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NOTE ON SOME CHARACTER FORMULAS

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Abstract. Chaudhary and Choi [7] presented 14 identities which reveal certain interesting interrelations among character formulas, combinatorial partition identities and continued partition identities. In this sequel, we aim to give slightly modified versions for 8 identities which are chosen among the above-mentioned 14 formulas.

1. Introduction and Preliminaries

We recall the following q-notation (see, *e.g.*, [10, Chapter 6]):

$$(a;q)_\infty := \prod_{k=0}^{\infty}\left(1-a\,q^k\right) = \prod_{k=1}^{\infty}\left(1-a\,q^{k-1}\right) \tag{1}$$

$$\left(a,\,q\in\mathbb{C};\;|q|<1;\;a\neq q^{-m}\;\;(m\in\mathbb{N}_0)\right).$$

It is noted that, when $a\neq 0$ and $|q| \geqq 1$, the infinite product in (1) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be tacitly assumed. Here and in the following, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, integers, and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

The following notation is also frequently used:

$$(a_1,\,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty\,(a_2;q)_\infty\cdots(a_m;q)_\infty \quad (m\in\mathbb{N}). \tag{2}$$

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Ramanujan defined the general theta function $f(a,b)$ as follows (see, for details, [2, p. 31, Eq.(18.1)]):

$$\begin{aligned} f(a,b) &= 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a) \quad (|ab|<1). \end{aligned} \tag{3}$$

We find from (3) that

$$f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b,a) \quad (n\in\mathbb{Z}). \tag{4}$$

Ramanujan also rediscovered the Jacobi's famous triple-product identity (see [2, p. 35, Entry 19]):

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{5}$$

which was first proved by Gauss.

Several q-series identities emerging from Jacobi's triple-product identity (5) are worthy of note here (see [2, pp. 36-37, Entry 22]):

$$\begin{aligned} \phi(q) := f(q,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} = 1+2\sum_{n=1}^{\infty} q^{n^2} \\ &= \{(-q;q^2)_\infty\}^2(q^2;q^2)_\infty = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}; \end{aligned} \tag{6}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{7}$$

$$\begin{aligned} f(-q) := f(-q,-q^2) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \end{aligned} \tag{8}$$

Equation (8) is known as Euler's *Pentagonal Number Theorem.* The following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{9}$$

provides the *analytic equivalence* of Euler's famous theorem: *The number of partitions of a positive integer* n *into distinct parts is equal to the number of partitions of* n *into odd parts.*

We also recall the Rogers-Ramanujan continued fraction of $R(q)$:

$$\begin{aligned} R(q) &:= q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = q^{\frac{1}{5}}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots \quad (|q|<1). \end{aligned} \tag{10}$$

Here $G(q)$ and $H(q)$ are widely investigated Roger-Ramanujan identities defined by

$$\begin{aligned} G(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{f(-q,-q^4)} \\ &= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}; \end{aligned} \tag{11}$$

$$\begin{aligned} H(q) &:= \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{f(-q^2,-q^3)} \\ &= \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}; \end{aligned} \tag{12}$$

and the functions $f(a,b)$ and $f(-q)$ are given in (3) and (8), respectively.

The following continued fractions are recalled (see, *e.g.*, [3, p. 5, Eq. (2.8)] and [4]): For $|q|<1$,

$$\begin{aligned} (q^2;q^2)_\infty(-q;q)_\infty &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\ &= \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-\cdots}; \end{aligned} \tag{13}$$

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots; \tag{14}$$

$$C(q) := \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots. \tag{15}$$

Andrews *et al.* [1] investigated new double summation hypergeometric q-series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,u,v,w;n), \tag{16}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{17}$$

Certain interesting special cases of (16) are recalled (see [1, p. 106, Theorem 3]; see also [3]-[7] and [9]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty; \tag{18}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty; \tag{19}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{20}$$

Chaudhary and Choi [7] presented 14 identities which reveal certain interesting interrelations among character formulas, combinatorial partition identities and continued partition identities. Here, in this paper, we give slightly modified versions for 8 identities which are chosen among the above-mentioned 14 formulas.

2. A Set of Preliminary Results

Here we recall certain interesting interrelations among character formulas, combinatorial partition identities and continued partition identities (see [7, Section 3]).

(21)

$$\begin{aligned} f(-q) = &- 4q\cdot\widehat{\Theta}_{12}^{-1}\left(q^{\frac{1}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\right) + 4q\widehat{\beta}_{12,1}(\tau) \\ &+ \frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2} \\ &\times [R(8,8,1,1,1,2)]^2\,[R(1,1,1,1,1,2)]^3 \\ &\times \left\{\frac{1}{1+}\;\frac{q^4}{1+}\;\frac{q^8}{1+}\;\frac{q^{12}}{1+}\;\frac{q^{16}}{1+}\;\frac{q^{20}}{1+}\;\frac{q^{24}}{1+}\cdots\right\}; \end{aligned}$$

(22)

$$\phi(q) = -\ 2q \cdot \widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)} q^{L_0} + q^{-\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)} q^{L_0}\big) +\ 2q\widehat{\beta}_{12,1}(\tau)$$
$$+ \frac{(q^2;q^2)_\infty (q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty \{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty \{(q^{16};q^{16})_\infty\}^2}$$
$$\times [R(8,8,1,1,1,2)]^2\, [R(1,1,1,1,1,2)]^3$$
$$\times \left\{ \frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots \right\};$$

(23)

$$\chi(-q) = -q \cdot \widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)} q^{L_0} + q^{-\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)} q^{L_0}\big)$$
$$+ q\widehat{\beta}_{12,1}(\tau) + \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty}{\{(q^{12};q^{12})_\infty\}^2 \{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}$$
$$\times [R(3,3,1,1,1,2)]\, [R(10,10,1,1,1,2)]^2$$
$$\times \left\{ \frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots \right\}^2$$
$$\times \left\{ 1 + \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots \right\}^2;$$

(24)

$$A(q^2) = q \cdot \widehat{\Theta}_8^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(2)};9)} q^{L_0}$$
$$- q \cdot \widehat{\eta}_{8,2}(\tau) - q(-q^2;q^2)_\infty (-q^4;q^4)_\infty\, R(4,4,1,1,1,2);$$

(25)

$$\mu(q^4) = -\ 2q \cdot \widehat{\Theta}_4^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(0)};\ 5)} q^{L_0} + 2q\,\widehat{\eta}_{4,0}(\tau) + \frac{12\,(q^8;q^8)_\infty R(1,1,1,1,1,2)}{(q;q)_\infty (q^2;q^4)_\infty};$$

(26)

$$\phi(q^4) = -\ 2q \cdot \widehat{\Theta}_{12}^{-1} \cdot \mathrm{r}_{L(\bigwedge_{(4)};\ 13)} q^{L_0} + 2q\,\widehat{\eta}_{12,4}(\tau)$$
$$+ \frac{(q^2,q^4,q^6;q^8)_\infty (q^{12};q^{24})_\infty (q^3;q^6)_\infty^2}{(q;q)_\infty} R(1,1,1,1,1,2)\, R(6,6,1,1,1,2);$$

(27)

$$X(-q^2) = -2q \cdot \widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(18)};\ 41)} q^{L_0} - \mathrm{tr}_{L(\bigwedge_{(2)};\ 41)} q^{L_0}\big) + 2q\,\widehat{\eta}_{40,18}(\tau)$$
$$-\ 2q\,\widehat{\eta}_{40,2}(\tau) + \frac{(j(-q^2,q^{20}) j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3)}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^8,q^{40})} R(2,2,1,1,1,2);$$

(28)

$$\chi(-q^2) = -2q^3 \cdot \widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(14)};\ 41)}q^{L_0} + q^2 \cdot \mathrm{tr}_{L(\bigwedge_{(6)};\ 41)}q^{L_0}\big) + 2q^3\widehat{\eta}_{40,14}(\tau)$$
$$+ 2q^5\widehat{\eta}_{40,6}(\tau) + q^2\frac{(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40}))}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^{16},q^{40})}\,R(2,2,1,1,1,2).$$

The general form of Kac-Wakimoto character formula (see [8, p. 442]) is given as follows:

$$\text{(29)}\quad \mathrm{tr}_{L(\bigwedge_{(s)};\ r+1)} \cdot q^{L_0} := 2q^{\frac{r-1}{24}-\frac{s}{2}} \cdot \frac{\eta^2(2\tau)}{\eta^{r+3}(\tau)} \cdot L_{r,s}(\tau) \quad (r \in \mathbb{N};\ s \in \mathbb{Z}),$$

where L_0 is the energy operator or Hamiltonian,

$$\text{(30)}\qquad L_{r,s}(\tau) := \sum_{k=(k_1,k_2,\ldots,k_r)\in\mathbb{Z}^r} \frac{q^{\frac{1}{2}}\sum_i k_i(k_i+1)}{1+q^{-s+\sum_i k^i}} \quad (r \in \mathbb{N};\ s \in \mathbb{Z}),$$

and the function $\eta(\tau)$ is the Dedekind η-function, a classical weight 1/2 modular form, defined by

$$\text{(31)}\qquad \eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)\,.$$

It is noted that $L(\bigwedge_{(s)};\ r+1)$ is the irreducible $s\ell(r+1,1)^\wedge$ module with highest weight $\bigwedge_{(s)}$.

The function $j(x;q)$ (see [8, p. 454, Table 8]) is defined by

$$\text{(32)}\qquad j(x;q) := (x;q)_\infty(x^{-1}q;q)_\infty(q;q)_\infty.$$

For the details of the other notations whose definitions are not given here, one may refer to the work [8].

3. Main Results

Here we present certain interesting interrelations among character formulas, combinatorial partition identities and continued partition identities

Theorem 3.1. *Each of the following identities holds true:*

(33)

$$f(-q) = -\ 4q\,\widehat{\Theta}_{12}^{-1}\ \left(q^{\frac{1}{2}}\,tr_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\,tr_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\right) +\ 4q\widehat{\beta}_{12,1}(\tau)$$
$$+\ \frac{(q^{16};q^{32})_\infty(q^4,q^{12};q^{16})_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty}{(q^4,q^{16};q^{20})_\infty(q^{16};q^{16})_\infty(q^{32};q^{32})_\infty}$$
$$\times\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^3$$
$$\times\, R(16,16,1,1,1,2)\ \{R(8,8,1,1,1,2)\}^2\ \{R(1,1,1,1,1,2)\}^3$$
$$\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\};$$

(34)

$$\phi(q) = -\ 2q\,\widehat{\Theta}_{12}^{-1}\left(q^{\frac{1}{2}}\,tr_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\,tr_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\right) +\ 2q\widehat{\beta}_{12,1}(\tau)$$
$$+\ \frac{(q^{16};q^{32})_\infty(q^4,q^{12};q^{16})_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty}{(q^4,q^{16};q^{20})_\infty(q^{16};q^{16})_\infty(q^{32};q^{32})_\infty}$$
$$\times\left\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\right\}^3$$
$$\times\, R(16,16,1,1,1,2)\,\{R(8,8,1,1,1,2)\}^2\,\{R(1,1,1,1,1,2)\}^3$$
$$\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\};$$

(35)

$$\chi(-q) = -\ q\,\widehat{\Theta}_{12}^{-1}\left(q^{\frac{1}{2}}\,tr_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\,tr_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\right) + q\widehat{\beta}_{12,1}(\tau)$$
$$+\ \frac{(q^4;q^4)_\infty(q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(q^{12};q^{12})_\infty(q^{24};q^{24})_\infty\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}$$
$$\times\, R(12,12,1,1,1,2)\,R(3,3,1,1,1,2)\,\{R(10,10,1,1,1,2)\}^2$$
$$\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2$$
$$\times\left\{1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2;$$

(36)
$$\begin{aligned}A(q^2) =& q\,\widehat{\Theta}_8^{-1}\,tr_{L(\bigwedge_{(2)};9)}q^{L_0} - q\,\widehat{\eta}_{8,2}(\tau)\\ &- q(-q^4;q^4)_\infty\,R(2,2,1,1,2,2)\,R(4,4,1,1,1,2);\end{aligned}$$

(37)
$$\mu(q^4) = -\ 2q\,\widehat{\Theta}_4^{-1} tr_{L(\bigwedge_{(0)};\ 5)} q^{L_0} + 2q\,\widehat{\eta}_{4,0}(\tau) + \frac{12\,(q^8;q^{16})_\infty R(8,8,1,1,1,2)\,R(1,1,1,1,1,2)}{(q;q^2)_\infty\ \{(q^2;q^4)_\infty\}^2\,(q^4;q^8)_\infty (q^{16};q^{16})_\infty};$$

(38)
$$\phi(q^4) = -\ 2q\,\widehat{\Theta}_{12}^{-1} tr_{L(\bigwedge_{(4)};\ 13)} q^{L_0} + 2q\,\widehat{\eta}_{12,4}(\tau) + \frac{(q^2,q^6;q^8)_\infty (q^{12};q^{24})_\infty (q^3;q^6)_\infty^2}{(q;q^2)_\infty (q^2;q^4)_\infty (q^8;q^8)_\infty} R(1,1,1,1,1,2)\,R(6,6,1,1,1,2);$$

(39)
$$X(-q^2) = 2q\,\widehat{\Theta}_{40}^{-1}\left(tr_{L(\bigwedge_{(2)};\ 41)} q^{L_0} - tr_{L(\bigwedge_{(18)};\ 41)} q^{L_0}\right) + 2q\left(\widehat{\eta}_{40,18}(\tau) - \widehat{\eta}_{40,2}(\tau)\right) + \left(\frac{j(-q^2,q^{20})j(q^{12},q^{40})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^8,q^{40})} + \frac{2q}{j(q^8,q^{40})} R(20,20,1,1,1,2)\right) \times R(2,2,1,1,1,2);$$

(40)
$$\chi(-q^2) = -\,2q^3\,\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(14)};\ 41)} q^{L_0} - 2q^5\,\widehat{\Theta}_{40}^{-1} tr_{L(\bigwedge_{(6)};\ 41)} q^{L_0} + 2q^3\widehat{\eta}_{40,14}(\tau) + 2q^5\widehat{\eta}_{40,6}(\tau) + q^2\left(\frac{2q\,R(20,20,1,1,1,2)}{j(q^{16},q^{40})} - \frac{j(-q^6,q^{20})^2 j(q^4,q^{40})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^{16},q^{40})}\right) \times R(2,2,1,1,1,2).$$

Proof. Using $(q^2;q^2)_\infty = (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16};q^{16})_\infty$ in (21) and (22). Then applying (20) with $m = 16$ is seen to yield (33) and (34).

Using $(q^6;q^6)_\infty = (q^6;q^{12})_\infty (q^{12};q^{24})_\infty (q^{24};q^{24})_\infty$ and $(q^{12};q^{12})_\infty = (q^{12};q^{24})_\infty$
$(q^{24};q^{24})_\infty$ in (23), and applying (20) with $m = 12$, we obtain the identity (35).

Using the combinatorial partition identity (19) in (24), we get (36).

Using the following identities:

$$\left(q^8;q^8\right)_\infty = \left(q^8;q^{16}\right)_\infty \left(q^{16};q^{16}\right)_\infty$$

and

$$(q;q)_\infty = \left(q;q^2\right)_\infty \left(q^2;q^4\right)_\infty \left(q^4;q^8\right)_\infty \left(q^8;q^{16}\right)_\infty \left(q^{16};q^{16}\right)_\infty$$

in (25) and making a suitable arrangement to apply (20) with $m = 8$, we obtain (37).

A similar argument as in getting (33)-(37) will establish the remaining three identities (38)-(40). So the detailed accounts of their proofs are omitted.

□

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