

**A DERIVATION OF TWO QUADRATIC  
TRANSFORMATIONS CONTIGUOUS TO THAT OF  
KUMMER VIA A DIFFERENTIAL EQUATION  
APPROACH**

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**Abstract.** The purpose of this note is to provide an alternative proof of two quadratic transformations contiguous to that of Kummer using a differential equation approach.

**1. Introduction**

Throughout this paper, let  $\mathbb{C}$ ,  $\mathbb{Z}$ , and  $\mathbb{N}$  be the sets of complex numbers, integers, and positive integers, respectively,  $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . The quadratic transformation for the Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$  which will be considered here and was originally obtained by Kummer [4] is given as follows:

$$(1) \quad (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b; \end{matrix} \frac{-2x}{1-x} \right] = {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} x^2 \right] \\ \left( |x| < 1, \left| \frac{-2x}{1-x} \right| < 1, 2b \in \mathbb{C} \setminus \mathbb{Z}_0^- \right).$$

Kim [2] re-derived this result (1) by using the following known hypergeometric identities:

$${}_2F_1 \left[ \begin{matrix} -2n, \alpha; \\ 2\alpha; \end{matrix} 2 \right] = \frac{(\frac{1}{2})_n}{(\alpha + \frac{1}{2})_n} \quad (n \in \mathbb{N}_0) \\ {}_2F_1 \left[ \begin{matrix} -2n - 1, \alpha; \\ 2\alpha; \end{matrix} 2 \right] = 0 \quad (n \in \mathbb{N}_0).$$

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Luke [4] established the transformation (1) by using the differential equation satisfied by  ${}_2F_1$ .

Kim *et al.* [3] presented two transformation formulas contiguous to (1) which are recalled in the following theorem.

**Theorem 1.1.** *The following transformation formulas hold true:*

$$(2) \quad \begin{aligned} & (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b+1; \end{matrix} \frac{-2x}{1-x} \right] \\ &= {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} x^2 \right] + \frac{ax}{2b+1} {}_2F_1 \left[ \begin{matrix} \frac{a}{2} + \frac{1}{2}, \frac{a}{2} + 1; \\ b + \frac{3}{2}; \end{matrix} x^2 \right] \end{aligned}$$

and

$$(3) \quad \begin{aligned} & (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b-1; \end{matrix} \frac{-2x}{1-x} \right] \\ &= {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \\ b - \frac{1}{2}; \end{matrix} x^2 \right] - \frac{ax}{2b-1} {}_2F_1 \left[ \begin{matrix} \frac{a}{2} + \frac{1}{2}, \frac{a}{2} + 1; \\ b + \frac{1}{2}; \end{matrix} x^2 \right] \end{aligned}$$

$$\left( |x| < 1, \left| \frac{-2x}{1-x} \right| < 1, 2b \pm 1 \in \mathbb{C} \setminus \mathbb{Z}_0^- \right).$$

Here we give an alternative proof of the quadratic transformations (2) and (3) by adopting the differential equation approach used by Luke [4]. It is worth remarking that these transformations cannot be completely derived by the hypergeometric differential equation, but that a related second-order differential equation has to be solved by the standard Frobenius method.

Before we give our alternative derivation of (2) and (3) in Section 3, we first give an outline of the arguments used by Luke [4] to establish the Kummer transformation (1).

## 2. Derivation of (1) by Luke

It is well known that the hypergeometric function  ${}_2F_1(a, b; c; z)$  satisfies the following differential equation (see, *e.g.*, [1, p. 75]; see also [5, Entry (15.10.1)]):

$$(4) \quad z(1-z) \frac{d^2w}{dz^2} + [c - (a+b+1)z] \frac{dw}{dz} - abw = 0.$$

If we put  $c = 2b$  and make the change of variable  $z = -2x/(1-x)$ , then Eq. (4) becomes

$$x(1+x)(1-x)^2 \frac{d^2w}{dx^2} + 2 \frac{dw}{dx} [x^3 - x^2(a+1) + x(a-b) + b] + 2abw = 0.$$

If we set  $w = (1-x)^a y$ , after a little simplification, we find

$$(5) \quad x(1-x^2) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} [x^2(a+1) - b] - a(a+1)xy = 0,$$

one of whose solutions is

$$(6) \quad y = (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b; \end{matrix} \frac{-2x}{1-x} \right].$$

The differential equation (5) is invariant under change of variable  $x$  to  $-x$ . Hence if we introduce the new independent variable  $\nu = x^2$  the equation describing  $y$  becomes

$$(7) \quad \nu(1-\nu) \frac{d^2y}{d\nu^2} + \left[ b + \frac{1}{2} - \nu \left( a + \frac{3}{2} \right) \right] \frac{dy}{d\nu} - \left( \frac{a^2}{4} + \frac{a}{4} \right) y = 0.$$

We observe that (5) is of the form as the hypergeometric differential equation (4) which therefore has in  $|\nu| < 1$  the general solution

$$(8) \quad y = A {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} \nu \right] + B \nu^{\frac{1}{2}-b} {}_2F_1 \left[ \begin{matrix} \frac{a}{2} - b + \frac{1}{2}, \frac{a}{2} - b + 1; \\ \frac{3}{2} - b; \end{matrix} \nu \right].$$

We observe that the differential equation (5) has the solution (6) valid in  $|-2x/1-x| < 1$  and if  $2b$  is neither zero nor a negative integer. At the same time, the equation (5) has the solution (8) with  $\nu = x^2$  valid in  $|x| < 1$ . Therefore subject to these conditions there exist constants  $A$  and  $B$  such that

$$(9) \quad (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b; \end{matrix} \frac{-2x}{1-x} \right] \\ = A {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \\ b + \frac{1}{2}; \end{matrix} x^2 \right] + B x^{1-2b} {}_2F_1 \left[ \begin{matrix} \frac{a}{2} - b + \frac{1}{2}, \frac{a}{2} - b + 1; \\ \frac{3}{2} - b; \end{matrix} x^2 \right].$$

The left-hand side and the first member on the right-hand side of (9) are both analytic at  $x = 0$ , but the remaining term is not, due to the presence of the factor  $x^{1-2b}$ . Hence  $B=0$  and by considering the terms at  $x = 0$  it is easily seen that  $A=1$ . This leads to the required quadratic transformation given in (1).

### 3. An alternative derivation of Theorem 1.1

In order to establish the result (2), setting  $c = 2b + 1$  in (4) and changing the variable  $z = -2x/(1 - x)$ , we obtain

$$(10) \quad x(1+x)(1-x)^2 \frac{d^2 w}{dx^2} + \frac{dw}{dx} (1-x)[x(2a-1) - 2x^2 + (2b+1)] + 2abw = 0,$$

which has a solution  $w = {}_2F_1(a, b; 2b+1; -2x/(1-x))$ . With a further change of the variable  $w = (1-x)^a y$ , after some simplification, we find

$$(11) \quad x(1-x^2) \frac{d^2 y}{dx^2} - \frac{dy}{dx} [x^2(2a+2) + x - (2b+1)] - ay[x(a+1) + 1] = 0.$$

A solution of (11) is consequently

$$y = (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b; \\ 2b+1; \end{matrix} \frac{-2x}{1-x} \right].$$

The differential equation (11) is not invariant under the change of variable  $x$  to  $-x$  and so we cannot reduce it to the hypergeometric equation (4). Inspection of (11) shows that the point  $x = 0$  is a regular singular point. Accordingly, we seek two linearly independent solutions of (11) by the Frobenius method. So let

$$(12) \quad y = x^\lambda \sum_{n=0}^{\infty} c_n x^n \quad (c_0 \neq 0),$$

where  $\lambda$  is the indicial exponent. Substitution of this form for  $y$  in (11) then, after a little simplification, leads to

$$(13) \quad \begin{aligned} & \sum_{n=0}^{\infty} c_n x^{n-1} (n+\lambda)(n+\lambda+2b) \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} [(n+\lambda)(n+\lambda-1) + (n+\lambda)(2a+2) + a^2 + a] \\ & \quad + \sum_{n=0}^{\infty} c_n x^n (n+\lambda+a). \end{aligned}$$

The coefficient of  $x^{-1}$  in (13) must vanish to yield the indicial equation:

$$\lambda(\lambda + 2b) = 0.$$

So we have  $\lambda = 0$  and  $\lambda = -2b$ . Equating the coefficient of  $x^n$  ( $n \in \mathbb{N}_0$ ) in (13), we obtain

$$c_1 = \frac{(\lambda + a)}{(\lambda + 1)(2b + \lambda + 1)} c_0$$

and

$$(14) \quad c_n = \frac{[(n-2+\lambda)(n+\lambda+2a-1) + a(a+1)]c_{n-2} + (n+\lambda+a-1)c_{n-1}}{(n+\lambda)(n+\lambda+2b)}$$

for  $n \in \mathbb{N} \setminus \{1\}$ . With the choice  $\lambda = 0$ , we have

$$c_1 = \frac{a}{2b+1} c_0$$

and

$$c_n = \frac{[(n-2)(n+2a-1) + a(a+1)]c_{n-2} + (n+a-1)c_{n-1}}{n(n+2b)} \quad (n \in \mathbb{N} \setminus \{1\}).$$

Solution of this three-term recurrence with the help of Mathematica generates the values given by

$$c_{2n} = \frac{\left(\frac{a}{2}\right)_n \left(\frac{a}{2} + \frac{1}{2}\right)_n}{n! \left(b + \frac{1}{2}\right)_n} c_0$$

and

$$c_{2n+1} = \frac{\left(\frac{a}{2} + \frac{1}{2}\right)_n \left(\frac{a}{2} + 1\right)_n}{n! \left(b + \frac{3}{2}\right)_n} c_1,$$

the general values of which can be established by induction. Substitution in (12) is easily seen to give one solution of (10) as follows:

$$y_1 = {}_2F_1 \left[ \begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2} \\ b + \frac{1}{2} \end{matrix}; x^2 \right] + \frac{ax}{2b+1} {}_2F_1 \left[ \begin{matrix} \frac{a}{2} + \frac{1}{2}, \frac{a}{2} + 1 \\ b + \frac{3}{2} \end{matrix}; x^2 \right]$$

when  $|x| < 1$ .

A second solution is obtained by taking  $\lambda = -2b$  in (14) to yield

$$c_1 = c_0 \frac{a-2b}{1-2b}$$

and

$$c_n = \frac{[(n-2-2b)(n-2b+2a-1) + a(a+1)]c_{n-2} + (n-2b+a-1)c_{n-1}}{(n-2b)n}$$

for  $n \in \mathbb{N} \setminus \{1\}$ . This generates the values

$$c_{2n} = \frac{(a-2b)_n (a-2b+1)_n}{n! \left(\frac{1}{2} - b\right)_n} c_0$$

and

$$c_{2n+1} = \frac{(a-2b+1)_n(a-2b+2)_n}{\left(\frac{3}{2}-b\right)_n n!} c_1.$$

A second solution of (11) is therefore given by

$$y_2 = c_0 x^{-2b} \left( {}_2F_1 \left[ \begin{matrix} a-2b, a-2b+1 \\ \frac{1}{2}-b \end{matrix}; x^2 \right] + \frac{(a-2b)x}{1-2b} {}_2F_1 \left[ \begin{matrix} a-2b+1, a-2b+2 \\ \frac{3}{2}-b \end{matrix}; x^2 \right] \right)$$

when  $|x| < 1$ .

It then follows, when  $|x| < 1$  and  $\left| \frac{-2x}{1-x} \right| < 1$  and provided  $2b+1$  is neither zero nor a negative integer, that there exist constants A and B such that

$$(15) \quad (1-x)^{-a} {}_2F_1 \left[ \begin{matrix} a, b \\ 2b+1 \end{matrix}; \frac{-2x}{1-x} \right] = Ay_1 + By_2.$$

Here we observe that the left-hand side of (15) and the solution  $y_1$  are both analytic at  $x=0$ , whereas the solution  $y_2$  is not analytic at  $x=0$  due to the presence of the factor  $x^{-2b}$ . Hence  $B=0$  and, by putting  $x=0$  in (15), it is easy to find  $A=1$ . This completes the proof of the result (2).

A similar procedure as in the proof of (2) can be employed to establish the quadratic transformation in (3). So its detailed account of proof is omitted to be left to the interested reader.

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