. This result will complete a circle of ideas initiated in [11] by Ishida. We also give some partial affirmative remarks toward the assertion that under certain condition our main result still holds to be true for two Bott manifolds just diffeomorphic, but not necessarily isomorphic, to each other.

1. Introduction and main results

Our main concern of this paper is a family of compact complex manifolds, called a Bott tower introduced first by Bott and Samelson in [2], which have recently attracted a great amount of attention in toric topology world (refer to [4] and [8]). They form a very nice family of toric manifolds, and possess several extra structures such as iterated fibrations and certain distinguished sections.

In order to construct such a family of compact complex manifolds, let \mathbb{L}_1 be a holomorphic complex line bundle over $B_1 = \mathbb{C}\mathbb{P}^1$. Then take its direct sum with the trivial complex line bundle $\underline{\mathbb{C}}$ and projectivize each fiber to obtain a complex manifold $B_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_1)$. B_2 is a fiber bundle over B_1 with a fiber $\mathbb{C}\mathbb{P}^1$, and is called a *Hirzebruch surface*. We can repeat this process n times, so that each B_j is a fiber bundle over B_{j-1} with a fiber $\mathbb{C}\mathbb{P}^1$.

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To be more precise, a *Bott tower* $\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^n$ of height n is inductively defined as a sequence of $\mathbb{C}\mathbb{P}^1$ -bundles starting from a point $*$:

$$B_n(\alpha_1, \dots, \alpha_n) \xrightarrow{\pi_n} B_{n-1}(\alpha_1, \dots, \alpha_{n-1}) \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} B_1(\alpha_1) \xrightarrow{\pi_1} B_0 = \{*\},$$

where $B_j(\alpha_1, \alpha_2, \dots, \alpha_j)$ denotes the projectivization $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_j)$ of the trivial complex line bundle $\underline{\mathbb{C}}$ and a complex line bundle \mathbb{L}_j over $B_{j-1}(\alpha_1, \dots, \alpha_{j-1})$ with the first Chern class α_j , and

$$\pi_j : B_j(\alpha_1, \alpha_2, \dots, \alpha_j) \rightarrow B_{j-1}(\alpha_1, \alpha_2, \dots, \alpha_{j-1})$$

denotes the projection for each j with $1 \leq j \leq n$. By definition, each $B_j(\alpha_1, \dots, \alpha_j)$ is a toric manifold which admits an effective algebraic action of the torus $(\mathbb{C} - \{0\})^n$ having it as an open dense orbit, and it is called a *j-step Bott manifold*, or just a *Bott manifold* (refer to [2] and [9] for more details). Note that by its construction α_1 always vanishes. Moreover, if all the first Chern classes α_i used to construct a Bott tower of height n are zero, then $B_n(\alpha_1, \alpha_2, \dots, \alpha_n)$ is diffeomorphic to $(\mathbb{C}\mathbb{P}^1)^n$, and $B_2(\alpha_1, \alpha_2)$ is simply a Hirzebruch surface. Analogously, a generalized Bott manifold of height n can be inductively defined also as a sequence of $\mathbb{C}\mathbb{P}^{n_i}$ -bundles with $n_i \geq 1$ starting from a point.

From now on, for the sake of simplicity, we will very often use the notation B_n for $B_n(\alpha_1, \alpha_2, \dots, \alpha_n)$, if there is no danger of any confusion. We shall also use the prime notation to denote any objects for the second Bott manifold $B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n)$.

Let x_j denote the first Chern class of the tautological line bundle γ_j over B_j . It follows from a formula of Borel and Hirzebruch in [1] that $H^*(B_j; \mathbb{Z})$ is a free module over $H^*(B_{j-1}; \mathbb{Z})$ through the projection $\pi_j^* : H^*(B_{j-1}; \mathbb{Z}) \rightarrow H^*(B_j; \mathbb{Z})$ with two generators 1 and x_j of degree 0 and 2, respectively. Thus, when $H^*(B_j; \mathbb{Z})$ is regarded as a subring of $H^*(B_n; \mathbb{Z})$ by using the pullback of the composition

$$\pi_{j+1} \circ \pi_{j+2} \circ \dots \circ \pi_n : B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_{j+2}} B_{j+1} \xrightarrow{\pi_{j+1}} B_j$$

of the projection maps $\pi_{j+1}, \pi_{j+2}, \dots$, and π_n , it can be shown that the cohomology ring $H^*(B_n; \mathbb{Z})$ of $B_n(\alpha_1, \alpha_2, \dots, \alpha_n)$ is given by

$$\mathbb{Z}[x_1, x_2, \dots, x_n] / \langle x_i^2 = \alpha_i x_i \mid i = 1, 2, \dots, n \rangle.$$

Since $\alpha_j \in H^2(B_{j-1}(\alpha_1, \dots, \alpha_{j-1}); \mathbb{Z})$, it follows from [7], Proposition 3.1 that we may write

$$(1.1) \quad \alpha_j = \sum_{i=1}^{j-1} c_j^i x_i, \quad c_j^i \in \mathbb{Z}.$$

Note also that $H^2(B_j; \mathbb{Z})$ is isomorphic to \mathbb{Z}^j and that \mathbb{Z}^j is bijective with the collection of isomorphism classes of holomorphic line bundles over B_j which are in turn classified by the first Chern classes ([9], Lemmas 2.11 and 2.14).

For the purposes of this paper, an *isomorphism* between two Bott towers is defined to be a collection $\{F_j\}_{j=1}^n$ of diffeomorphisms

$$F_j : B_j(\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \alpha_j) \rightarrow B_j(\alpha'_1, \alpha'_2, \dots, \alpha'_{j-1}, \alpha'_j)$$

which commute with the projection maps π_j and π'_j in that the following diagram commutes:

$$\begin{array}{ccc} B_j(\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \alpha_j) & \xrightarrow{F_j} & B_j(\alpha'_1, \alpha'_2, \dots, \alpha'_{j-1}, \alpha'_j) \\ \pi_j \downarrow & & \downarrow \pi'_j \\ B_{j-1}(\alpha_1, \alpha_2, \dots, \alpha_{j-2}, \alpha_{j-1}) & \xrightarrow{F_{j-1}} & B_{j-1}(\alpha'_1, \alpha'_2, \dots, \alpha'_{j-2}, \alpha'_{j-1}). \end{array}$$

Note that this definition is weaker than that given in [9], Definition 2.6.

It is well-known from a result of Hirzebruch in [10]¹ that $B_2(\alpha_1, \alpha_2)$ is diffeomorphic to $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, if $\alpha_2 \equiv 0 \pmod 2$, while $B_2(\alpha_1, \alpha_2)$ is diffeomorphic to $\mathbb{C}\mathbb{P}^1 \# \overline{\mathbb{C}\mathbb{P}^1}$, otherwise. That is, we have the following theorem (see, e.g., [16], p. 16 or [14], Example 2.2).

Theorem 1.1. *$B_2(\alpha_1, \alpha_2)$ is isomorphic (or diffeomorphic) to $B_2(\alpha_1, \alpha'_2)$ if and only if $\alpha_2 \equiv \alpha'_2 \pmod 2$ holds.*

In particular, this implies that $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ is graded ring isomorphic to $H^*(B_2(\alpha_1, \alpha'_2); \mathbb{Z})$ if and only if both α_2 and α'_2 are equal to 0 mod 2, or equivalently $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ is graded ring isomorphic to $H^*(B_2(\alpha_1, \alpha'_2); \mathbb{Z})$ if and only if neither α_2 nor α'_2 are equal to 0 mod 2.

Related to these observations, the following conjecture has been posed (refer to [14] and [15]).

Conjecture 1.2. *Let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \dots, \alpha_n); \mathbb{Z}) \rightarrow H^*(B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n); \mathbb{Z})$$

be a graded ring isomorphism over integers. Then ψ is actually given by the pullback of a diffeomorphism from $B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n)$ to $B_n(\alpha_1, \alpha_2, \dots, \alpha_n)$.

This conjecture is usually called a *strong cohomological rigidity conjecture for Bott manifolds*. It is still open in its full generality, and seems to be a very delicate and also difficult problem, even though there are some partial affirmative answers for certain special cases (refer to, e.g., [5], [6], and [13]). Conjecture 1.2 is also closely related to the well-known Petrie's conjecture, and there are some recent results for Bott manifolds, related to the Petrie's conjecture (see, e.g., [6], [12], [17] and [18]).

Our primary aim of this paper is not to directly deal with Conjecture 1.2, but rather to generalize the theorem of Hirzebruch for the complex 2-dimensional Bott manifolds $B_2(\alpha_1, \alpha_2)$ to more general Bott towers of height n . In view of

¹It needs to be remarked that the paper [10] contains many more interesting results other than this.

the very definitions of Bott towers or Bott manifolds, we think that this kind of generalization is more appropriate and more useful in understanding the Bott towers or Bott manifolds.

One of the key problems which will play an important role in resolving the above conjecture is to classify complex vector bundles of rank two over a Bott manifold. Our first main result is to give a complete classification of those rank two complex vector bundles over a Bott manifold in terms of the total Chern classes, as follows.

Theorem 1.3. *Let \mathbb{E}_1 and \mathbb{E}_2 be two complex vector bundles of rank 2 over a Bott manifold B_j ($j \geq 2$) such that the total Chern class $c(\mathbb{E}_1)$ coincides with the total Chern class $c(\mathbb{E}_2)$. Then \mathbb{E}_1 is isomorphic to \mathbb{E}_2 as complex vector bundles.*

Theorem 1.3 extends a result of Ishida in [11] where he proved Theorem 1.3 only for decomposable vector bundles of rank two over a Bott manifold. Note, however, that our proof is completely different from that of [11], Theorem 3.1.

As an application of Theorem 1.3, we can give the following theorem for Bott towers.

Theorem 1.4. *Let a Bott tower $\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^{n-1}$ be isomorphic to*

$$\{(B_j(\alpha'_1, \dots, \alpha'_j), \pi'_j)\}_{j=1}^{n-1}$$

by a family $\{F_j\}_{j=1}^{n-1}$ of diffeomorphisms, and let α_n and α'_n be two elements of

$$H^2(B_{n-1}(\alpha_1, \dots, \alpha_{n-1}); \mathbb{Z}) \text{ and } H^2(B_{n-1}(\alpha'_1, \dots, \alpha'_{n-1}); \mathbb{Z}),$$

respectively. Then the following three statements are equivalent.

- (a) $(B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n), \pi_n)$ is diffeomorphic to $(B_n(\alpha'_1, \dots, \alpha'_{n-1}, \alpha'_n), \pi'_n)$ by a diffeomorphism F_n which commutes with π_n and π'_n , so that two Bott towers $\{(B_j(\alpha_1, \alpha_2, \dots, \alpha_j), \pi_j)\}_{j=1}^n$ and

$$\{(B_j(\alpha'_1, \alpha'_2, \dots, \alpha'_j), \pi'_j)\}_{j=1}^{n-1} \cup \{(B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_{n-1}, \alpha'_n), \pi'_n)\}$$

are isomorphic to each other.

- (b) *Two cohomology rings*

$$H^*(B_n(\alpha_1, \dots, \alpha_n); \mathbb{Z}) \text{ and } H^*(B_n(\alpha'_1, \dots, \alpha'_n); \mathbb{Z}),$$

as $H^(B_{n-1}; \mathbb{Z})$ -algebras are isomorphic to each other by the isomorphism F_n^* .*

- (c) *Two elements α_n and α'_n satisfy both conditions:*

$$\alpha_n \equiv F_{n-1}^*(\alpha'_n) \pmod{2} \text{ and } \alpha_n^2 = (F_{n-1}^*(\alpha'_n))^2.$$

Remark 1.5. (a) Since F_{n-1}^* is a graded ring isomorphism, the condition that $\alpha_n \equiv F_{n-1}^*(\alpha'_n) \pmod{2}$ is equivalent to saying that either both of α_2 and α'_2 are equal to 0 mod 2 or neither of α_2 and α'_2 are equal to 0 mod 2. This fits with the condition in Theorem 1.1.

- (b) Ishida also proved in [11], Corollary 4.3 that Theorem 1.4(a) is equivalent to Theorem 1.4(b) by a conceptually similar but technically different method.

As an immediate consequence of Theorem 1.4, we have the following corollary which can be regarded as a higher dimensional analogue of a result of Hirzebruch in [10] (Theorem 1.1).

Corollary 1.6. *Given two Bott manifolds*

$$B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n) \text{ and } B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha'_n),$$

$B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ is isomorphic to $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n)$ as Bott towers if and only if both

$$(1.2) \quad \alpha_n \equiv \alpha'_n \pmod{2}, \quad \text{and} \quad \alpha_n^2 = (\alpha'_n)^2$$

hold.

Remark 1.7. In case of Hirzebruch surfaces, every complex vector bundle of rank 2 over $B_1(\alpha_1) = \mathbb{C}P^1$ is classified by the *first* Chern class only, so we do not need the second condition $\alpha_2^2 = (\alpha'_2)^2$ in Theorem 1.1, contrary to Theorem 1.3 or Corollary 1.6. In fact, in this case the extra condition $\alpha_2^2 = (\alpha'_2)^2$ automatically holds, since they always vanish, due to the degree reason.

Next we provide infinitely many non-trivial examples which satisfy two conditions in (1.2).

Example 1.8. Consider the 2-step Bott manifold $B_2(\alpha_1, \alpha_2)$ with $c_2^1 = -(l+1)$ for any integer l . Let α_3 and α'_3 be two elements of degree 2 of

$$H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z}) = \mathbb{Z}[x_1, x_2] / \langle x_1^2 = 0, x_2^2 = \alpha_2 x_2 \rangle, \quad \alpha_2 = c_2^1 x_1$$

such that

$$\alpha_3 = x_1 + lx_2, \quad \text{and} \quad \alpha'_3 = (1 + 2l)x_1 + (l + 2)x_2.$$

Then clearly $\alpha_3 \equiv \alpha'_3 \pmod{2}$, and it is also easy to show that

$$\alpha_3^2 = -l(l - 1)(l + 2)x_1x_2 = (\alpha'_3)^2,$$

as required. Therefore, in this case two Bott manifolds

$$B_3(\alpha_1, \alpha_2, \alpha_3) \quad \text{and} \quad B_3(\alpha_1, \alpha_2, \alpha'_3)$$

are isomorphic to each other by Corollary 1.6.

It is a natural question to ask whether or not Theorem 1.4 or Corollary 1.6 is also true just for two diffeomorphic, but not necessarily isomorphic, Bott manifolds, at least up to some permutation in the symmetric group S_n on the index set $\{1, 2, \dots, n\}$. In Section 4, we give some affirmative remark for this question, as follows (refer to Section 4, Lemma 4.4 for more on the notation r_i).

Theorem 1.9. *Let $B_n(\alpha_1, \dots, \alpha_n)$ be diffeomorphic to $B_n(\alpha'_1, \dots, \alpha'_n)$ by a diffeomorphism F , and let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \dots, \alpha_n); \mathbb{Z}) \rightarrow H^*(B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n); \mathbb{Z})$$

be the graded ring isomorphism over integers given by the pullback of F^{-1} . If all of r_i ($1 \leq i \leq n$) are equal to ± 1 , then there is some permutation $\sigma \in S_n$ such that we have

$$\psi(\alpha_i) \equiv \alpha'_{\sigma(i)} \pmod{2}$$

for all $1 \leq i \leq n$.

We organize this paper, as follows. In Section 2, we prove an important result that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes (Theorem 2.1). In Section 3, we give a detailed proof of Theorem 1.4 by essentially using the mathematical induction on n . In this section, we give a proof of a simpler version of Theorem 1.4 which is easily shown to imply the full version of Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.9 as well as some other related discussions.

2. Proof of Theorem 1.3

The aim of this section is to give a proof of Theorem 1.3 which will be crucially used in the proof of Theorem 1.4. As mentioned in Section 1, this theorem says that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes.

Theorem 2.1. *Let \mathbb{E}_1 and \mathbb{E}_2 be two complex vector bundles of rank 2 over a Bott manifold B_j ($j \geq 2$). Then \mathbb{E}_1 is isomorphic to \mathbb{E}_2 as complex vector bundles if and only if the total Chern class $c(\mathbb{E}_1)$ coincides with the total Chern class $c(\mathbb{E}_2)$.*

Proof. To prove it, note first from [5], Lemma 3.4 that every complex vector bundle of rank 2 over a Bott manifold B_2 is classified by the total Chern class. Thus we need to prove the theorem only for $j \geq 3$. To do so, consider the following exact sequence

$$(2.1) \quad [B_j, U(\infty)/U(2)] \rightarrow [B_j, BU(2)] \xrightarrow{c_2} [B_j, BU(\infty)]$$

induced from the fibration

$$U(\infty)/U(2) \rightarrow BU(2) \rightarrow BU(\infty),$$

where $U(\infty) = \bigcup_{n=1}^{\infty} U(n)$ is obtained as the telescoping construction given by the standard embedding (see, e.g., [3], p.241). It can be shown as in [5], Lemma 3.4 that

$$[B_1, U(\infty)/U(2)] = [B_2, U(\infty)/U(2)] = 0,$$

since B_1 and B_2 are of the real dimension at most 4, and $U/U(2)$ is 4-connected. Recall that $U(\infty)/U(2)$ is 4-connected, since $U(3)/U(2)$ is diffeomorphic to S^5 and $U(3)/U(2)$ is embedded into $U(\infty)/U(2)$ as a telescoping subspace in the natural way (see, e.g., [19], p. 127).

Note also that there is an exact sequence

$$(2.2) \quad [B_{j-1}, U(\infty)/U(2)] \rightarrow [B_j, U(\infty)/U(2)] \rightarrow [B_1, U(\infty)/U(2)]$$

induced from the fibration

$$B_1 = \mathbb{C}P^1 \hookrightarrow B_j \rightarrow B_{j-1}.$$

Since $[B_1, U(\infty)/U(2)] = [B_2, U(\infty)/U(2)] = 0$, it follows inductively from the exact sequence (2.2) that all $[B_j, U(\infty)/U(2)] = 0$ for all $j \geq 3$. Thus, by the exact sequence (2.1) the Chern class map c is injective.

Since $H^{\text{odd}}(B_j; \mathbb{Z}) = 0$, $[B_j, BU(\infty)]$ is torsion free. This together with the fact that c is injective implies that all elements of $[B_j, BU(2)]$ can be classified by their Chern classes. This completes the proof of Theorem 2.1. \square

3. Proof of Theorem 1.4

The aim of this section is to give a proof of Theorem 1.3. To do so, we begin with the following theorem which clearly proves that Theorem 1.4(c) implies Theorem 1.4(a).

Theorem 3.1. *Let a Bott tower $\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^{n-1}$ be isomorphic to itself by a family $\{F_j\}_{j=1}^{n-1}$ of diffeomorphisms, and let α_n and α'_n be two elements of $H^2(B_{n-1}(\alpha_1, \dots, \alpha_{n-1}); \mathbb{Z})$ such that*

- (a) $\alpha_n \equiv F_{n-1}^*(\alpha'_n) \pmod{2}$, and
- (b) $\alpha_n^2 = (F_{n-1}^*(\alpha'_n))^2$.

Then $(B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n), \pi_n)$ is diffeomorphic to $(B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n), \pi'_n)$ by a diffeomorphism F_n which commutes with π_n and π'_n , so that two Bott towers $\{(B_j(\alpha_1, \alpha_2, \dots, \alpha_j), \pi_j)\}_{j=1}^n$ and

$$\{(B_j(\alpha_1, \alpha_2, \dots, \alpha_j), \pi_j)\}_{j=1}^{n-1} \cup \{(B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha'_n), \pi'_n)\}$$

are isomorphic to each other.

Proof. To prove it, we first assume that both $\alpha_n \equiv F_{n-1}^*(\alpha'_n) \pmod{2}$ and $\alpha_n^2 = (F_{n-1}^*(\alpha'_n))^2$ hold in $H^2(B_{n-1}(\alpha_1, \dots, \alpha_{n-1}); \mathbb{Z})$, and then use the induction on n . Note that Theorem 3.1 holds for $n = 2$ case, due to the validity of Theorem 1.1.

So suppose that Theorem 3.1 holds for any $n - 1 \geq 2$. Then consider the following commutative diagram:

$$\begin{array}{ccc} B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n)) & \xrightarrow{\tilde{F}_n} & B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha'_n) \\ \tilde{\pi}_n \downarrow & & \downarrow \pi'_n \\ B_{n-1}(\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}) & \xrightarrow{F_{n-1}} & B_{n-1}(\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}). \end{array}$$

Here the Bott manifold $B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ is nothing but the projectivization $\mathbb{P}(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n)$ of the trivial complex line bundle $\underline{\mathbb{C}}$ and a complex line bundle $\tilde{\mathbb{L}}_n$ with the first Chern class $F_{n-1}^*(\alpha'_n)$ over $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$, and \tilde{F}_n

is a bundle isomorphism between $B_n(\alpha_1, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ and $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n)$. Thus, in particular, $B_n(\alpha_1, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ and $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n)$ are diffeomorphic to each other by \tilde{F}_n .

Next we want to show that $B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ is actually diffeomorphic to $B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n)$ by a diffeomorphism G which commutes with π_n and $\tilde{\pi}_n$. To do so, note first that, since $F_{n-1}^*(\alpha'_n) \equiv \alpha_n \pmod{2}$, there are some integers b_n^j such that

$$(3.1) \quad F_{n-1}^*(\alpha'_n) = \alpha_n + 2 \sum_{j=1}^{n-1} b_n^j x_j.$$

We then recall the following well-known fact (refer to, e.g., [5], Lemma 2.1).

Lemma 3.2. *Let $\pi : \mathbb{E} \rightarrow B$ be a complex vector bundle over a smooth manifold B and let $\mathbb{P}(\mathbb{E})$ be the projectivization of \mathbb{E} . Let \mathbb{L} be a complex line bundle over B . We denote by \mathbb{E}^* the complex vector bundle dual to \mathbb{E} . Then both $\mathbb{P}(\mathbb{E}^*)$ and $\mathbb{P}(\mathbb{E} \otimes \mathbb{L})$ are isomorphic to $\mathbb{P}(\mathbb{E})$ as fiber bundles, and, in particular, they are diffeomorphic to each other.*

Let γ be a complex line bundle over $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$ defined by

$$\gamma = \gamma_1^{-b_n^1} \otimes \gamma_2^{-b_n^2} \otimes \dots \otimes \gamma_{n-1}^{-b_n^{n-1}},$$

where γ_j denotes the tautological line bundle over B_j , and here is regarded as a complex line bundle over the same Bott manifold B_{n-1} through the obvious pullback map.

Lemma 3.3. *The first Chern class $c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma)$ of a complex vector bundle $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ of rank 2 over $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$ is equal to α_n that is the first Chern class of a complex line bundle \mathbb{L}_n over the same Bott manifold $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$.*

Proof. It is easy to see

$$\begin{aligned} c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) &= c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n \otimes \gamma) = 2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n) \\ &= -2 \sum_{j=1}^{n-1} b_n^j x_j + F_{n-1}^*(\alpha'_n) \stackrel{(3.1)}{=} \alpha_n, \end{aligned}$$

as desired. □

With the help of Theorem 2.1, we can prove the following corollary.

Corollary 3.4. *Two complex vector bundles $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ and $\underline{\mathbb{C}} \oplus \mathbb{L}_n$ of rank 2 over $B_{n-1}(\alpha_1, \dots, \alpha_{n-1})$ is isomorphic to each other, as complex vector bundles of rank 2.*

Proof. To prove it, recall first that by assumption we have the following identities:

$$(3.2) \quad \alpha_n^2 = (F_{n-1}^*(\alpha'_n))^2 = c_1(\tilde{\mathbb{L}}_n)^2 \text{ in } H^*(B_{n-1}; \mathbb{Z}).$$

Since by Lemma 3.3 we have

$$(3.3) \quad \alpha_n = 2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n)$$

and so

$$\alpha_n^2 = 4c_1(\gamma)^2 + 4c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) + c_1(\tilde{\mathbb{L}}_n)^2,$$

it follows from (3.2) that we have

$$(3.4) \quad 4(c_1(\gamma)^2 + c_1(\gamma)c_1(\tilde{\mathbb{L}}_n)) = \alpha_n^2 - c_1(\tilde{\mathbb{L}}_n)^2 = 0 \text{ in } H^*(B_{n-1}; \mathbb{Z}).$$

Since there is no torsion in $H^*(B_{n-1}; \mathbb{Z})$, this implies that

$$c_1(\gamma)^2 + c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) = 0.$$

It is also easy to show that in $H^*(B_{n-1}; \mathbb{Z})$ we have

$$\begin{aligned} c((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) &= 1 + c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) + c_2((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) \\ &= c(\gamma)c(\tilde{\mathbb{L}}_n \otimes \gamma) = (1 + c_1(\gamma))(1 + c_1(\tilde{\mathbb{L}}_n) + c_1(\gamma)) \\ &= 1 + (2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n)) + (c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) + c_1(\gamma)^2) \\ &= 1 + \alpha_n, \quad \text{by (3.3) and (3.4)} \\ &= c(\underline{\mathbb{C}} \oplus \mathbb{L}_n). \end{aligned}$$

Therefore, it follows from Theorem 2.1 that two vector bundles $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ and $\underline{\mathbb{C}} \oplus \mathbb{L}_n$ are isomorphic to each other, as claimed. \square

Finally, we are ready to finish the proof of Theorem 3.1, as follows. Since $B_n(\alpha_1, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ is diffeomorphic to $\mathbb{P}(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n)$, it follows from Lemma 3.2 that it is also diffeomorphic to $\mathbb{P}((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma)$. On the other hand, the total Chern class of $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ coincides with that of $\underline{\mathbb{C}} \oplus \mathbb{L}_n$, and so they are isomorphic to each other by Corollary 3.4. Hence clearly their projectivizations $B_n(\alpha_1, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n))$ and $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ should be diffeomorphic by a diffeomorphism G . Notice also that by its construction G commutes with the projection maps π_n and $\tilde{\pi}_n$, as follows.

$$\begin{array}{ccc} B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n) & \xrightarrow{G} & B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, F_{n-1}^*(\alpha'_n)) \\ \pi_n \downarrow & & \downarrow \tilde{\pi}_n \\ B_{n-1}(\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}) & \xlongequal{\quad} & B_{n-1}(\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}). \end{array}$$

It is now easy to see that the composition $F_n := \tilde{F}_n \circ G$ of two diffeomorphisms G and \tilde{F}_n gives rise to a diffeomorphism between two Bott manifolds $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ and $B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n)$ which commutes with π_n and π'_n . Therefore, two Bott towers $\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^n$ and

$$\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^{n-1} \cup \{(B_n(\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha'_n), \pi'_n)\}$$

are isomorphic to each other. This implies that by the induction on n Theorem 3.1 holds for any $n \geq 2$, which completes the proof. \square

Note that clearly Theorem 1.4(a) implies Theorem 1.4(b). Next we show that Theorem 1.4(b) implies Theorem 1.4(c), as follows.

Theorem 3.5. *Assume that two cohomology rings*

$$H^*(B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha_n); \mathbb{Z}) \text{ and } H^*(B_n(\alpha_1, \dots, \alpha_{n-1}, \alpha'_n); \mathbb{Z}),$$

as $H^(B_{n-1}; \mathbb{Z})$ -algebras are isomorphic to each other by the isomorphism F_n^* . Then two elements α_n and α'_n satisfy both conditions:*

$$\alpha_n \equiv F_{n-1}^*(\alpha'_n) \pmod{2} \text{ and } \alpha_n^2 = (F_{n-1}^*(\alpha'_n))^2.$$

Proof. For simplicity, let $\Psi = F_n^*$, $x = x_n$, and $y = x'_n$. To prove the theorem, note also that it suffices to prove it under the assumption that F_{n-1}^* is the identity map.

To begin with the proof, we let

$$\Psi(x) = \pm y + \beta, \quad \beta \in H^*(B_{n-1}; \mathbb{Z}).$$

Since $\Psi(x^2 - \alpha_n x) = 0$ and $y^2 - \alpha'_n y = 0$, we have

$$\begin{aligned} 0 &= \Psi(x^2 - \alpha_n x) = \Psi(x)^2 - \alpha_n \Psi(x) = (\pm y + \beta)^2 - \alpha_n(\pm y + \beta) \\ &= (\alpha'_n \mp \alpha_n \pm 2\beta)y + (\beta^2 - \alpha_n \beta). \end{aligned}$$

Thus we obtain

$$(3.5) \quad \alpha'_n \mp \alpha_n \pm 2\beta = 0 \text{ and } \beta^2 - \alpha_n \beta = 0.$$

The first equation of (3.5) implies that we have

$$\alpha'_n \equiv \alpha_n \pmod{2}.$$

On the other hand, by plugging $2\beta = \alpha_n \pm \alpha'_n$ into the second equation $\beta^2 - \alpha_n \beta = 0$ of (3.5), it is easy to see that we have $\alpha_n^2 = (\alpha'_n)^2$, as desired. \square

4. Further remarks

As mentioned in Section 1, it is a natural question to ask whether or not Theorem 1.3 or Corollary 1.6 still holds for two diffeomorphic Bott manifolds by a more general diffeomorphism which does not necessarily come from Bott towers. The aim of this section is to discuss this question in more detail.

To do so, we first want to recall an alternative construction of Bott towers or Bott manifolds (see [9] for more details). It is easy to see that a one-step Bott tower $B_1(\alpha_1) = \mathbb{C}P^1$ can be obtained as a quotient of $(\mathbb{C}^2 - \{0\})/\mathbb{C}^*$, where \mathbb{C}^* denotes the complex numbers \mathbb{C} minus the origin, and \mathbb{C}^* acts on $(\mathbb{C}^*)^2$ diagonally. Then take a complex line bundle $\mathbb{L}_1 = (\mathbb{C}^*)^2 \times_{\mathbb{C}^*} \mathbb{C}$ over B_1 , where $g_1 \in \mathbb{C}^*$ acts on \mathbb{C} by $g_1 \cdot v = g_1^{c_2^1} v$ for some $c_2^1 \in \mathbb{Z}$ and in \mathbb{L}_1 we have

$$[(z_1, w_1), v] = [(z_1 g_1, w_1 g_1), g_1^{-c_2^1} v].$$

Let $\alpha_2 = c_1(\mathbb{L}_1) \in H^2(B_1(\alpha_1); \mathbb{Z})$. Then the 2-step Bott tower $B_2(\alpha_1, \alpha_2) = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_1)$ can be written as the quotient $(\mathbb{C}^2 - \{0\})^2 / (\mathbb{C}^*)^2$, where $(g_1, g_2) \in (\mathbb{C}^*)^2$ acts on $(\mathbb{C}^2 - \{0\})^2$ by

$$(g_1, g_2) \cdot ((z_1, w_1), (z_2, w_2)) = ((z_1 g_1, w_1 g_1), (z_2 g_2, g_1^{-c_2^1} w_2 g_2)).$$

We can continue this process to construct higher dimensional Bott towers

$$\{(B_j(\alpha_1, \dots, \alpha_j), \pi_j)\}_{j=1}^n$$

as the quotient of $(\mathbb{C}^2 - \{0\})^n / (\mathbb{C}^*)^n$ by the free action of $(\mathbb{C}^*)^n$ given by

$$\begin{aligned} & (g_1, g_2, \dots, g_j, \dots, g_n) \cdot ((z_1, w_1), (z_2, w_2), \dots, (z_j, w_j), \dots, (z_n, w_n)) \\ &= ((z_1 g_1, w_1 g_1), (z_2 g_2, g_1^{-c_2^1} w_2 g_2), \dots, \\ & \quad (z_j g_j, \left(\prod_{i=1}^{j-1} g_i^{-c_j^i}\right) w_j g_j), \dots, (w_n g_n, \left(\prod_{i=1}^{n-1} g_i^{-c_n^i}\right) w_n g_n)), \end{aligned}$$

where $\{c_j^i\}_{1 \leq i < j \leq n}$ denotes any collection of $n(n-1)/2$ integers. The obvious projections given by

$$\begin{aligned} & ((z_1, w_1), \dots, (z_n, w_n)) \xrightarrow{\pi_n} ((z_1, w_1), \dots, (z_{n-1}, w_{n-1})) \xrightarrow{\pi_{n-1}} \dots \\ & \dots \xrightarrow{\pi_3} ((z_1, w_1), (z_2, w_2)) \xrightarrow{\pi_2} (z_1, w_1) \xrightarrow{\pi_1} \{*\} \end{aligned}$$

then induces the Bott tower. As mentioned in Section 1, (1.1), it is interesting to note that

$$\alpha_j = \sum_{i=1}^{j-1} c_j^i x_i, \quad c_j^i \in \mathbb{Z}.$$

Let \mathcal{N}_n be the set of integral strictly upper triangular square matrices of size n . Then it has been shown in [9], Section 2.3 that the map from \mathcal{N}_n to the collection of isomorphism classes of n -step Bott towers is bijective. In other words, the Bott tower is also determined by a matrix $C \in \mathcal{N}_n$, and so we may write the n -step Bott tower as $B_n(C)$ for some $C \in \mathcal{N}_n$. If C is the zero matrix in \mathcal{N}_n , then clearly $B_n(C)$ is diffeomorphic to $(\mathbb{C}\mathbb{P}^1)^n$, and the converse is also true ([14]).

However, in general $B_n(C)$ and $B_n(D)$ may be diffeomorphic, even if C and D are different. In particular, let P denote the permutation matrix of a permutation σ in S_n on n letters $\{1, 2, \dots, n\}$ whose (i, j) -entry is 1 if $\sigma(j) = i$, and 0 otherwise. When both C and PCP^{-1} are elements of \mathcal{N}_n , $B_n(C)$ and $B_n(PCP^{-1})$ are diffeomorphic (refer to [6], Lemma 6.1). In view of these observations, the converse of Corollary 1.6 might not be true in that form, but might be true at least up to some permutation of the index set $\{1, 2, \dots, n\}$.

For the rest of this section, we will make this discussion more precise and deal with a more general case of Theorem 1.3 and Corollary 1.6. To do so, we

first need to collect some preliminary results. As in the paper [6], it will be useful to let

$$(4.1) \quad y_i = x_i - \frac{1}{2}\alpha_i, \quad 1 \leq i \leq n.$$

Then it is easy to see from (4.1) that the following lemma holds.

Lemma 4.1. *For each $1 \leq i \leq n$, we have*

$$(4.2) \quad x_i = \sum_{j=1}^i b_i^j y_j,$$

where all of b_i^j are elements of \mathbb{Q} and $b_i^i = 1$.

Proof. To prove it, we will use the mathematical induction on n . For $n = 1$, clearly $x_1 = y_1$, since $\alpha_1 = 0$. Next suppose that the lemma holds for any $n - 1 \geq 1$. Then we have

$$\begin{aligned} x_n &= y_n + \frac{1}{2}\alpha_n = y_n + \frac{1}{2} \sum_{j=1}^{n-1} c_n^j x_j = y_n + \frac{1}{2} \sum_{j=1}^{n-1} \left(c_n^j \left(\sum_{k=1}^j b_j^k y_k \right) \right) \\ &= y_n + \frac{1}{2} \sum_{k=1}^{n-1} \left(\sum_{j=k}^{n-1} c_n^j b_j^k \right) y_k, \end{aligned}$$

which completes the proof of the identity of the equation (4.2) for any n , as desired. \square

Note that Lemma 4.1 actually implies that x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n satisfy the following linear equation

$$(4.3) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ * & * & * & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix}_{n \times n} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}.$$

Since the coefficient matrix of the above linear equation (4.3) is lower triangular and unipotent, we can also express y_1, y_2, \dots, y_n in terms of x_1, x_2, \dots, x_n in such a way that we have

$$y_i = \sum_{j=1}^i \tilde{b}_i^j x_j,$$

where all of \tilde{b}_i^j are elements of \mathbb{Q} and $\tilde{b}_i^i = 1$.

The following lemma immediately follows from (4.1), so that its proof will be left to the reader.

Lemma 4.2. *If $x_i^2 = \alpha_i x_i$ holds, then $y_i^2 = \frac{1}{4}\alpha_i^2$ holds. Conversely, if $y_i^2 = \frac{1}{4}\alpha_i^2$ holds, then $x_i^2 = \alpha_i x_i$ holds.*

For the sake of notational simplicity, let

$$P := \mathbb{Q}[x_1, x_2, \dots, x_n] / \langle x_i^2 = \alpha_i x_i \mid i = 1, 2, \dots, n \rangle,$$

$$Q := \mathbb{Q}[y_1, y_2, \dots, y_n] / \langle y_i^2 = \frac{1}{4}\alpha_i^2 \mid i = 1, 2, \dots, n \rangle.$$

Then, it is straightforward to see from the identities (1.1) and (4.1), and Lemmas 4.1 and 4.2 that P is isomorphic to Q . It is often convenient to use Q instead of P in order to show some important properties of the cohomology ring $H^*(B_n; \mathbb{Q})$, although we do not use this fact very much in this paper.

We need the following lemma whose proof can be found in [5], Corollary 3.1.

Lemma 4.3. *Let ω be a primitive element of $H^2(B_n(\alpha_1, \alpha_2, \dots, \alpha_n); \mathbb{Z})$ whose square is zero. Then ω is either one of the following forms: $\pm(x_i - \frac{1}{2}\alpha_i)$, if $\alpha_i \equiv 0 \pmod{2}$, and $\pm(2x_i - \alpha_i)$, otherwise.*

The following lemma from [6], Proposition 4.1 also plays an important role in the proof of Theorem 4.5.

Lemma 4.4. *Let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \dots, \alpha_n); \mathbb{Z}) \rightarrow H^*(B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n); \mathbb{Z})$$

be a graded ring isomorphism over integers. Then, for each $1 \leq i \leq n$ there is $r_i \in \mathbb{Q}$ such that $\psi(y_i) = r_i y'_{\sigma(i)}$, where σ is a permutation in S_n .

Proof. For the sake of reader's convenience, we will provide a proof of the lemma, essentially due to that of [6], Proposition 4.1.

For the proof, we shall use the induction on i with $1 \leq i \leq n$. For $i = 1$, note that

$$\psi(y_1)^2 = \psi(y_1^2) = \psi(0) = 0,$$

where we used the identities $y_i^2 = \frac{1}{4}\alpha_i y_i$ and $\alpha_1 = 0$. Then it follows from Lemma 4.3 that there is some $r_1 \in \mathbb{Q}$ such that

$$\psi(y_1) = r_1 y'_{\sigma(1)},$$

where σ is an element of the symmetric group S_n . So, we are done with the case of $i = 1$.

Next, suppose that the lemma holds for any $i < k$, i.e.,

$$\psi(y_i) = r_i y'_{\sigma(i)}, \quad r_i \in \mathbb{Q}, \quad 1 \leq i < k,$$

where σ is an element of the symmetric group S_n . We then may assume without loss of generality that $\sigma(i) = i$ for any $1 \leq i < k$. It is also easy to obtain

(4.4)

$$\begin{aligned} \psi(y_k)^2 &= \psi(y_k^2) = \psi\left(\frac{1}{4}\alpha_k^2\right) \stackrel{(1.1)}{=} \psi\left(\frac{1}{4}\left(\sum_{i=1}^{k-1} c_k^i x_i\right)^2\right) \\ &\stackrel{(\text{Lem. 3.2})}{=} \frac{1}{4}\left(\psi\left(\sum_{i=1}^{k-1} \sum_{l=1}^i c_k^i b_i^l y_l\right)\right)^2 = \frac{1}{4}\left(\psi\left(\sum_{l=1}^{k-1} \left(\sum_{i=l}^{k-1} c_k^i b_i^l\right) y_l\right)\right)^2 \\ &= \frac{1}{4}\left(\psi\left(\sum_{l=1}^{k-1} d_l y_l\right)\right)^2 = \frac{1}{4} \sum_{t,s=1}^{k-1} d_t d_s \psi(y_t) \psi(y_s) \\ &= \frac{1}{4} \sum_{t,s=1}^{k-1} (d_t r_t)(d_s r_s) y'_t y'_s, \quad \psi(y_t) = r_t y'_t, \quad \psi(y_s) = r_s y'_s, \end{aligned}$$

where $d_l = \sum_{i=l}^{k-1} c_k^i b_i^l$.

Now, let

$$\psi(y_k) = \sum_{m=1}^n e_m y'_m, \quad e_m \in \mathbb{Q}.$$

It is not difficult to see from (4.4) that $e_m = 0$ at least for all $m \geq k + 1$. Thus we have

$$\psi(y_k) = \sum_{m=1}^k e_m y'_m.$$

We then claim that all of e_m are zero for $1 \leq m \leq k - 1$. To see it, suppose that there is some non-zero coefficient e_m for $1 \leq m \leq k - 1$, then there should be a term of the form $A y'_m y'_k$ ($A \in \mathbb{Q}$) in the expression of $\psi(y_k)^2$. But this contradicts the equation (4.4). Therefore, we have $\psi(y_k) = e_k y'_k$, which implies that, in general, for a permutation $\sigma \in S_n$ we have

$$\psi(y_k) = e_k y'_{\sigma(k)}, \quad 1 \leq k \leq n.$$

This completes the proof of Lemma 4.4. □

Note that r_i in Lemma 4.4 is either $\pm\frac{1}{2}$, or ± 2 , or ± 1 (see, e.g., [6], Lemma 4.1). With this understood, we give a partial result, related to Corollary 1.6, for two diffeomorphic Bott manifolds, as follows.

Theorem 4.5. *Let $B_n(\alpha_1, \dots, \alpha_n)$ be diffeomorphic to $B_n(\alpha'_1, \dots, \alpha'_n)$ by a diffeomorphism F , and let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \dots, \alpha_n); \mathbb{Z}) \rightarrow H^*(B_n(\alpha'_1, \alpha'_2, \dots, \alpha'_n); \mathbb{Z})$$

be the graded ring isomorphism over integers given by the pullback of F^{-1} . If all of r_i ($1 \leq i \leq n$) are equal to ± 1 , then there is some permutation $\sigma \in S_n$

such that we have

$$\psi(\alpha_i) \equiv \alpha'_{\sigma(i)} \pmod{2}$$

for all $1 \leq i \leq n$.

Proof. To prove it, recall first that $\alpha_1 = \alpha'_1 = 0$ by definition. So the theorem trivially holds for $n = 1$.

For the case of $n = 2$, by assumption two Bott manifolds $B_2(\alpha_1, \alpha_2)$ and $B_2(\alpha'_1, \alpha'_2)$ are diffeomorphic. Thus there is a graded ring isomorphism ψ from $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ and $H^*(B_2(\alpha'_1, \alpha'_2); \mathbb{Z})$. It then follows from Lemma 4.4 that there is a permutation $\sigma \in S_2$ such that

$$\psi(y_1) = \pm y'_{\sigma(1)}, \text{ and } \psi(y_2) = \pm y'_{\sigma(2)}.$$

In other words, we have

$$\psi(2x_1 - \alpha_1) = \pm(2x'_{\sigma(1)} - \alpha'_{\sigma(1)}), \text{ and } \psi(2x_2 - \alpha_2) = \pm(2x'_{\sigma(2)} - \alpha'_{\sigma(2)}).$$

This implies that

$$\psi(\alpha_1) \equiv \alpha'_{\sigma(1)}, \text{ and } \psi(\alpha_2) \equiv \alpha'_{\sigma(2)} \pmod{2},$$

as claimed. It should be now clear that exactly same arguments apply to any general $n \geq 3$. So we are done. \square

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DEPARTMENT OF MATHEMATICS EDUCATION
CHOSUN UNIVERSITY
GWANGJU 61452, KOREA
E-mail address: jinhkim11@gmail.com