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On Crossing Changes for Surface-Knots

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Abstract. In this paper, we discuss the crossing change operation along exchangeable double curves of a surface-knot diagram. We show that under certain condition, a finite sequence of Roseman moves preserves the property of those exchangeable double curves. As an application for this result, we also define a numerical invariant for a set of surface-knots called *du*-exchangeable set.

1. Introduction

A *surface-knot* F is an orientable connected, closed surface smoothly embedded in the Euclidean 4-space $\mathbb{R}^4$. It is is called an *unknotted* or *trivial* if it is isotopic to the boundary of a handlebody embedded in $\mathbb{R}^3 \times \{0\}$. To describe a surface-knot F, we consider the image of the surface-knot under the orthogonal projection $p : \mathbb{R}^4 \to \mathbb{R}^3$ that is defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. We may slightly perturb F by an isotopy so that its projection image in $\mathbb{R}^3$ is a *generic* surface [1]; that is, its singularity set consists of at most three types: double points, isolated triple points or isolated branch points. The *surface-knot diagram*, or simply the *diagram* of a surface-knot F, denoted by D, is the generic projection image of F in 3-space with crossing information.
The crossing change in classical knot theory, that is defined by exchanging an upper arc and a lower arc at a crossing point in a knot diagram, can be generalized to theory of surface-knots. When the crossing change operation is defined, it is natural to consider the following problem.

Problem 1. *Let $p(F)$ be the projection of a surface-knot F in 3-space. Is there an unknotted surface-knot F_0 such that $p(F) = p(F_0)$?*

In other words, can we transform any surface-knot diagram into a diagram of

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the trivial surface-knot by making suitable crossing changes? The crossing change operation is called an unknotting operation for a surface-knot F if an unknotted surface-knot is obtained from F by a finite sequence of crossing changes. Although some projections of surface-knots are known to be projections of an unknotted surface-knot [11], it is still an open problem whether all projections satisfy this property; that is whether the crossing change is an unknotting operation for any surface-knot. The answer for the problem is affirmative in the case of classical knot theory and this fact is used to compute the invariants that are defined by skein relations, e.g., [8]. Moreover an invariant, the unknotting number is defined.

When we explore into the literature related to Problem 1, we see that some partial results were found. A special kind of crossing changes was defined [16], but this kind can not be applied to any surface-knot diagram. A special surface braid diagram is shown to be unknotted by crossing changes ([4], [6]). K. Tanaka in [13] proved that any pseudo-ribbon surface-knot diagram can be deformed by crossing changes into a diagram with knot group isomorphic to $\mathbb{Z}$. K. Yoshida [17] verified that the projection p of an S^n-knot ($n \neq 3,\ 4$) is the projection of a trivial n-knot provided that the singularity set of p consists of only double points and is homeomorphic to a disjoint union of $(n-1)$-spheres. E. Ogasa [9] proved that there exists a projection of an S^n-knot which is not the projection of any trivial knot, for $n \geq 3$.

In this paper, we define a set of surface-knots called the du-exchangeable set that contains du-exchangeable surface-knots. We show that for a du-exchangeable surface-knot, there exists a finite sequence of surface-knot diagrams each of which can be unknotted by the crossing changes. As an application for this result, we also define a numerical invariant for the du-exchangeable set.

The rest of the paper is organized as follows. In section 2, we give some basics about surface-knots to facilitate later discussions. In section 3, we recall Roseman moves. In section 4, we define a finite sequence for a surface-knot represented by its surface-knot diagrams, called a t-descendent sequence. In section 5, we review the crossing change operation and section 6 is devoted to stating and proving the main result. Finally, section 7 introduces an invariant for a set of surface-knots called the du-exchangeable set.

2. Preliminaries

Let F be a surface-knot and let $h : \mathbb{R}^4 \to \mathbb{R}$ be the height function, $h(x_1, x_2, x_3, x_4) = x_4$. Let $\mathrm{cl}(S)$ stand for the closure of the set S.

The closure of the set

$$\{x \in F : \#p^{-1}(p(x)) \geq 2\}$$

can be regarded as the image of compact 1-dimensional manifold immersed into F. It is divided into two families $\mathcal{S}_a = \{s_a^1, \dots, s_a^n\}$ and $\mathcal{S}_b = \{s_b^1, \dots, s_b^n\}$ of immersed closed intervals or simple closed curves in F such that $h(x) > h(x')$ holds for any

$x \in s_a^i$ and $x' \in s_b^i$ $(i = 1, 2, \ldots, n)$. Let $S_a = \cup_{i=1}^n \mathrm{cl}(s_a^i)$ and let $S_b = \cup_{i=1}^n \mathrm{cl}(s_b^i)$. The union $S_a \cup S_b$ is called the *double decker set* (see [1] and [2] for details).

Double points in a generic projection image of a surface-knot in 3-space form a disjoint union of 1-manifolds that may appear as open arcs or simple closed curves. We say that such an open arc is called a *double edge*. Let D be a surface-knot diagram of a surface-knot F. Let $e_1, \ldots, e_n, e_{n+1} = e_1$ be double edges and let $T_1, \ldots, T_n, T_{n+1} = T_1$ be triple points of D. For $i = 1, \ldots, n$, assume that the boundary points of each of e_i are T_i and T_{i+1} and that e_i and e_{i+1} are in opposition to each other at T_{i+1}. The closure of the union $e_1 \cup e_2 \cup \ldots \cup e_n$ forms a circle component called a *double point circle* of the diagram.

Similarly, let $e_1, \ldots, e_n$ be double edges, $T_1, \ldots, T_{n-1}$ be triple points of D and suppose b_1 and b_n are branch points of D. Assume the boundary points of e_1 are the triple point T_1 and the branch point b_1. Assume also that the double edge e_n is bounded by T_{n-1} and b_n. For $i = 2, \ldots, n-1$, the double edge e_i is bounded by T_{i-1} and T_i. If e_i and e_{i+1} are in opposition to each other at T_i $(1, 2, \ldots, n-1)$, then the closure of the union $e_1 \cup e_2 \cup \ldots \cup e_n$ forms an arc component called a *double point interval* of the diagram. By a *double curve*, we refer to a double point circle or a double point interval. Let T be a triple point of D and $B(T)$ a 3-ball neighbourhood of T. The intersection of $B(T)$ and the double edges consists of six short arcs. We call them the *branches* of double edges at T. A branch at T is called a *b/m-, b/t- or m/t-branch* if it is the intersection between bottom and middle or bottom and top or middle and top sheets, respectively.

3. Roseman Moves

D. Roseman introduced analogues of the Reidemeister moves as local moves to surface-knot diagrams. Let D and D' be surface-knot diagrams of F and F', respectively. It is known that F and F' are equivalent if and only if there exists a finite sequence of surface-knot diagrams $D = D_0 \to D_1 \to \ldots \to D_n = D'$ such that for all $i = 0, 1, \ldots, n-1$, D_i and D_{i+1} differ by one of seven Roseman moves [10]. We will write $D \sim D'$ to indicate that D and D' present the same surface-knot. T. Yashiro [14] showed that the Roseman's seven moves can be described by six moves depicted in Figure 1 (see also [7]). We call these moves also Roseman moves. Each move from left to right is denoted by R-X^+ and right to left by R-X^- except R-6. The move R-X^- is called the *reverse* of R-X^+. Note that the information on height has not been specified in Figure 1.

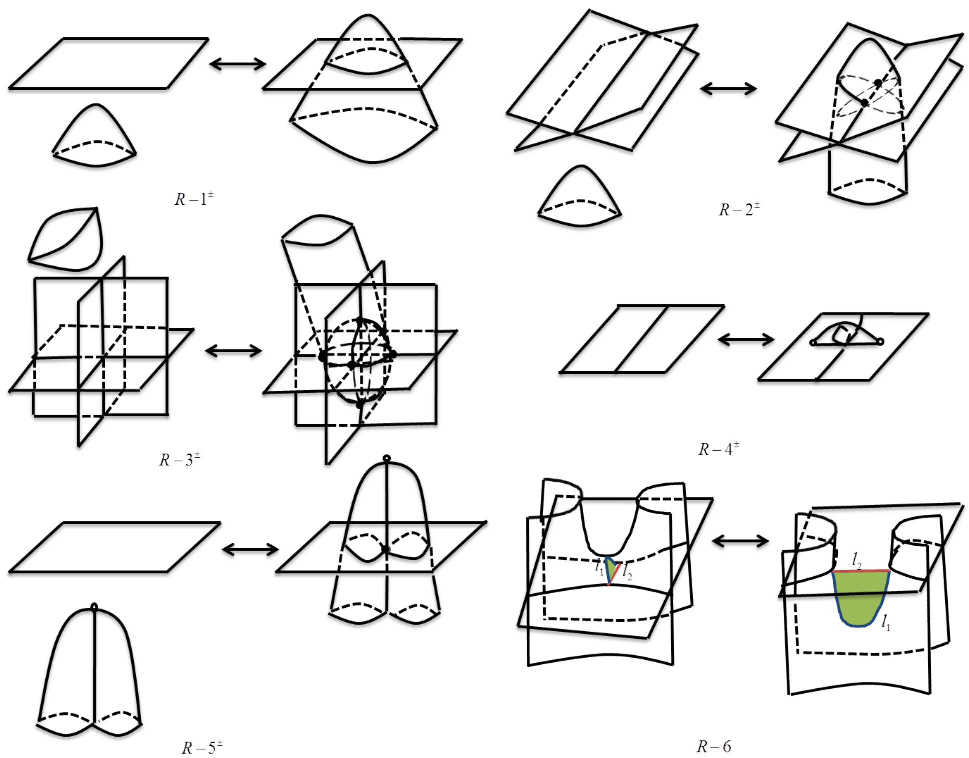

Figure 1: Roseman moves

Here we describe the R-6 move. Type R-6 move consists of deformations of two disks P_1 and P_2 in a surface-knot diagram D. The disk P_1 forms a saddle point and the other disk P_2 passing through the saddle point of P_1 in a direction perpendicular to the tangent plane at the saddle point. The two disks P_1 and P_2 intersect at two double edges, say e_1 from a_1 to a_2 and e_2 from b_1 to b_2, where a_i and b_i $(i = 1, 2)$ are boundary points of P_2. We give an order to these boundary points such that the four points are ordered as $\{a_1, a_2, b_1, b_2\}$ with respect to the orientation of the boundary. As P_2 passes through the saddle point of P_1, the two double edges e_1 and e_2 get closer and join at the middle point of each double edge. As a result, the new double edges e'_1 from a_1 to b_2 and e'_2 from a_2 to b_1 appear.

Assume that the R-6 move is applied to the pair of disks P_1 and P_2. In the notation above, let a be the middle point of the double segment e_1 and let b be the middle point of the double segment e_2. Then we can find a disk P, in a closure of one of complementary open regions of D, satisfying the following properties:

(1) $\partial P = l_1 \cup l_2$, where l_1 and l_2 are two simple arcs in D, each of which is

terminated by a and b;

(2) One of l_i $(i = 1, 2)$ is on P_1 and the other one is on P_2;

(3) The pre-images of l_i $(i = 1, 2)$ do not meet with $S_a \cup S_b$ other than their end points.

(4) The two endpoints of one of the pre-images l_i $(i = 1, 2)$ are on S_a and both endpoints of the other one are on S_b.

The disk P is called a *descendent disk* of D [3]. Conversely, if a descendent disk exists, then R-6 can be applied. In this sense, the two descendent disks involved in the left and right of R-6 are said to be *dual to each other*.

4. t-descendent Sequences

Let $D = D_0 \to D_1 \to \ldots \to D_n = D'$ be a finite sequence of surface-knot diagrams satisfying the following condition:

(*) A transition from D_i to D_{i+1} $(i = 0, \ldots, n-1)$ is done by one of Roseman moves which can be realized by an isotopy of the surface-knot in $\mathbb{R}^4$ without creating a triple point in the projection.

In other words, the above condition says that the finite sequence of Roseman moves that connects D and D' does not include the Roseman moves R-i^+ $(i = 2, 3, 5)$. Note that the reverse moves of these prohibited moves are allowed.

Definition 4.1. A finite sequence of surface-knot diagrams satisfying the condition (*) is called a *t-descendent sequence*.

The terminology "t-descendent" is used to clarify that the number of triple points will not increase under the sequence.

Remark 4.2. We point out here that M. Jabonowski published a paper [5] in which he provided an example of two equivalent surface-knot diagrams which can not be connected by a t-descendent sequence. Both diagrams in the example are with singularity set consisting of only closed 1-manifolds.

5. The Crossing Change Operation

A crossing change operation is a local operation for a diagram of a surface-knot which has a natural analogy to the crossing changes of classical knots.

Definition 5.1. Let D be a surface-knot diagram of a surface-knot. Let $\Gamma = \cup_{j=1}^{r} \gamma_j$ be a union of double curves in D. Γ is *exchangeable* if a surface-knot diagram is obtained from D by changing the upper/lower information along the double curves of Γ simultaneously. This operation is called the *crossing change operation* along Γ, and we denote by $D(\Gamma)$ the surface-knot diagram obtained from D by the operation.

We do not assume that D and $D(\Gamma)$ present distinct surface-knots.

Let T be a triple point of D. The figure below shows all possible cases of changing the crossing information around the triple point T, where the branches at T contained in Γ are bold lines for each possible case.

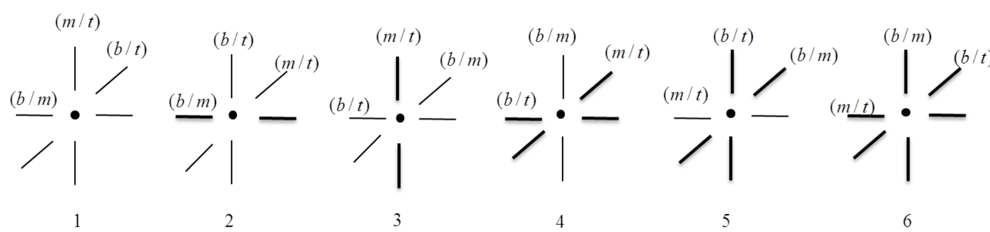

Figure 2: Changing the crossing information around a triple point

Remark 5.2. Let D be a surface-knot diagram and γ be a double curve in D. Assume $\gamma \subset \Gamma$ where Γ is exchangeable. Suppose γ has an edge which contains a b/t-branch at a triple point. Then, the b/m-branches or m/t-branches at T are subsets of a double curve in D that is contained in Γ.

Definition 5.3. Let Γ be an exchangeable union of double curves of a surface-knot diagram D. We say that Γ satisfies the *descendent disk condition* for D if exchanging the crossing information along the double curves of Γ preserves all descendent disks of D. We say that Γ satisfies the *unknotting condition* for D if the surface-knot diagram $D(\Gamma)$ presents an unknotted surface-knot.

Definition 5.4. A surface-knot diagram D is called *du-exchangeable* if it presents an unknotted surface-knot or it has an exchangeable union of double curves Γ satisfying both the descendent disk condition and the unknotting condition for D.

Definition 5.5. A surface-knot F is *du-exchangeable* if there is a surface-knot diagram D presenting F such that D is *du*-exchangeable.

Remark 5.6. It is not difficult to see that any surface-knot diagram of a surface-knot has exchangeable double curves satisfying the descendent disk condition. For example, the union of all double curves of a surface-knot diagram is exchangeable and it satisfies the descendent disk condition.

6. The Main Result

In this section we prove the main theorem in this paper (Theorem 6.4.). In particular, the proof is divided into three lemmas.

Let D be a surface-knot diagram of a surface-knot F. Let $\Gamma = \cup_{j=1}^{r} \gamma_j$ be an exchangeable union of double curves in D. The surface-knot diagram obtained after cross-change operation along Γ applied is denoted by $D(\Gamma)$. Throughout the proof of the three lemmas in this section, $\Gamma^{(s)}$ denotes the union of double curves $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_{(s-1)} \cup \hat{\gamma}_s \cup \gamma_{(s+1)} \cup \ldots \cup \gamma_r$ that is obtained from Γ by deleting the double curve γ_s. Similarly, we define $\Gamma^{(s,w)}$ to be the union of double curves that is obtained from Γ by deleting the double curves γ_s and γ_w.

Lemma 6.1. *Suppose that D is transformed into D' by one of the Roseman moves of R-i^- $(i = 2, 3, 5)$. For any exchangeable union Γ of double curves of D, there is an exchangeable union Γ' of double curves of D' such that $D(\Gamma) \sim D'(\Gamma')$. Moreover, if Γ satisfies the descendent disk condition, then we may assume that Γ' also satisfies it.*

Proof. Let $\Gamma = \cup_{j=1}^{r}\gamma_j$ be an exchangeable union of double curves in D satisfying the descendent disk condition for D. We need to define an exchangeable union of double curves Γ' in D' that satisfies the assertion of the lemma for each Roseman move R-i^- $(i = 2, 3, 5)$.

The Roseman move R-2^- can be viewed that it takes the paraboloid away from the double edge so that the paraboloid does not meet with the two intersecting disks (see R-$2^{\pm}$ in Figure 1). Let γ_s be a double point curve in D containing branches formed by the two disks. By applying R-2^-, γ_s is restricted to a double curve γ'_s of D' that has less triple points by two. The exchangeable union of double curves Γ' of D' is defined as follows.

$$\Gamma' = \begin{cases} \Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\ \Gamma^{(s)} \cup \gamma'_s & \text{if } \gamma_s \subset \Gamma. \end{cases}$$

We see that the diagram $D'(\Gamma')$ differs from $D(\Gamma)$ by the move R-2^- and thus they are equivalent.

The move R-3^- can be viewed that it takes the paraboloid away from the triple point T so that it does not meet with the three intersecting disks (see R-$3^{\pm}$ in Figure 1). This leads to elimination of six triple points and three double point circles. Assume that the b/m- branches at T are subsets of the double curve γ_s of D, the m/t- branches at T are subsets of the double curve γ_w of D and the b/t-branches at T are subsets of the double curve γ_k of D. By applying R-3^-, γ_s is transformed to a double curve γ'_s of D' such that γ'_s has less double edges than γ_s by two. Similarly, γ_w and γ_k of D are transformed to γ'_w and γ'_k in D', respectively. We define Γ' by

$$\Gamma' = \begin{cases} \Gamma & \text{if } \gamma_s, \gamma_w, \gamma_k \not\subseteq \Gamma \\ \Gamma^{(s)} \cup \gamma'_s & \text{if } \gamma_s \subset \Gamma, \gamma_w, \gamma_k \not\subseteq \Gamma \\ \Gamma^{(w)} \cup \gamma'_w & \text{if } \gamma_w \subset \Gamma, \gamma_s, \gamma_k \not\subseteq \Gamma \\ \Gamma^{(k,s)} \cup \gamma'_s \cup \gamma'_k & \text{if } \gamma_s \text{ and } \gamma_k \subset \Gamma, \gamma_w \not\subseteq \Gamma \\ \Gamma^{(k,w)} \cup \gamma'_w \cup \gamma'_k & \text{if } \gamma_w \text{ and } \gamma_k \subset \Gamma, \gamma_s \not\subseteq \Gamma \\ \Gamma^{(k,s,w)} \cup \gamma'_s \cup \gamma'_w \cup \gamma'_k & \text{if } \gamma_s, \gamma_w, \gamma_k \subset \Gamma \end{cases}$$

From the definition of Γ', we obtain that the diagram $D'(\Gamma')$ differs from $D(\Gamma)$ by the move R-3^-. It follows that they are equivalent.

The move R-5^- can be described as moving a disk with a branch point away from the second disk (see R-$5^\pm$ in Figure 1) and thus a triple point T is cancelled. Let the branch of T whose other endpoint is a branch point be a subset of a double curve γ_s of D. By applying R-5^- to D, γ_s is restricted to the double curve γ'_s of D' that has less double edges by one. Define

$$\Gamma' = \begin{cases} \Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\ \Gamma^{(s)} \cup \gamma'_s & \text{if } \gamma_s \subset \Gamma \end{cases}$$

It follows that the transition from $D(\Gamma)$ to $D'(\Gamma')$ is done by R-5^-. We obtain that $D(\Gamma) \sim D'(\Gamma')$.

It remains to prove that Γ' defined above for each Roseman move R-i^- ($i = 2, 3, 5$) satisfies the descendent disk condition. Note that none of the three moves R-i^- ($i = 2, 3, 5$) create new double edges and thus no new descendent disks are involved in D'. Since D satisfies the descendent disk condition, we can assume that D' also does. □

Lemma 6.2. *Suppose that D is transformed into D' by one of the Roseman moves of R-i^- and R-i^+ $(i = 1, 4)$. For any exchangeable union Γ of double curves of D, there is an exchangeable union Γ' of double curves of D' such that $D(\Gamma) \sim D'(\Gamma')$. Moreover, if Γ satisfies the descendent disk condition, then we may assume that Γ' also satisfies it.*

Proof. We prove the moves R-1^+ and R-1^-. The moves R-4^+ and R-4^- are similarly proved. The Roseman move R-1^+ has the affect of adding a simple double point circle, denoted by γ' , that is independent from the other double curves of the diagram. The resulting double curve might be involved in a descendent disk where the other involved double edge is in D. Assume that the new descendent disk created is P and that each of γ' and γ_s contains a boundary point of P, where γ_s is a double curve of D. We define Γ' in this case such that

$$\Gamma' = \begin{cases} \Gamma & \text{if } \gamma_s \not\subseteq \Gamma \\ \Gamma \cup \gamma' & \text{if } \gamma_s \subset \Gamma \end{cases}$$

If there is no such a descendent disk, we can assume that $\Gamma' = \Gamma$. It is not hard to see that $D(\Gamma) \sim D'(\Gamma')$ in both cases and that Γ' satisfies the descendent disk condition by the definition. The move R-1^- is the reverse of R-1^+. The double edge γ' will be eliminated as a result. Denote γ' by γ_w for this move. Then, Γ' of D' that satisfy the assertion of the lemma can be defined such that

$$\Gamma' = \begin{cases} \Gamma & \text{if } \gamma_w \not\subseteq \Gamma \\ \Gamma^{(w)} & \text{if } \gamma_w \subset \Gamma \end{cases}$$

The lemma follows. □

Lemma 6.3. *Suppose that D is transformed into D' by Roseman move of R-6. Let Γ be an exchangeable union of double curves of D satisfying the descendent disk condition. Then there is an exchangeable union Γ' of double curves of D' such that $D(\Gamma) \sim D'(\Gamma')$ and that Γ' also satisfies the descendent disk condition.*

Proof. Let P be a descendent disk of D and suppose R-6 move is applied along P to obtain the surface-knot diagram D'. Let e_1 and e_2 be double edges of D each of which contains a boundary point of P. Assume that $e_1 \subset \gamma_s$ and $e_2 \subset \gamma_w$, where γ_s and γ_w are double curves of D. Since Γ satisfies the descendent disk condition, either the upper/lower information of both γ_s and γ_w are exchanged or neither. In the latter case, the result follows by letting $\Gamma' = \Gamma$. On the other hand, let γ_s and γ_w be subsets of Γ. Apply R-6 to obtain the surface-knot diagram D'. The connection between the double edges e_1 and e_2 is changed so that we obtain new double edges, say e_1' and e_2' in D'. Assume that e_1' and e_2' are subsets of double curves γ_s' and γ_w' of D', respectively. Exchanging the upper/lower information of

$$\Gamma' = \Gamma^{(s,w)} \cup \gamma_s' \cup \gamma_w'$$

in D' gives a surface-knot diagram equivalent to $D(\Gamma)$. □

Theorem 6.4. *Let F be a du-exchangeable surface-knot and D be a du-exchangeable surface-knot diagram of F. There exists a finite t-descendent sequence $D = D_0 \to D_1 \to \ldots \to D_n = D'$ such that for each $i = 1, \ldots, n$, D_i is du-exchangeable.*

Proof. Without loss of generality, we can assume that D' is obtained from D by applying a single Roseman move of one of the possible types in a t-descendent sequence. The theorem then follows from Lemma 6.1., Lemma 6.2. and Lemma 6.3. □

Example 6.5. The *triple point number* of a surface-knot, denoted by $t(F)$ is defined to be the minimal number of triple points over all possible diagrams of the surface-knot. Let K be a knot in $\mathbb{R}^3$ such that K has a knot diagram D_K with c crossings in which there is a full twisted trivial tangle with 2-strings. S. Satoh and A. Shima estimated in [12] the upper bound of triple point number for the m-twist spun of the knot K, $\tau^m(K)$. In fact, they showed that $t\big(\tau^m(K)\big) \le 2(c-2)m$. T. Yashiro in [15] gave an improved upper bound for some family of twist-spun knots including the m-twist spun of the knot K described above. Both upper bounds can be obtained by deforming surface-knot diagrams that are obtained by following Satoh's construction [11]. In particular, the deformations applied in both cases are done by t-descendent sequences.

7. The *du*-exchange Index

Let F be a du-exchangeable surface-knot. The du-exchange index $du(F)$ is defined as follows.

Definition 7.1. The *du-exchange index* $du(F)$ of a du-exchangeable surface-knot

F is the minimum number of double curves required, taken over all du-exchangeable diagrams representing F, to convert F into a trivial surface-knot.

The du-exchange index is an invariant for du-exchangeable surface-knots.

Example 7.2. S. Satoh and A. Shima in [12] gave an estimate for a lower bound of the triple point number for tri-colourable surface-knots. Using this estimate, they showed that the 2-twist spun trefoil has the triple point number four. Let K be the trefoil knot and let D_K be a knot diagram of K with 3 crossings. Let $T(K)$ be the one string tangle obtained from D_K by removing a small 1-dim trivial neighbourhood of a point on the diagram D_K. By following Satoh's construction of diagrams of twist-spun knots [11], we obtain a surface-knot diagram of the 2-twist spun trefoil with twelve triple points. In particular, Satoh's diagram can be deformed into one with four triple points by a t-descendent sequence involving Roseman moves R-2^-, R-5^- and R-6. Figure 3 depicts a schematic picture of the double curves of a t-minimal diagram of the 2-twist spun trefoil showing the types of double branches at each triple point.

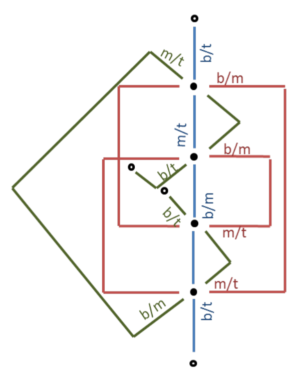

Figure 3: Schematic picture of the double curves of a t-minimal diagram of the 2-twist spun trefoil

From the figure, we see that there are three double curves and two of them are double point intervals. Note that exchanging the crossing information of the double point circle gives a trivial 2-knot diagram and this shows that the 2-twist spun trefoil has the du-exchange index equal to one.

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References

[1] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, **55**, Amer. Math. Soc., Providence, RI, 1998.

[2] J. S. Carter and M. Saito, *Surfaces in 3-space that do not lift to embeddings in 4-space*, Knot Theory, Banach Center Proceedings, 42Polish Academy of Science, Warsaw, **4**(1998), 29–47.

[3] T. Homma, *Elementary deformations on homotopy 3-spheres*, Topology and Computer Science, Kinokuniya, Tokyo, (1987), 21–27.

[4] M. Iwakiri, *Unknotting singular surface braids by crossing changes*, Osaka J. Math., **45(1)**(2008), 61–84.

[5] M. Jabonowski, *Knotted surfaces and equivalencies of their diagrams without triple points*, J. Knot Theory Ramifications, **21**(2012), pp. 6.

[6] S. Kamada, *Crossing changes for singular 2-dimensional braids without branch points*, Kobe J. Math., **13**(1996), 177–182.

[7] K. Kawamura, *On relationship between seven types of Roseman moves*, Topology Appl., **196(B)**(2015), 551–557.

[8] R. Morton and G. Lukac, *Homfly polynomial of decorated Hopf link*, J. Knot Theory Ramifications, **12**(2003), 395–416.

[9] E. Ogasa, *The projections of n-knots which are not the projection of any unkotted knot*, J. Knot Theory Ramifications, **10**(2001), 121132.

[10] D. Roseman, *Reidemeister-type moves for surfaces in four dimensional space*, Banach Center Publ., **42**(1998), 347–380.

[11] S. Satoh, *Surface diagrams of twist-spun 2-knots*, J. Knot Theory Ramifications, **11**(2002), 413–430.

[12] S. Satoh and A. Shima, *The 2-twist-spun trefoil has the triple point number four*, Trans. Amer. Math. Soc., **356**(2004), 1007–1024.

[13] K. Tanaka, *Crossing changes for pseudo-ribbon surface-knots*, Osaka J. Math., **41**(2004), 877–890.

[14] T. Yashiro, *A note on Roseman moves*, Kobe J. Math., **22**(2005), 31–38.

[15] T. Yashiro, *Triple point numbers of twist spun knots*, J. Knot Theory Ramifications, **14**(2005), 831–840.

[16] T. Yashiro, *Crossing distances of surface-knots*, Topology Appl., **154**(2007), 1532–1539.

[17] K. Yoshida, *Unknotting of pseudo-ribbon n-knots*, J. Knot Theory Ramifications, **13**(2004), 259–275.