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SUPERCONVERGENCE OF HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR SECOND-ORDER ELLIPTIC EQUATIONS

MINAM MOON[1†] AND YANG HWAN LIM[1]

[1]DEPARTMENT OF MATHEMATICS, KOREA MILITARY ACADEMY, SOUTH KOREA
E-mail address: hereandnow@kma.ac.kr

ABSTRACT. We propose a projection-based analysis of a new hybridizable discontinuous Galerkin method for second order elliptic equations. The method is more advantageous than the standard HDG method in a sense that the new method has higher-order accuracy and lower computational cost, and is more flexible. Notable distinctions of our new method, when compared to the standard HDG emthod, are that our method uses L^2-projection and suitable stabilization parameter depending on a mesh size for superconvergence. We show that the error for the solution of the equation converges with order $p+2$ when we only use polynomials of degree $p+1$ as a finite element space without postprocessing. After establishing the theory, we carry out numerical tests to demonstrate and ensure that the proposed method is effective and accurate in practice.

1. INTRODUCTION

In this paper, we develop and analyze a new hybridizable discontinuous Galerkin (HDG) method for second-order elliptic equations. To present the main idea of the method, we consider the following elliptic equation with the homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\nabla\cdot(\kappa(x)\nabla u) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polyhedron in $\mathbb{R}^k$ with its boundary $\partial\Omega$, $f\in L_2(\Omega)$, $\kappa(x)\in L_2(\Omega)$, and $\kappa(x)\geq\kappa_0>0$.

Finite element method(FEM) can be used as a efficient numerical technique for obtaining physically relevant solution of the problem. Recently, we are interested in finite element method with local mass conservation. The discontinuous Galerkin (DG) method has been a popular choice for conservative schemes [1, 2, 3]. Above this, the DG method has several advantages over the continuous Galerkin (CG) method, see [4]. The DG method can be implemented on general meshes and polynomials of arbitrary degree. The method also easily handles

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† Corresponding author.

adaptivity (both mesh size h and degree of polynomials p) in [5, 6] and leads to efficient parallelization in [7]. The DG method can be used to produce highly accurate discretization for convection-diffusion equations in [8, 9] and can be applied for problems with unambiguous boundary conditions. However, despite the listed advantages, the DG method has some practical shortcoming. The main issue is that the DG method gives larger globally coupled degrees of freedom for the same mesh, since the boundary of element does not share the degree of freedom. Thus, the DG method is more computationally costly compared to the continuous Galerkin method and/or finite difference schemes.

The standard hybridizable discontinuous Galerkin (HDG) method was recently introduced and developed to overcome this issue. The HDG method results in an algebraic system that involves only the degrees of freedom associated with the numerical traces of the field variables. Since the numerical traces are only defined on the inter-element boundaries, degrees of freedom are substantially reduced. As a result, the HDG method can significantly save computational cost.

The standard HDG method was first introduced for second order elliptic problems in [10, 11]. The error estimates based on a spacial projection were developed for elliptic problems in [12]. Optimal convergence order for HDG methods were established in the L^2-norm of $p+1$ if polynomials of degree p are used and the exact solution is smooth enough [13]. The choices of stabilization parameter were numerically presented and analyzed in the sense of the optimal convergence order of numerical solutions [14, 15]. Based on the optimal convergence and superconvergence of HDG methods, local postprocessing was developed to get $p+2$ convergence order of numerical solutions [13].

On the other hand, our HDG method improves the standard HDG method. First, our method obtains $(p+2)$th convergence order by changing the value of the stabilization parameter τ with polynomials of degree $p+1$ as a finite element space, without using the local postprocessing. For a given triangulation $\mathcal{T}_h$ with maximum diameter $h > 0$, we can define the stability parameter $\tau = \mathcal{O}(h^{-1})$ and derive the convergence order of $p+2$ of the solution of the elliptic problem with polynomials of degree $p+1$. Globally coupled degrees of freedom is higher than those derived by the standard HDG method because the method requires polynomials with degree one higher than those needed for the standard HDG method. However, we can save time for computation for the local processing. Secondly, we can easily derive error estimates for the problem with local L^2-projection. In the standard HDG method, we use a local HDG-projection with some assumptions to derive the error estimates. Our approach is more general and easier to understand. Furthermore, by using the method we can derive the errors of solution. Based on these projection, we derive the error estimates for the solution (pressure) u and velocity $\boldsymbol{q}$.

The elliptic problem with constant coefficient κ are numerically tested on the proposed HDG method. With this, we can observe the robustness of the convergence based on our new HDG method. We also confirm that the convergence order of our HDG method equals to the order derived from theoretical analysis, i.e., the convergence order of the pressure u is $p+2$ when we use polynomials of degree $p+1$ and the convergence order of velocity $\boldsymbol{q}$ is $p+1$ when we use polynomials of degree p.

The paper is organized in the following way. In Section 2, we introduce some notations, a new hybridizable discontinuous Galerkin method. We will also derive the solvability of the method. In Section 3, we drive the error estimates for our method by using L^2 projection, not HDG-projection. In Section 4, we present a numerical experiment and confirm our theoretical analysis by documenting the error of our method. In the final section, we discuss the analysis and the numerical result. The theoretical results are derived under the condition that the order of convergence depends on the stabilization parameter τ. Based on the analysis and numerical results we can conclude that our method gives practically valid results in using the stabilization parameter depending on mesh size.

2. Notation, HDG Method, and Solvability

We begin this section with presenting basic notations and hypotheses of meshes. We then introduce our new HDG method for the problem (1.1). Finally, we derive the solvability of the proposed method.

2.1. **Notation.** Let $\mathcal{T}_h$ be a conforming, shape-regular simplicial triangulation of Ω. For any element $T \in \mathcal{T}_h$, ∂T is defined to be the set of the edges of T when $\dim(T) = 2$, and the set of the faces of T when $\dim(T) = 3$, and denoted by F. Let $\partial\mathcal{T}_h = \cup_{T\in\mathcal{T}_h}\partial T$. Let $\mathcal{E}_h$ denote the set of all edges/faces of the triangulation $\mathcal{T}_h$, and $\mathcal{E}_h^0$ be the set of all interior faces of the triangulation. For any element $T \in \mathcal{T}_h$, let h_T be the diameter of element T, and let $h = \max_{T\in\mathcal{T}_h} h_T$.

Throughout the paper, we will use the standard notations for Sobolev spaces and their norms on the domain Ω and their boundaries. For example, $||v||_{s,\Omega}$, $|v|_{s,\Omega}$, $||v||_{s,\partial\Omega}$, $|v|_{s,\partial\Omega}$, $s > 0$, denote the Sobolev norms and semi-norms on Ω and its boundary $\partial\Omega$. For an integer s, the Sobolev spaces are Hilbert spaces and the norms are defined by the L^2-norms of their weak derivatives up to order s. For a non-integer s, the spaces are defined by interpolation [16]. When $s = 0$ we will use $||v||_\Omega$ instead of $||v||_{0,\Omega}$.

2.2. **HDG method.** To apply the new Hybridizable discontinuous Galerkin method, we consider the model problem (1.1) with homogeneous boundary condition in a mixed form:

$$\begin{aligned} \alpha\boldsymbol{q} + \nabla u &= 0 \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{q} &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\boldsymbol{q} = -\kappa(x)\nabla u$, and $\alpha(x) = \kappa(x)^{-1}$.

For any element $T \in \mathcal{T}_h$ and any face $F \in \mathcal{E}_h$, we define

$$\boldsymbol{V}(T) := (\mathcal{P}^p(T))^k,\ W(T) := \mathcal{P}^{p+1}(T),\ M(F) := \mathcal{P}^p(F),$$

where $\mathcal{P}^p(D)$ denotes the set of polynomials of degree at most p on a D. Now, we consider the following finite element spaces:

$$\begin{aligned} \boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_T \in \boldsymbol{V}(T) \text{ for all } T \in \mathcal{T}_h\}, \\ W_h &:= \{w \in L^2(\Omega) : w|_T \in W(T) \text{ for all } T \in \mathcal{T}_h\}, \\ M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in M(F) \text{ for all } F \in \mathcal{E}_h\}, \end{aligned}$$

where $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^k$ and $L^2(\mathcal{E}_h) := \Pi_{F\in\mathcal{E}_h} L^2(F)$.

Remark 2.1. *Note that unlike the standard HDG method, we define the finite element space $W(T)$ with polynomials of degree at most $p+1$. As we will soon see, this will yield higher accuracy ($p+2$ instead of $p+1$) without any postprocessing.*

For vector-valued functions $\boldsymbol{u}, \boldsymbol{v} \in (L^2(D))^k$, we define $(\boldsymbol{u}, \boldsymbol{v})_D = \int_D \boldsymbol{u}\cdot\boldsymbol{v}$. For scalar-valued functions $u, v \in L^2(D)$, let $(u,v)_D = \int_D uv$, if the domain D is a subset of $\mathbb{R}^k$. If ∂D is in $\mathbb{R}^{k-1}$, we define $\langle u, v\rangle_{\partial D} = \int_{\partial D} uvds$. Then, we introduce the following notation:

$$(w,v)_{\mathcal{T}_h} = \textstyle\sum_{T\in\partial\mathcal{T}_h} (w,v)_T, \quad \langle w, v\rangle_{\partial\mathcal{T}_h} = \textstyle\sum_{\partial T\in\partial\mathcal{T}_h} \langle w, v\rangle_{\partial T}. \tag{2.2}$$

With these finite element spaces and notations, we can get the following HDG formulation: Find $(u_h, \boldsymbol{q}_h, \hat{u}_h) \in W_h \times \boldsymbol{V}_h \times M_h$ such that

$$\begin{array}{rcll} (\alpha\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &=& 0 & \forall \boldsymbol{v}\in\boldsymbol{V}_h, \\ -(\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} &=& (f,w)_{\mathcal{T}_h} & \forall w\in W_h, \\ \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &=& 0 & \forall \mu\in M_h, \\ \hat{u}_h &=& 0 & \text{on } \partial\Omega. \end{array} \tag{2.3}$$

We define the normal component of the numerical trace as follows :

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(P_\partial u_h - \widehat{u}_h) \tag{2.4}$$

where P_∂ is the L^2-projection operator on the space M_h.

Remark 2.2. *In our method, to define the nomal component of the numerical trace* (2.4)*, we consider the L^2-projection operator on the space M_h because of $W(T)|_F \neq M(F)$.*

2.3. **Matrix formulation.** For implementation, we insert the normal component of the numerical trace in the third equation of (2.3), after some manipulations, obtain that $(u_h, \boldsymbol{q}_h, \hat{u}_h) \in W_h \times \boldsymbol{V}_h \times M_h$ is the solution of the following weak formulation.

$$\begin{array}{rcl} a(\boldsymbol{q}_h, \boldsymbol{v}) - b(u_h, \boldsymbol{v}) + c(\widehat{u}_h, \boldsymbol{v}) &=& 0, \\ -b(w, \boldsymbol{q}_h) - d(P_\partial u_h, w) + e(\widehat{u}_h, w) &=& -f(w), \\ c(\mu, \boldsymbol{q}_h) + e(\mu, P_\partial u_h) - g(\mu, \widehat{u}_h) &=& 0, \end{array} \tag{2.5}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$. Here, the bilinear forms and the linear functional are defined by

$$\begin{gathered} a(\boldsymbol{q}, \boldsymbol{v}) = (\alpha\boldsymbol{q}\,,\, \boldsymbol{v})_{\mathcal{T}_h}, \quad b(u, \boldsymbol{v}) = (u\,,\, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h}, \\ c(\widehat{u}, \boldsymbol{v}) = \langle \widehat{u}\,,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \quad d(u,w) = \langle w\,,\, \tau u\rangle_{\partial\mathcal{T}_h}, \\ e(\mu, u) = \langle \mu\,,\, \tau u\rangle_{\partial\mathcal{T}_h}, \quad g(\mu, \widehat{u}) = \langle \mu\,,\, \tau\widehat{u}\rangle_{\partial\mathcal{T}_h}, \quad f(w) = (f\,,\, w)_{\mathcal{T}_h}, \end{gathered} \tag{2.6}$$

for all $(u, \boldsymbol{q}, \widehat{u})$ and $(w, \boldsymbol{v}, \mu)$ in $W_h \times \boldsymbol{V}_h \times M_h$.

The discretization of the system of equations (2.5) gives rise to the following matrix equation :

$$\begin{bmatrix} A & -B^T & C^T \\ -B & -D & E \\ C & E^T & G \end{bmatrix} \begin{bmatrix} Q \\ U \\ \widehat{U} \end{bmatrix} = - \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix} \tag{2.7}$$

Here Q, U and $\widehat{U}$ are vectors of degrees of freedom for $\boldsymbol{q}_h, u_h$, and $\widehat{u}_h$, respectively. The matrices in (2.7) correspond to the bilinear forms in (2.6) in the order they appear in the system of equation (2.5).

Since the HDG method produces a final system in terms of globally coupled degrees of freedom of the numerical trace $\widehat{u}_h$ (or $\widehat{U}$) only, the first and second equation of (2.3) can be used to remove both $\boldsymbol{q}_h$ and u_h in an element by element sense. Then, we obtain a reduced globally coupled matrix equation just for $\widehat{U}$:

$$\mathbb{K}\widehat{U} = \mathbb{F} \tag{2.8}$$

where

$$\mathbb{K} = - \begin{bmatrix} C & E^T \end{bmatrix} \begin{bmatrix} A & -B^T \\ -B & -D \end{bmatrix}^{-1} \begin{bmatrix} C^T \\ E \end{bmatrix} + G,$$

and

$$\mathbb{F} = \begin{bmatrix} C & E^T \end{bmatrix} \begin{bmatrix} A & -B^T \\ -B & -D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F \end{bmatrix}$$

By solving the matrix equation (2.8), we get the value of $\widehat{U}$. Plugging $\widehat{U}$ back to the equation (2.7), we obtain Q and U as below :

$$\begin{bmatrix} Q \\ U \end{bmatrix} = \begin{bmatrix} A & -B^T \\ -B & -D \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ -F \end{bmatrix} - \begin{bmatrix} C^T \\ E \end{bmatrix} \widehat{U} \right).$$

2.4. **Solvability of the HDG method.** Next, we discuss the stability and solvability of the HDG method. Under some conditions for the stabilization parameter τ, we derive the solvability of the method.

Theorem 2.3. *If $\tau > 0$ on ∂T for all T, then for any f, the system* (2.3) *has a unique solution.*

Proof. Note that the system (2.3) is the square system by (2.7). It is enough to show that the homogeneous system(i.e., $f = 0$) only has a trivial solution. Take $(w, \boldsymbol{v}, \mu) = (u_h, \boldsymbol{q}_h, \widehat{u}_h)$ and by adding all equations, we get after some algebraic manipulation, we get

$$(\alpha \boldsymbol{q}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} - \langle \boldsymbol{q}_h \cdot \boldsymbol{n} - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} = 0.$$

By the definition of the numerical traces (2.4) and the property of local projection operator P_∂, we have

$$(\alpha \boldsymbol{q}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} + \langle \tau (P_\partial u_h - \widehat{u}_h), P_\partial u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} = 0$$

and since $\tau > 0$ we get

$$\boldsymbol{q}_h = 0, \quad P_\partial u_h - \widehat{u}_h = 0. \tag{2.9}$$

Also, the first equation of (2.3) becomes

$$-(u_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}_h.$$

Now we take this over an element T and by integrating by parts and using the property of local projection operator P_∂, we get

$$(\nabla u_h, \boldsymbol{v})_T + \langle \widehat{u}_h - P_\partial u_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T} = 0, \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(T).$$

Since $P_\partial u_h - \widehat{u}_h = 0$, this becomes

$$(\nabla u_h, \boldsymbol{v})_T = 0 \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(T).$$

By taking $\boldsymbol{v} = \nabla u_h$, we conclude that u_h is piecewise constant on each T. Since any interior edge F is shared by two neighboring elements T^+ and T^-, $u_h = C$. Since $\widehat{u}_h = 0$ on $\partial\Omega$ and $P_\partial u_h - \widehat{u}_h = 0$, $u_h, \widehat{u}_h$, and C should all be zero. □

3. Error analysis

On each element T, there exist local L^2-projection operators

$$\Pi_W : H^1(T) \to W(T) \quad \text{and} \quad \boldsymbol{\Pi}_V : H_{div}(T) \to \boldsymbol{V}(T)$$

defined by:

$$\begin{aligned} (u, w)_T &= (\Pi_W u, w)_T \quad \text{for all} \quad w \in W(T), \\ (\boldsymbol{q}, \boldsymbol{v})_T &= (\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{v})_T \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(T). \end{aligned}$$

3.1. **Error equations.** We begin by obtaining the error equations that will be used for the error analysis. The main idea is to work with the following projection errors:

$$\boldsymbol{e}_q := \boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h, \quad e_u := \Pi_W u - u_h, \quad e_{\widehat{u}} := P_\partial u - \widehat{u}_h.$$

Lemma 3.1. *We have*

$$\begin{aligned} (\alpha \boldsymbol{e}_q, \boldsymbol{v})_{\mathcal{T}_h} - (e_u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle e_{\widehat{u}}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ -(\boldsymbol{e}_q, \nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{q} \cdot \boldsymbol{n} - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial \mathcal{T}_h} &= 0, \\ \langle \boldsymbol{q} \cdot \boldsymbol{n} - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h} &= 0, \\ \langle e_{\widehat{u}}, \mu \rangle_{\partial \mathcal{T}_h} &= 0, \end{aligned} \tag{3.1}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$.

Proof. The exact solution $(u, \boldsymbol{q})$ obviously satisfies

$$\begin{aligned} (\alpha \boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h} - (u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle u, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ -(\boldsymbol{q}, \nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{q} \cdot \boldsymbol{n}, w \rangle_{\partial \mathcal{T}_h} &= (f, w), \\ \langle \boldsymbol{q} \cdot \boldsymbol{n}, \mu \rangle_{\partial \mathcal{T}_h} &= 0, \\ \langle u, \mu \rangle_{\partial \mathcal{T}_h} &= 0, \end{aligned}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$.

By the definition of the projections $(\boldsymbol{\Pi}_V, \Pi_W, P_\partial)$, we can get

$$\begin{aligned}(\alpha\boldsymbol{\Pi}_V\boldsymbol{q},\boldsymbol{v})_{\mathcal{T}_h}-(\Pi_W u,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h}+\langle P_\partial u,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\ -(\boldsymbol{\Pi}_V\boldsymbol{q},\nabla w)_{\mathcal{T}_h}+\langle\boldsymbol{q}\cdot\boldsymbol{n},w\rangle_{\partial\mathcal{T}_h} &= (f,w),\\ \langle\boldsymbol{q}\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h} &= 0,\\ \langle P_\partial u,\mu\rangle_{\partial\mathcal{T}_h} &= 0,\end{aligned}$$

for all $(w,\boldsymbol{v},\mu)\in W_h\times\boldsymbol{V}_h\times M_h$. Subtracting these equations defining the weak formulation (2.3) from the above equation, respectively, we obtain the error equations of Lemma 3.1. □

Lemma 3.2. *We have*

$$\boldsymbol{e}_q\cdot\boldsymbol{n}-(\boldsymbol{q}\cdot\boldsymbol{n}-\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n})=\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}-\boldsymbol{q}\cdot\boldsymbol{n}-\tau(P_\partial e_u-e_{\widehat{u}})+\tau P_\partial(\Pi_W u-u). \quad (3.2)$$

Proof. By the normal component of the numerical trace (2.4), we have

$$\begin{aligned}\boldsymbol{e}_q\cdot\boldsymbol{n}-(\boldsymbol{q}\cdot\boldsymbol{n}-\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}) &= \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}-\boldsymbol{q}_h\cdot\boldsymbol{n}-\boldsymbol{q}\cdot\boldsymbol{n}+\boldsymbol{q}_h\cdot\boldsymbol{n}+\tau(P_\partial u_h-\widehat{u}_h)\\ &= \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}-\boldsymbol{q}\cdot\boldsymbol{n}+\tau(P_\partial u_h-\widehat{u}_h).\end{aligned}$$

Also, we get

$$\begin{aligned}P_\partial u_h-\widehat{u}_h &= P_\partial u_h+P_\partial u-P_\partial u+P_\partial\Pi_W u-P_\partial\Pi_W u-\widehat{u}_h\\ &= (P_\partial u-\widehat{u}_h)-P_\partial(\Pi_W u-u_h)+P_\partial(\Pi_W u-u)\\ &= e_{\widehat{u}}-P_\partial e_u+P_\partial(\Pi_W u-u).\end{aligned}$$

Therefore, we get the equation (3.2). □

3.2. **Estimate for $\boldsymbol{q}-\boldsymbol{q}_h$.** We define the weighted L^2-norm as following:

$$|||\boldsymbol{v}|||^2_{\alpha,\Omega}:=(\alpha\boldsymbol{v},\boldsymbol{v})_{\mathcal{T}_h}\quad\text{and}\quad |||\,w|||^2_{\tau,\partial\mathcal{T}_h}:=\langle\tau w,w\rangle_{\partial\mathcal{T}_h}\,. \quad (3.3)$$

First we note the following standard estimates for the error on any triangle T and edgy/face $F\subset\partial T$.

Proposition 3.3. *If u and $\boldsymbol{q}$ are smooth functions, then we have*

$$\begin{aligned}\|u-\Pi_W u\|_T &\le Ch^{k+2}|u|_{k+2,T},\\ \|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_T &\le Ch^{k+1}|\boldsymbol{q}|_{k+1,T},\\ \|\boldsymbol{q}-P_\partial\boldsymbol{q}\|_F &\le Ch^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,T}.\end{aligned} \quad (3.4)$$

Proof. See [17]. □

Lemma 3.4. *We have*

$$|||\boldsymbol{e}_q\,|||^2_{\alpha,\Omega}+|||\,P_\partial e_u-e_{\widehat{u}}|||^2_{\tau,\partial\mathcal{T}_h}\le\mathbb{S}_1+\mathbb{S}_2+\mathbb{S}_3, \quad (3.5)$$

where

$$\begin{aligned}\mathbb{S}_1 &:= C\tau^{\frac{1}{2}}h^{-\frac{1}{2}}\|u-\Pi_W u\|_\Omega\cdot|||P_\partial e_u-e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ \mathbb{S}_2 &:= C\tau^{-\frac{1}{2}}h^{-\frac{1}{2}}\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_\Omega\cdot|||P_\partial e_u-e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ \mathbb{S}_3 &:= Ch^{\frac{1}{2}}\|\boldsymbol{q}-P_\partial\boldsymbol{q}\|_{\partial\mathcal{T}_h}\cdot\|\nabla e_u\|_\Omega.\end{aligned}$$

Proof. Take $\boldsymbol{v} = \boldsymbol{e}_q, w = e_u$, and $\mu = e_{\widehat{u}}$ in error equations (3.1) and by Lemma 3.2, we have

$$\begin{aligned} |||\boldsymbol{e}_q|||^2_{\alpha,\Omega} + |||P_\partial e_u - e_{\widehat{u}}|||^2_{\tau,\partial\mathcal{T}_h} = & \ \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \\ & + \langle \tau(P_\partial(u - \Pi_W u)), e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Observe that

$$\begin{aligned} \langle \tau(P_\partial(u-\Pi_W u)), e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} &= \langle \tau(P_\partial(u-\Pi_W u)), P_\partial e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \\ &= \left\langle \tau^{\frac12}(u-\Pi_W u), \tau^{\frac12}(P_\partial e_u - e_{\widehat{u}})\right\rangle_{\partial\mathcal{T}_h} \\ &\le C\tau^{\frac12}h^{-\frac12}\|u-\Pi_W u\|_\Omega \cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}. \end{aligned}$$

Also, we have

$$\begin{aligned} \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} = & \ \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, P_\partial e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \\ & + \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, e_u - P_\partial e_u\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Note that

$$\langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, P_\partial e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \le C\tau^{-\frac12}h^{-\frac12}\|\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q}\|_\Omega \cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}$$

and

$$\begin{aligned} \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}, e_u - P_\partial e_u\rangle_{\partial\mathcal{T}_h} &= \langle \boldsymbol{q}\cdot\boldsymbol{n} - P_\partial(\boldsymbol{q}\cdot\boldsymbol{n}), e_u - P_\partial e_u\rangle_{\partial\mathcal{T}_h} \\ &\le Ch^{-\frac12}\|\boldsymbol{q} - P_\partial\boldsymbol{q}\|_{\partial\mathcal{T}_h}\cdot\|e_u - P_\partial e_u\|_\Omega \\ &\le Ch^{\frac12}\|\boldsymbol{q} - P_\partial\boldsymbol{q}\|_{\partial\mathcal{T}_h}\cdot\|\nabla e_u\|_\Omega. \end{aligned}$$

Therefore, we get the inequality (3.5). □

Lemma 3.5. *We have*

$$\|\nabla e_u\|_\Omega \le C\left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + \tau^{-\frac12}h^{-\frac12}\|P_\partial e_u - e_{\widehat{u}}\|_{\tau,\partial\mathcal{T}_h}\right). \tag{3.6}$$

Proof. By the first equation of error equation (3.1) and integration by parts, we have

$$(\alpha\boldsymbol{e}_q, \boldsymbol{v})_{\mathcal{T}_h} + (\nabla e_u, \boldsymbol{v})_{\mathcal{T}_h} + \langle e_{\widehat{u}} - e_u, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0.$$

Take $\boldsymbol{v} = \nabla e_u$. Then, we get

$$\begin{aligned} \|\nabla e_u\|^2_\Omega &= -(\alpha\boldsymbol{e}_q, \nabla e_u)_{\mathcal{T}_h} + \langle e_u - e_{\widehat{u}}, \nabla e_u\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\ &= -(\alpha\boldsymbol{e}_q, \nabla e_u)_{\mathcal{T}_h} + \langle P_\partial e_u - e_{\widehat{u}}, \nabla e_u\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\ &\le C\,|||\boldsymbol{e}_q|||_{\alpha,\Omega}\,\|\nabla e_u\|_\Omega + \tau^{-\frac12}\,|||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\,Ch^{-\frac12}\|\nabla e_u\|_\Omega. \end{aligned}$$

Therefore, we have

$$\|\nabla e_u\|_\Omega \le C\,|||\boldsymbol{e}_q|||_{\alpha,\Omega} + C\tau^{-\frac12}h^{-\frac12}\,|||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}.$$

□

Theorem 3.6. *We have*

$$\begin{aligned} |||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h} \le & \ C\tau^{\frac12}h^{k+\frac32}|u|_{k+2,\Omega} \\ & + C\tau^{-\frac12}h^{k+\frac12}|\boldsymbol{q}|_{k+1,\Omega} + Ch^{k+1}|\boldsymbol{q}|_{k+1,\Omega}. \end{aligned} \tag{3.7}$$

Proof. By Lemmas 3.4, 3.5 and Proposition 3.3, we have

$$\begin{aligned}\mathbb{S}_1 &= C\tau^{\frac{1}{2}}h^{-\frac{1}{2}}\|u-\Pi_W u\|_\Omega \cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ &\le C\tau^{\frac{1}{2}}h^{k+\frac{3}{2}}|u|_{k+2,\Omega}\cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ &\le C\tau^{\frac{1}{2}}h^{k+\frac{3}{2}}|u|_{k+2,\Omega}\cdot \left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\right),\end{aligned}$$

$$\begin{aligned}\mathbb{S}_2 &= C\tau^{-\frac{1}{2}}h^{-\frac{1}{2}}\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_\Omega \cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ &\le C\tau^{-\frac{1}{2}}h^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,\Omega}\cdot |||P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\\ &\le C\tau^{-\frac{1}{2}}h^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,\Omega}\cdot \left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}_3 &= Ch^{\frac{1}{2}}\|\boldsymbol{q}-P_\partial\boldsymbol{q}\|_{\partial\mathcal{T}_h}\cdot\|\nabla e_u\|_\Omega\\ &\le Ch^{k+1}|\boldsymbol{q}|_{k+1,\Omega}\cdot\left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + \tau^{-\frac{1}{2}}h^{-\frac{1}{2}}\, |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\right)\\ &\le Ch^{k+1}|\boldsymbol{q}|_{k+1,\Omega}\cdot\left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\right)\\ &\quad + C\tau^{-\frac{1}{2}}h^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,\Omega}\cdot\left(|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h}\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h} \le\ & C\tau^{\frac{1}{2}}h^{k+\frac{3}{2}}|u|_{k+2,\Omega} + C\tau^{-\frac{1}{2}}h^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,\Omega}\\ &+ Ch^{k+1}|\boldsymbol{q}|_{k+1,\Omega}.\end{aligned}$$

□

Corollary 3.7. *If $\tau = \mathcal{O}\left(h^{-1}\right)$, then we have*

$$|||\boldsymbol{e}_q|||_{\alpha,\Omega} + |||\, P_\partial e_u - e_{\widehat{u}}|||_{\tau,\partial\mathcal{T}_h} \le Ch^{k+1}\left(|u|_{k+2,\Omega} + |\boldsymbol{q}|_{k+1,\Omega}\right). \tag{3.8}$$

Remark 3.8. *It is important to note that the stabilization parameter τ of the above Corollary depends on a mesh size, i.e., $\tau = \mathcal{O}(h^{-1})$. When we choose the value depending on a mesh size $\mathcal{O}(h^{-1})$ as the stabilization parameter τ, we achieve the consistent order of convergence. If this is not the case, the order of convergence is inconsistent since the order of the stabilization parameter τ varies according to right-hand side terms from Theorem 3.6.*

3.3. **Estimate for $u - u_h$.** Next, we derive the result regarding the error $u - u_h$. It is valid under a typical elliptic regularity property. Let $(\boldsymbol{\theta}, \phi)$ be the solution of the following dual problem :

$$\begin{aligned}\alpha\boldsymbol{\theta} + \nabla\phi &= 0, && \text{in } \Omega\\ \nabla\cdot\boldsymbol{\theta} &= e_u, && \text{in } \Omega\\ \phi &= 0 && \text{on } \partial\Omega.\end{aligned} \tag{3.9}$$

We assume that we have full H^2-regularity,

$$\|\phi\|_{2,\Omega} + \|\boldsymbol{\theta}\|_{1,\Omega} \le C\|e_u\|_\Omega, \tag{3.10}$$

where C only depends on the domain Ω.

Lemma 3.9. *Assume that the H^2-regularity* (3.10) *holds. Then, we have*

$$(e_u, e_u)_{\mathcal{T}_h} = \mathbb{T}_1 + \mathbb{T}_2 - \mathbb{T}_3, \tag{3.11}$$

where

$$\begin{aligned}
\mathbb{T}_1 &= -(\alpha \boldsymbol{e}_q, \boldsymbol{\theta} - \boldsymbol{\Pi}_V \boldsymbol{\theta})_{\mathcal{T}_h}, \\
\mathbb{T}_2 &= \langle e_u - e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}, \\
\mathbb{T}_3 &= \langle \boldsymbol{e}_q\cdot\boldsymbol{n} - (\boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}), P_\partial\phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

Proof. By the second equation of dual problem (3.9) and integrating by parts, we get

$$\begin{aligned}
(e_u, e_u)_{\mathcal{T}_h} &= (e_u, \nabla\cdot\boldsymbol{\theta})_{\mathcal{T}_h} \\
&= (e_u, \nabla\cdot\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{e}_q, \alpha\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{e}_q, \nabla\phi)_{\mathcal{T}_h} \\
&= (e_u, \nabla\cdot\boldsymbol{\Pi}_V\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{e}_q, \alpha\boldsymbol{\Pi}_V\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{e}_q, \nabla\Pi_W\phi)_{\mathcal{T}_h} \\
&\quad + (e_u, \nabla(\boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta}))_{\mathcal{T}_h} - (\boldsymbol{e}_q, \alpha(\boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta}))_{\mathcal{T}_h} - (\boldsymbol{e}_q, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h} \\
&= \langle e_{\widehat{u}}, \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} + (e_u, \nabla\cdot(\boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta}))_{\mathcal{T}_h} \\
&\quad - (\alpha\boldsymbol{e}_q, \boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{e}_q, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h}.
\end{aligned}$$

We observe that

$$\langle e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0 \quad \text{and} \quad \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, P_\partial\phi\rangle_{\partial\mathcal{T}_h} = 0.$$

Integrating by parts and the above result lead us to conclude that

$$\begin{aligned}
\mathbb{T}_2 - \mathbb{T}_3 &= \langle e_{\widehat{u}}, \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} \\
&\quad + (e_u, \nabla\cdot(\boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta}))_{\mathcal{T}_h} - (\boldsymbol{e}_q, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h} \\
&= \langle e_{\widehat{u}}, \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle e_u, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{e}_q\cdot\boldsymbol{n}, P_\partial\phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h} \\
&= \langle e_u - e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle \boldsymbol{e}_q\cdot\boldsymbol{n} - (\boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}), P_\partial\phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

□

Lemma 3.10. *Assume that the H^2-regularity* (3.10) *holds. Then, we have*

$$\begin{aligned}
|\mathbb{T}_1| &\le Ch \,|||\, \boldsymbol{e}_q \,|||_{\alpha,\Omega} \cdot \|e_u\|_\Omega, \\
|\mathbb{T}_2| &\le Ch \,|||\, \boldsymbol{e}_q \,|||_{\alpha,\Omega} \cdot \|e_u\|_\Omega + C\tau^{-\frac{1}{2}} h^{\frac{1}{2}} \,|||\, P_\partial e_u - e_{\widehat{u}} \,|||_{\tau,\partial\mathcal{T}_h} \cdot \|e_u\|_\Omega, \\
|\mathbb{T}_3| &\le Ch^{k+2}|\boldsymbol{q}|_{k+1,\Omega} \cdot \|e_u\|_\Omega + C\tau^{\frac{1}{2}} h^{\frac{3}{2}} \,|||\, P_\partial e_u - e_{\widehat{u}} \,|||_{\tau,\partial\mathcal{T}_h} \cdot \|e_u\|_\Omega \\
&\quad + C\tau h^{k+3}|u|_{k+2,\Omega} \cdot \|e_u\|_\Omega.
\end{aligned} \tag{3.12}$$

Proof. We will estimate $|\mathbb{T}_1|$, $|\mathbb{T}_2|$, and $|\mathbb{T}_3|$ separately, by using results from Lemma 3.9. First we estimate $|\mathbb{T}_1|$.

$$\begin{aligned}
|\mathbb{T}_1| &= |(\alpha\boldsymbol{e}_q, \boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta})_{\mathcal{T}_h}| \le C \,|||\, \boldsymbol{e}_q \,|||_{\alpha,\Omega} \cdot \|\boldsymbol{\theta} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\|_\Omega \\
&\le Ch \,|||\, \boldsymbol{e}_q \,|||_{\alpha,\Omega} \cdot \|\boldsymbol{\theta}\|_{1,\Omega} \le Ch \,|||\, \boldsymbol{e}_q \,|||_{\alpha,\Omega} \cdot \|e_u\|_\Omega.
\end{aligned}$$

Next, we estimate of $|\mathbb{T}_2|$.

$$\begin{aligned}
|\mathbb{T}_2| &= \left|\langle e_u - e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\right| \\
&\le \left|\langle P_\partial e_u - e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\right| + \left|\langle e_u - P_\partial e_u, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\right|
\end{aligned}$$

Observe that

$$\begin{aligned}\left|\langle P_\partial e_u - e_{\widehat{u}}, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\right| &\le C\tau^{-\frac{1}{2}}h^{-\frac{1}{2}} \,|||\, P_\partial e_u - e_{\widehat{u}} \,|||_{\tau,\partial\mathcal{T}_h}\cdot\|\boldsymbol{\theta}-\boldsymbol{\Pi}_V\boldsymbol{\theta}\|_\Omega\\ &\le C\tau^{-\frac{1}{2}}h^{\frac{1}{2}} \,|||\, P_\partial e_u - e_{\widehat{u}} \,|||_{\tau,\partial\mathcal{T}_h}\cdot\|\boldsymbol{\theta}\|_{1,\Omega}\\ &\le C\tau^{-\frac{1}{2}}h^{\frac{1}{2}} \,|||\, P_\partial e_u - e_{\widehat{u}} \,|||_{\tau,\partial\mathcal{T}_h}\cdot\|e_u\|_\Omega,\end{aligned}$$

and

$$\begin{aligned}\left|\langle e_u - P_\partial e_u, \boldsymbol{\theta}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_V\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\right| &= \left|\langle e_u - P_\partial e_u, \boldsymbol{\theta}\cdot\boldsymbol{n} - P_\partial(\boldsymbol{\theta}\cdot\boldsymbol{n})\rangle_{\partial\mathcal{T}_h}\right|\\ &\le Ch^{-1}\|e_u - P_\partial e_u\|_\Omega\cdot\|\boldsymbol{\theta}-\boldsymbol{\Pi}_V\boldsymbol{\theta}\|_\Omega\\ &\le C\|e_u - P_\partial e_u\|_\Omega\cdot\|\boldsymbol{\theta}\|_{1,\Omega}\\ &\le Ch\|\nabla e_u\|_\Omega\cdot\|e_u\|_\Omega\\ &\le Ch\,|||\,\boldsymbol{e}_q\,|||_{\alpha,\Omega}\cdot\|e_u\|_\Omega\\ &\quad +C\tau^{-\frac{1}{2}}h^{\frac{1}{2}}\,|||\, P_\partial e_u - e_{\widehat{u}}\,|||_{\tau,\partial\mathcal{T}_h}\cdot\|e_u\|_\Omega.\end{aligned}$$

Then, we get the second inequality. By Lemma 3.2, we have

$$\begin{aligned}\mathbb{T}_3 &= \langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n} - \tau(P_\partial e_u - e_{\widehat{u}}) + \tau(\Pi_W u - u), P_\partial\phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\\ &= \langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n} - \tau(P_\partial e_u - e_{\widehat{u}}) + \tau(\Pi_W u - u), \phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\\ &\quad + \langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n} - \tau(P_\partial e_u - e_{\widehat{u}}) + \tau(\Pi_W u - u), P_\partial\phi - \phi\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Observe that

$$\begin{aligned}\left|\langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n}, \phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\right| &= \left|\langle P_\partial(\boldsymbol{q}\cdot\boldsymbol{n}) - \boldsymbol{q}\cdot\boldsymbol{n}, \phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\right|\\ &\le \|\boldsymbol{q} - P_\partial\boldsymbol{q}\|_{\partial\mathcal{T}_h}\cdot\|\phi - \Pi_W\phi\|_{\partial\mathcal{T}_h}\\ &\le Ch^{k+\frac{1}{2}}|\boldsymbol{q}|_{k+1,\Omega}\cdot h^{\frac{3}{2}}\|\phi\|_{2,\Omega}\\ &\le Ch^{k+2}|\boldsymbol{q}|_{k+1,\Omega}\cdot\|e_u\|_\Omega,\end{aligned}$$

$$\begin{aligned}\left|\langle\tau(P_\partial e_u - e_{\widehat{u}}), \phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\right| &\le C\tau^{\frac{1}{2}}\,|||\, P_\partial e_u - e_{\widehat{u}}\,|||_{\tau,\partial\mathcal{T}_h}\cdot\|\phi - \Pi_W\phi\|_{\partial\mathcal{T}_h}\\ &\le C\tau^{\frac{1}{2}}h^{\frac{3}{2}}\,|||\, P_\partial e_u - e_{\widehat{u}}\,|||_{\tau,\partial\mathcal{T}_h}\cdot\|e_u\|_\Omega,\end{aligned}$$

and

$$\begin{aligned}\left|\langle\tau(\Pi_W u - u), \phi - \Pi_W\phi\rangle_{\partial\mathcal{T}_h}\right| &\le C\tau\|\Pi_W u - u\|_{\partial\mathcal{T}_h}\cdot\|\phi - \Pi_W\phi\|_{\partial\mathcal{T}_h}\\ &\le C\tau h^{k+\frac{3}{2}}|u|_{k+2,\Omega}\cdot h^{\frac{3}{2}}\|\phi\|_{2,\Omega}\\ &\le C\tau h^{k+3}|u|_{k+2,\Omega}\cdot\|e_u\|_\Omega.\end{aligned}$$

Since $\|P_\partial\phi - \phi\|_{\partial\mathcal{T}_h} \le h^{\frac{3}{2}}\|e_u\|_\Omega$, by combining the above estimates, we get the desired result. □

Theorem 3.11. *Assume that the H^2-regularity* (3.10) *holds. Then, we have*

$$\begin{aligned}\|e_u\|_\Omega &\le C\left(\tau^{\frac{1}{2}}h^{k+\frac{5}{2}} + \tau h^{k+3} + h^{k+2}\right)|u|_{k+2,\Omega}\\ &\quad +C\left(\tau^{-\frac{1}{2}}h^{k+\frac{3}{2}} + \tau^{-1}h^{k+1} + \tau^{\frac{1}{2}}h^{k+\frac{5}{2}} + h^{k+2}\right)|\boldsymbol{q}|_{k+1,\Omega}.\end{aligned} \tag{3.13}$$

Proof. By the Lemma 3.9, 3.10 and Theorem 3.6, we automatically get the inequality. □

As a consequence of Theorem 3.11, we immediately have the following estimate for e_u:

Corollary 3.12. *Assume that the H^2-regularity (3.10) holds. If $\tau = \mathcal{O}\left(h^{-1}\right)$, then we have*

$$\|e_u\|_\Omega \leq Ch^{k+2}\left(|u|_{k+2,\Omega} + |\boldsymbol{q}|_{k+1,\Omega}\right). \tag{3.14}$$

Remark 3.13. *Note that the main advantage of the proposed HDG method is superconvergence of problems without a local postprocessing. Achievement of this goal depends on the value of stabilization parameter τ in Corollary 3.12. When we choose the value depending on a mesh size, $\mathcal{O}(h^{-1})$, as the stabilization parameter τ, we achieve the $(p+2)$th convergence order of the problem without the local postprocessing. However, when other values are choosen, the superconvergence may not be guaranteed.*

4. Numerical results

In this section, we present a numerical example to demonstrate the accuracy and efficiency of the proposed HDG method. We are mainly interested on the order of convergence when a mesh size h is refined. We study the error behavior originated from selecting of the stabilization parameter τ.

We consider a numerical example in two dimensions. Similarly, extending the result to three dimensions is simple. We also generate a structured triangulation with nth subintervals in each coordinate direction. We consider the following finite element spaces in order to apply the proposed HDG method. $\boldsymbol{V}_h$ consists of piecewise linear, discontinuous functions on $\mathcal{T}_h$, M_h of piecewise linear, discontinuous functions in $\mathcal{E}_h$, and W_h of piecewise quadratic, discontinuous on $\mathcal{T}_h$.

Example 4.1. *We consider the domain $\Omega = (0,1)^2$, with $\alpha = 1$ and $\tau = h^{-1}$. The source term $f(x,y) = -2x(x-1) - 2y(y-1)$ is selected in such a way that*

$$u(x,y) = xy(x-1)(y-1)$$

is the exact solution of (1.1).

Table 1 shows the convergence rates of the Example 4.1. The order of convergence for velocity $\boldsymbol{q}$ in weighted L^2-norm is two and the those for pressure u in L^2-norm is three. This matches well with the prediction in Corollaries 3.7 and 3.12.

5. Conclusion

In this paper, we introduce a new hybrid discontinuous Galerkin method for solving elliptic equations. Comparing with the standard HDG method, the proposed HDG method is advantageous in three different aspects. First of all, under the framework of weak formulation, weak formulation in our method just changes the finite element space W_h with polynomials with degree one higher than those used in the standard HDG method and the numerical component of the numerical trace considering projection operator on the M_h. Despite this advantage, we

TABLE 1. The order of convergence for $\tau = h^{-1}$

h	$\|\boldsymbol{q} - \boldsymbol{q}_h\|_\Omega$ Error	Order	$\|u - u_h\|_\Omega$ Error	Order
$1/8$	0.0227	—–	0.0023	—–
$1/16$	0.0056	2.019	2.8434e-04	3.015
$1/32$	0.0014	2.000	3.5487e-05	3.002
$1/64$	3.4963e-04	2.001	4.4330e-06	3.000
$1/128$	8.7377e-05	2.000	5.5397e-07	3.000

retain the merit of the standard HDG method such as reduction of degree of freedom and flexibility since the changes we made do not affect the advantage of the standard HDG method. Secondly, in high order accuracy, the standard HDG method requires a local postprocessing for superconvergence, while our HDG method only needs to change the stabilization parameter. Finally, error estimates can be derive by using local L^2-projecton when we perform error analysis. In short, projection based analysis in our proposed method is both easier and more convenient than the HDG-projection for the standard HDG method. Our numerical examples and error analysis further demonstrate high order accuracy.

REFERENCES

[1] B.S, Kim and J.T. Oden, *hp-version discontinuous Galerkin methods for hyperbolic conservation laws*, Computer Methods in Applied Mechanics and Engineering, **133(3)** (1996), 259–286.

[2] B. Cockburn and C-W. Shu, *The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems*, Journal of Computational Physics, **141(2)** (1998), 199–224.

[3] L. Krivodonova, J. Xin, J-F. Remacle, N. Chevaugeon, and J.E. Flaherty, *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*, Applied Numerical Mathematics, **48(3)** (2004), 323–338.

[4] T.J. Hughes, G. Engel, L. Mazzei, and M.G. Larson, *A comparison of discontinuous and continuous Galerkin methods based on error estimates, conservation, robustness and efficiency*, Discontinuous Galerkin Methods, (2000), 135–146.

[5] P. Houston and E. Süli, *hp-adaptive discontinuous Galerkin finite element methods for first-order hyperbolic problems*, SIAM Journal on Scientific Computing, **23(4)** (2001), 1226–1252.

[6] R. Hartmann and P. Houston, *Adaptive discontinuous Galerkin finite element methods for the compressible Euler equations*, Journal of Computational Physics, **183(2)** (2002), 508–532.

[7] R. Biswas, K.D. Devine, and J.E. Flaherty, *Parallel, adaptive finite element methods for conservation laws*, Applied Numerical Mathematics, **14** (1994), 255–283.

[8] C.E. Baumann and J.T. Oden, *A discontinuous hp finite element method for convection-diffusion problems*, Computer Methods in Applied Mechanics and Engineering, **175(3)** (1999), 311–341.

[9] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab, *Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems*, Mathematics of Computation, **71** (2002), 455–478.

[10] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM Journal on Numerical Analysis, **39** (2001), 1749–1779.

[11] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems*, SIAM Journal on Numerical Analysis, **47** (2009), 1319–1365.
[12] B. Cockburn, J. Gopalakrishnan, and F-J. Sayas, *A projection-based error analysis of HDG methods*, Mathematics of Computation, **79** (2010), 1351–1367.
[13] B. Cockburn, W. Qiu, and K. Shi, *Conditions for superconvergence of HDG methods for second-order elliptic problems*, Mathematics of Computation, **81** (2012), 1327–1353.
[14] N.C. Nguyen, J. Peraire, and B. Cockburn, *An implicit high-order hybridizable discontinuous Galerkin method for linear convection–diffusion equations*, Journal of Computational Physics, **228** (2009), 3232–3254.
[15] N.C. Nguyen, J. Peraire, and B. Cockburn, *An implicit high-order hybridizable discontinuous Galerkin method for nonlinear convection–diffusion equations*, Journal of Computational Physics, **228** (2009), 8841–8855.
[16] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, MA, 1985.
[17] P.G. Ciarlet, *The finite element method for elliptic problems*, Elsevier North-Holland, New York, 1978.