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Structure Jacobi Operators of Real Hypersurfaces with Constant Mean Curvature in a Complex Space Form

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Abstract. Let M be a real hypersurface with constant mean curvature in a complex space form $M_n(c), c \neq 0$. In this paper, we prove that if the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector field ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the structure tensor field ϕ, then M is a homogeneous real hypersurface of Type A.

1. Introduction

Let $M_n(c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $4c \neq 0$, and let J be its complex structure. Complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ for $c > 0$ or $c < 0$, respectively.

Let M be a conected submanifold of $M_n(c)$ with real codimension 1. We refer to this simply as a real hypersurface below.

For a local unit normal vector field N of M, we define the structure vector field ξ of M by $\xi = -JN$. The structure vector ξ is said to be principal if $A\xi = \alpha\xi$ is satisfied for some functuion α, where A is the shape operator of M.

A real hypersurface M is said to be a Hopf hypersurface if the structure vector ξ of M is principal.

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Hopf hypersurfaces is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map (see Cecil and Ryan [2]). By making use of those results and the mentioned work of Takagi ([15], [16]), Kimura [11] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties.

The structure vector field ξ plays an important role in the theory of real hypersurfaces in a complex space form $M_n(c)$. Related to the structure vector field ξ the Jacobi operator R_ξ defined by $R_\xi = R(\cdot, \xi)\xi$ for the curvature tensor R on a real hypersurface M in $M_n(c)$ is said to be a *structure Jacobi operator* on M. The structure Jacobi operator has a fundamental role in contact geometry. In [3], Cho and second author started the study on real hypersurfaces in complex space form by using the operator R_ξ. In particular the structure Jacobi operator has been studied under the various commutative condition ([3], [7], [10], [14]). For example, Pérez *et al.* [14] called that real hypersurfaces M has commuting structure Jacobi operator if $R_\xi R_X = R_X R_\xi$ for any vector field X on M, and proved that there exist no real hypersurfaces in $M_n(c)$ with commuting structure Jacobi operator. On the other hand Ortega *et al.* [12] have proved that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field X on M. More generally, such a result has been extended by [13]. In this situation, if naturally leads us to be consider another condition weaker than parallelness. In the preceding work, we investigate the weaker condition ξ-parallelness, that is, $\nabla_\xi R_\xi = 0$ (cf. [4], [8], [9]). Moreover some works have studied several conditions on the structure Jacobi operator R_ξ and given some results on the classification of real hypersurfaces of Type A in complex space form ([3], [5], [8] and [9]). The following facts are used in this paper without proof.

Theorem 1.1.([5]) *Let M be a real hypersurface in a nonflat complex space form $M_n(c), c \neq 0$ which satisfies $R_\xi(A\phi - \phi A) = 0$. Then M is a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces:*

(I) *In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(II) *In cases $M_n(c) = H_n\mathbb{C}$,*

- (A_0) *a horosphere;*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

In continuing work [10] Nagai, Takagi and the first author proved the following:

Theorem 1.2.(Ki, Nagai and Takagi [10]) *Let M be a real hypersurface in a nonflat complex space form $M_n(c), c \neq 0$ If M satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$. Then M is the same types as those in Theorem 1.1, where S denotes the Ricci tensor of M.*

In [7], the authors started the study on real hypersurfaces in a complex space form with $\phi\nabla_\xi\xi$-parallel structure Jacobi operator R_ξ, that is, $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ for the vector $\phi\nabla_\xi\xi$ orthogonal to ξ. In previous paper [6], Kim and two of present authors prove that if the structure Jacobi operator R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the structure tensor ϕ, then M is homogeneous real hypersurfaces of Type A provided that $\mathrm{Tr}R_\xi$ is constant. The main purpose of the present paper is to prove that if the structure Jacobi operator is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the structure tensor field ϕ, then the real hypersurfaces M with constant mean curvature is homogeneous real hypersurfaces of Type A.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c), c \neq 0$ with almost complex structure J, and N be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

where g denotes the Riemannian metric of M induced from that of $M_n(c)$ and A denotes the shape operator of M in direction N. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We call ξ the structure vector field (or the Reeb vector field) and its flow also denoted by the same latter ξ. The Reeb vector field ξ is said to be principal if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$.

A real hypersurface M is said to be a Hopf hypersurface if the Reeb vector field ξ is principal. It is known that the aggregate (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \ \phi\xi = 0, \ \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. From Kähler condition $\tilde{\nabla}J = 0$, and taking account of above equations, we see that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.2}$$

for any vector fields X and Y tangent to M.

Since we consider that the ambient space is of constant holomorphic sectional curvature $4c$, equations of the Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z = \; & c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & -2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\} \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$ and $h = \mathrm{Tr}A$, and for a function f we denote by ∇f the gradient vector field of f.

From the Gauss equation (2.3), the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.5}$$

for any vector field X on M.

Now, we put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W is a unit vector field orthogonal to ξ. In the sequel, we put $U = \nabla_\xi \xi$, then by (2.1) we see that

$$U = \mu\phi W \tag{2.7}$$

and hence U is orthogonal to W. So we have $g(U,U) = \mu^2$. Using (2.7), it is clear that

$$\phi U = -A\xi + \alpha\xi, \tag{2.8}$$

which shows that $g(U,U) = \beta - \alpha^2$. Thus it is seen that

$$\mu^2 = \beta - \alpha^2. \tag{2.9}$$

Making use of (2.1), (2.7) and (2.8), it is verified that

$$\mu g(\nabla_X W, \xi) = g(AU, X), \tag{2.10}$$

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.11}$$

because W is orthogonal to ξ.

Now, differentiating (2.8) covariantly and taking account of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.12}$$

which together with (2.4) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{2.13}$$

Applying (2.12) by ϕ and making use of (2.11), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.14}$$

which connected to (2.1), (2.9) and (2.13) gives

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.15}$$

Using (2.3), the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = R(X,\xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.16}$$

for any vector field X on M. Differentiating this covariantly along M, we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) = {} & g(\nabla_X(R_\xi Y) - R_\xi(\nabla_X Y), Z) \\ = {} & -c(\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)) \\ & +(X\alpha)g(AY, Z) + \alpha g((\nabla_X A)Y, Z) \\ & -\eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\} \\ & -\eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}. \end{aligned} \tag{2.17}$$

From (2.5) and (2.16), we have

$$\begin{aligned} (R_\xi S - SR_\xi)(X) = {} & -\eta(AX)A^3\xi + \eta(A^3X)A\xi - \eta(A^2X)(hA\xi - c\xi) \\ & +(h\eta(AX) - c\eta(X))A^2\xi - ch\{\eta(AX)\xi - \eta(X)A\xi\}. \end{aligned} \tag{2.18}$$

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M; A\xi - \alpha\xi \neq 0\}.$$

At each point of Ω, the Reeb vector field ξ is not principal. That is, ξ is not an eigenvector of the shape operator A of M if $\Omega \neq \emptyset$.

In what follows we assume that Ω is not an empty set in order to prove our main theorem by reductio ad absurdum, unless otherwise stated, all discussion concerns the set Ω.

3. Real Hypersurfaces Satisfying $R_\xi\phi = \phi R_\xi$

Let M be a real hypersurface in $M_n(c), c \neq 0$. We suppose that $R_\xi\phi = \phi R_\xi$, which means that the eigenspace R_ξ is invariant by the structure tensor ϕ. Then by using (2.16) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi. \tag{3.1}$$

Using (3.1), it is clear that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Because of (2.9), we have

$$\mu^2 = \alpha\lambda - \alpha^2. \tag{3.2}$$

Replacing X by U in (3.1) and taking account of (2.8), we find

$$\phi AU = \lambda A\xi - A^2\xi, \tag{3.3}$$

which enable us to obtain

$$\phi A^2\xi = AU + \lambda U \tag{3.4}$$

because U is orthogonal to $A\xi$. From this and (2.6) we have

$$\mu\phi AW = AU + (\lambda - \alpha)U, \tag{3.5}$$

which together with (2.7) yields

$$g(AW, U) = 0. \tag{3.6}$$

Using (2.6) and (3.3), we can write (2.15) as

$$\nabla_\xi U = (3\lambda - 2\alpha)A\xi - 3\mu AW - \alpha\lambda\xi + \phi\nabla\alpha. \tag{3.7}$$

Since $\alpha \neq 0$ on Ω, (3.1) reformed as

$$(\phi A - A\phi)X = \eta(X)U + u(X)\xi + \tau(u(X)W + w(X)U), \tag{3.8}$$

where a 1-form u is defined by $u(X) = g(U, X)$ and w by $w(X) = g(W, X)$, where we put

$$\alpha\tau = \mu,\ \ \lambda - \alpha = \mu\tau. \tag{3.9}$$

Differentiating (3.8) covariantly and taking the inner product with any vector field Z, we find

$$\begin{aligned}
&g(\phi(\nabla_Y A)X, Z) + g(\phi(\nabla_Y A)Z, X)\\
&= -\eta(AX)g(AY, Z) - g(AX, Y)\eta(AZ)\\
&\quad + g(A^2X, Y)\eta(Z) + \eta(X)g(A^2Y, Z)\\
&\quad + (\eta(X) + \tau w(X))g(\nabla_Y U, Z)\\
&\quad + g(\nabla_Y U, X)(\eta(Z) + \tau w(Z))\\
&\quad + u(X)g(\nabla_Y \xi, Z) + g(\nabla_Y \xi, X)u(Z)\\
&\quad + (Y\tau)(u(X)w(Z) + u(Z)w(X))\\
&\quad + \tau(u(X)g(\nabla_Y W, Z) + g(\nabla_Y W, X)u(Z))
\end{aligned} \tag{3.10}$$

because of (2.1) and (2.2). From this, taking the skew-symmetric part with respect to X and Y, and making use of the Codazzi equation (2.4), we find

$$\begin{aligned}
&c(\eta(X)g(Y,Z)-\eta(Y)g(X,Z))\\
&+g((\nabla_X A)\phi Y,Z)-g((\nabla_Y A)\phi X,Z)\\
&=-\eta(AX)g(AY,Z)+\eta(AY)g(AX,Z)\\
&\quad+\eta(X)g(A^2Y,Z)-\eta(Y)g(A^2X,Z)\\
&\quad+(\eta(X)+\tau w(X))g(\nabla_Y U,Z)-(\eta(Y)+\tau w(Y))g(\nabla_X U,Z)\\
&\quad+(g(\nabla_Y U,X)-g(\nabla_X U,Y))(\eta(Z)+\tau w(Z))\\
&\quad+u(X)g(\nabla_Y\xi,Z)-u(Y)g(\nabla_X\xi,Z)\\
&\quad+(g(\nabla_Y\xi,X)-g(\nabla_X\xi,Y))u(Z)\\
&\quad+(Y\tau)(u(X)w(Z)+u(Z)w(X))-(X\tau)(u(Y)w(Z)+u(Z)w(Y))\\
&\quad+\tau\{u(X)g(\nabla_Y W,Z)-u(Y)g(\nabla_X W,Z)\}\\
&\quad+\tau\{(g(\nabla_Y W,X)-g(\nabla_X W,Y))u(Z)\}.
\end{aligned}\tag{3.11}$$

Interchanging Y and Z in (3.10), we obtain

$$\begin{aligned}
&g(\phi(\nabla_Z A)X,Y)+g(\phi(\nabla_Z A)Y,X)\\
&=-\eta(AX)g(AY,Z)-g(AX,Z)\eta(AY)\\
&\quad+g(A^2X,Z)\eta(Y)+\eta(X)g(A^2Y,Z)\\
&\quad+(\eta(X)+\tau w(X))g(\nabla_Z U,Y)+g(\nabla_Z U,X)(\eta(Y)+\tau w(Y))\\
&\quad+u(X)g(\nabla_Z\xi,Y)+g(\nabla_Z\xi,X)u(Y)\\
&\quad+(Z\tau)(u(X)w(Y)+u(Y)w(X))\\
&\quad+\tau(u(X)g(\nabla_Z W,Y)+g(\nabla_Z W,X)u(Y)),
\end{aligned}$$

which connected to (2.4) and (3.11)

$$\begin{aligned}
&2g((\nabla_Y A)\phi X,Z)+2c(\eta(Z)g(X,Y)-\eta(X)g(Y,Z))\\
&+2\eta(X)g(A^2Z,Y)-2\eta(AX)g(AZ,Y)\\
&+(g(\nabla_Z U,X)-g(\nabla_X U,Z))(\eta(Y)+\tau w(Y))\\
&+(g(\nabla_Y U,X)-g(\nabla_X U,Y))(\eta(Z)+\tau w(Z))\\
&+(g(\nabla_Z U,Y)+g(\nabla_Y U,Z))(\eta(X)+\tau w(X))\\
&+(g(\nabla_Z\xi,X)-g(\nabla_X\xi,Z))u(Y)+(g(\nabla_Y\xi,X)-g(\nabla_X\xi,Y))u(Z)\\
&+(g(\nabla_Z\xi,Y)+g(\nabla_Y\xi,Z))u(X)+(Y\tau)(u(X)w(Z)+u(Z)w(X))\\
&+(Z\tau)(u(X)w(Y)+u(Y)w(X))-(X\tau)(u(Y)w(Z)+u(Z)w(Y))\\
&+\tau\{u(X)(g(\nabla_Z W,Y)+g(\nabla_Y W,Z))\\
&+u(Z)(g(\nabla_X W,Y)-g(\nabla_Y W,X))\\
&+u(Y)(g(\nabla_Z W,X)-g(\nabla_X W,Z))\}=0.
\end{aligned}\tag{3.12}$$

If we put $X = \xi$ in (3.12), then we have

$$\begin{aligned} &g(\nabla_Y U, Z) + g(\nabla_Z U, Y) + 2c(\eta(Z)\eta(Y) - g(Z, Y)) \\ &+2g(A^2Y, Z) - 2\alpha g(AY, Z) - du(\xi, Z)(\eta(Y) + \tau w(Y)) \\ &-du(\xi, Y)(\eta(Z) + \tau w(Z)) - 2u(Y)u(Z) \\ &-(\xi\tau)(u(Y)w(Z) + u(Z)w(Y)) \\ &-\tau\{u(Z)dw(\xi, Y) + u(Y)dw(\xi, Z)\} = 0, \end{aligned} \tag{3.13}$$

where d denotes the operator of the exterior derivative.

4. Real Hypersurfaces Satisfying $R_\xi\phi = \phi R_\xi$ and $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$

We will continue our discussions under the same hypothesis $R_\xi\phi = \phi R_\xi$ as in Section 3. Further, suppose that $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ and then $\nabla_W R_\xi = 0$ since we assume that $\mu \neq 0$. In the following, arguments discussed on [6] are reviewed. Replacing X by W in (2.17), we find

$$\begin{aligned} &(W\alpha)g(AY, Z) - c\{\eta(Z)g(\phi AW, Y) + \eta(Y)g(\phi AW, Z)\} \\ &+\alpha g((\nabla_W A)Y, Z) - \eta(AZ)\{g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)\} \\ &-\eta(AY)\{g((\nabla_W A)\xi, Z) + g(A\phi AW, Z)\} = 0 \end{aligned} \tag{4.1}$$

by virtue of $\nabla_W R_\xi = 0$. Putting $Y = \xi$ in this and making use of (2.13) and (3.6), we obtain

$$\alpha A\phi AW + c\phi AW = 0 \tag{4.2}$$

because U and W are mutually orthogonal. From this and (2.16), it is seen that $R_\xi \phi AW = 0$ by virtue of (3.6), and hence $R_\xi AW = 0$ which together with (2.16) implies that

$$\alpha A^2 W = -cAW + c\mu\xi + \mu(\alpha + g(AW, W))A\xi, \tag{4.3}$$

which unables us to obtain

$$\alpha g(A^2W, W) = (\mu^2 - c)g(AW, W) + \alpha\mu^2. \tag{4.4}$$

Since $\alpha \neq 0$, $\beta = \alpha\lambda$ and (3.2), it is clear that

$$g(A^2W, W) = \left(\lambda - \alpha - \frac{c}{\alpha}\right) g(AW, W) + \mu^2. \tag{4.5}$$

Combining (3.5) to (4.2), we get

$$\alpha A^2 U = -(\mu^2 + c)AU - c(\lambda - \alpha)U. \tag{4.6}$$

If we apply μW to (3.3) and make use of (2.6), then we find

$$g(AU, U) = \mu^2(g(AW, W) + \alpha - \lambda). \tag{4.7}$$

Using (4.2), we see from (4.1)

$$\alpha(\nabla_W A)X = -(W\alpha)AX + \eta(AX)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi - \frac{c}{\alpha}\mu(w(X)\phi AW + g(\phi AW, X)W)$$

for any vector field X, which together with (3.5) yields

$$\alpha(\nabla_W A)X = -(W\alpha)AX + \eta(AX)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi - \frac{c}{\alpha}\{w(X)AU + u(AX)W + (\lambda - \alpha)(w(X)U + u(X)W)\}. \tag{4.8}$$

Now, if we put $X = W$ in (2.12), and make use of (3.5) and (4.2), then we find

$$(\nabla_W A)\xi = -\phi\nabla_W U + (W\alpha)\xi + \frac{1}{\mu}\left(\alpha + \frac{c}{\alpha}\right)\{AU + (\lambda - \alpha)U)\}. \tag{4.9}$$

Also, if we take the inner product (2.12) with $A\xi$ and take account of (2.6), (3.2) and (3.4), then we obtain

$$\alpha(X\alpha) + \mu(X\mu) = g(\alpha\xi + \mu W, (\nabla_X A)\xi) - g(A^2U + \lambda AU, X),$$

which connected to (2.4), (2.13) and (4.6) yields

$$\mu(\nabla_W A)\xi = -\left(\alpha + \frac{c}{\alpha}\right)AU - \frac{c}{\alpha}(\lambda + \alpha)U + \mu\nabla\mu. \tag{4.10}$$

If we take the inner product (4.10) with ξ and make use of (2.13), then we find

$$W\alpha = \xi\mu \tag{4.11}$$

because AU and W are mutually orthogonal. Using (4.10), we can write (4.8) as

$$\begin{aligned}
&\alpha(\nabla_W A)X + (W\alpha)AX \\
&+\frac{1}{\mu}\eta(AX)\left\{\left(\alpha + \frac{c}{\alpha}\right)AU + \frac{c}{\alpha}(\lambda + \alpha)U - \mu\nabla\mu\right\} \\
&+\frac{1}{\mu}\left\{\left(\alpha + \frac{c}{\alpha}\right)u(AX) + \frac{c}{\alpha}(\lambda + \alpha)u(X) - \mu(X\mu)\right\}A\xi \\
&+\frac{c}{\alpha}\{w(X)AU + u(AX)W + (\lambda - \alpha)(w(X)U + u(X)W)\} = 0.
\end{aligned} \tag{4.12}$$

Putting $X = W$ in this, we get

$$\alpha(\nabla_W A)W + (W\alpha)AW - (W\mu)A\xi + \left(\alpha + \frac{2c}{\alpha}\right)AU + \frac{2c\lambda}{\alpha}U - \mu\nabla\mu = 0. \tag{4.13}$$

Combining (4.9) to (4.10), we obtain

$$\mu\phi\nabla_W U - \mu(W\alpha)\xi + \mu\nabla\mu = 2\left(\alpha + \frac{c}{\alpha}\right)AU + \left(\mu^2 + \frac{2c}{\alpha}\lambda\right)U.$$

If we apply ϕ to this and make use of (2.8), (2.11) and (3.3), then we find

$$\begin{aligned}&-\mu\nabla_W U-\mu^2 g(AW,W)\xi+\mu\phi\nabla\mu\\&=2\left(\alpha+\frac{c}{\alpha}\right)(\lambda A\xi-A^2\xi)-\mu\left(\mu^2+\frac{2c}{\alpha}\lambda\right)W,\end{aligned}$$

which together with (2.6) yields

$$\begin{aligned}\mu\nabla_W U=\ &\mu\phi\nabla\mu+(2c-\mu^2)A\xi+2\mu\left(\alpha+\frac{c}{\alpha}\right)AW\\&-(\alpha\mu^2+2c\lambda+\mu^2 g(AW,W))\xi.\end{aligned}\tag{4.14}$$

Now, we can take a orthonormal frame field $\{e_0=\xi,\ e_1=W,\ e_2,\ldots,e_n,\ e_{n+1}=\phi e_1=(1/\mu)U,\ e_{n+2}=\phi e_2,\ldots,e_{2n}=\phi e_n\}$ of M. Differentiating (2.6) covariantly and making use of (2.1), we find

$$(\nabla_X A)\xi+A\phi AX=(X\alpha)\xi+\alpha\phi AX+(X\mu)W+\mu\nabla_X W,\tag{4.15}$$

which implies

$$\mu\mathrm{div}W=\mu\sum_{i=0}^{2n}g(\nabla_{e_i}W,e_i)=\xi h-\xi\alpha-W\mu.\tag{4.16}$$

Taking the inner product with Y to (4.15) and taking the skew-symmetric part, we have

$$\begin{aligned}&-2cg(\phi X,Y)+2g(A\phi AX,Y)\\&=(X\alpha)\eta(Y)-(Y\alpha)\eta(X)+\alpha g((\phi A+A\phi)X,Y)\\&\quad+(X\mu)w(Y)-(Y\mu)w(X)\\&\quad+\mu(g(\nabla_X W,Y)-g(\nabla_Y W,X)).\end{aligned}\tag{4.17}$$

Replacing X by ξ in this and using (2.10) and (4.11), we have

$$\mu\nabla_\xi W=3AU-\alpha U+\nabla\alpha-(\xi\alpha)\xi-(W\alpha)W.\tag{4.18}$$

Putting $X=\mu W$ in (4.15) and taking account of (4.10), we get

$$\begin{aligned}&-\left(\alpha+\frac{c}{\alpha}\right)AU-\frac{c}{\alpha}(\lambda+\alpha)U+\mu\nabla\mu+\mu A\phi AW\\&=\mu(W\alpha)\xi+\mu(W\mu)W+\mu\alpha\phi AW+\mu^2\nabla_W W,\end{aligned}$$

or, using (3.5) and (4.2),

$$\mu^2\nabla_W W=-2\left(\alpha+\frac{c}{\alpha}\right)AU-\left(\mu^2+\frac{2c}{\alpha}\lambda\right)U+\mu\nabla\mu-\mu(W\alpha)\xi-\mu(W\mu)W.\tag{4.19}$$

Now, putting $X = U$ in (4.17) and making use of (2.6) and (3.3), we have

$$\begin{aligned}&\mu(g(\nabla_U W, Y) - g(\nabla_Y W, U))\\&= (2c\mu - U\mu)w(Y) - (U\alpha)\eta(Y)\\&\quad + \mu^2\eta(AY) + 2\lambda\mu w(AY) - 2\mu w(A^2 Y),\end{aligned}$$

which together with (4.3) gives

$$\begin{aligned}\mu dw(U,Y) = &\ (2c\mu - U\mu)w(Y) - \{U\alpha + 2c(\lambda-\alpha)\}\eta(Y)\\&-\{\mu^2 + 2(\lambda-\alpha)g(AW,W)\}\eta(AY) + 2\mu\left(\lambda + \frac{c}{\alpha}\right)w(AY).\end{aligned} \tag{4.20}$$

Because of (2.10) and (4.18), it is seen that

$$\mu dw(\xi, X) = 2u(AX) - \alpha u(X) - (\xi\alpha)\eta(X) - (W\alpha)w(X) + X\alpha. \tag{4.21}$$

Using (2.11) and (3.7), we obtain

$$du(\xi, X) = (3\lambda - 2\alpha)\eta(AX) - 2\mu w(AX) - \alpha\lambda\eta(X) + g(\phi\nabla\alpha, X). \tag{4.22}$$

Using above two equations, (3.13) is reduced to

$$\begin{aligned}&g(\nabla_X U, Y) + g(\nabla_Y U, X))\\&= 2c(g(X,Y) - \eta(X)\eta(Y)) - 2g(A^2X, Y) + 2\alpha g(AX, Y)\\&\quad +(\xi\tau)(u(X)w(Y) + u(Y)w(X))\\&\quad +\frac{1}{\alpha}\{2u(AX) + X\alpha - (\xi\alpha)\eta(X) - (W\alpha)w(X)\}u(Y)\\&\quad +\frac{1}{\alpha}\{2u(AY) + Y\alpha - (\xi\alpha)\eta(Y) - (W\alpha)w(Y)\}u(X)\\&\quad +\{(3\lambda - 2\alpha)\eta(AX) - 2\mu w(AX)\\&\qquad -\alpha\lambda\eta(X) + g(\phi\nabla\alpha, X)\}(\eta(Y) + \tau w(Y))\\&\quad +\{(3\lambda - 2\alpha)\eta(AY) - 2\mu w(AY)\\&\qquad -\alpha\lambda\eta(Y) + g(\phi\nabla\alpha, Y)\}(\eta(X) + \tau w(X)),\end{aligned} \tag{4.23}$$

where we have used (4.21) and (4.22). Taking the trace of this and using (4.7), we find

$$\mathrm{div}U = 2c(n-1) + \alpha h - \mathrm{Tr}A^2 + \lambda(\lambda - \alpha). \tag{4.24}$$

Replacing X by U in (4.23) and using (4.6) and (4.7), we find

$$\begin{aligned}&g(\nabla_U U, Y) + g(\nabla_Y U, U)\\&= (\lambda-\alpha)(Y\alpha) + 2\left(2\lambda - \alpha + \frac{c}{\alpha}\right)u(AY)\\&\quad + \left\{\frac{U\alpha}{\alpha} + \frac{2c\lambda}{\alpha} + 2(\lambda-\alpha)(g(AW,W) + \alpha - \lambda)\right\}u(Y)\\&\quad + \{\mu(W\alpha) - (\lambda-\alpha)\xi\alpha\}\eta(Y) + \mu^2(\xi\tau)w(Y).\end{aligned}$$

Since $g(\nabla_X U, U) = \mu(X\mu)$, it follows that

$$\begin{aligned} du(U,X) = & -2\mu(X\mu) + (\lambda-\alpha)(X\alpha) + 2\left(2\lambda - \alpha + \frac{c}{\alpha}\right)u(AX) \\ & + \left\{\frac{U\alpha}{\alpha} + \frac{2c\lambda}{\alpha} + 2(\lambda-\alpha)(g(AW,W)+\alpha-\lambda)\right\}u(X) \\ & + \{\mu(W\alpha) - (\lambda-\alpha)\xi\alpha\}\eta(X) + \mu^2(\xi\tau)w(X), \end{aligned} \tag{4.25}$$

which implies that

$$du(U,W) = -2\mu(W\mu) + (\lambda-\alpha)W\alpha + \mu^2(\xi\tau). \tag{4.26}$$

Because of (2.1), (2.11) and (3.3), it is seen that

$$d\eta(U,X) = (\lambda-\alpha)\eta(AX) - 2\mu w(AX).$$

Putting $Z = U$ in (3.12) and using this and (2.4), we obtain

$$\begin{aligned} & -2\mu g((\nabla_W A)Y, X) - 2c(\eta(Y)u(X) + \eta(X)u(Y)) \\ & - du(U,X)(\eta(Y) + \tau w(Y)) - du(U,Y)(\eta(X) + \tau w(X)) \\ & + \mu^2((X\tau)w(Y) + (Y\tau)w(X)) \\ & - (U\tau)(u(X)w(Y) + u(Y)w(X)) \\ & - \{(\lambda-\alpha)\eta(AX) - 2\mu w(AX)\}u(Y) \\ & - \{(\lambda-\alpha)\eta(AY) - 2\mu w(AY)\}u(X) \\ & + \mu^2(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)) \\ & + \tau\{\mu^2(g(\nabla_X W, Y) + g(\nabla_Y W, X)) \\ & \quad - dw(U,Y)u(X) - dw(U,X)u(Y)\} = 0. \end{aligned}$$

Substituting (4.20) into this, we obtain

$$\begin{aligned} & 2\mu g((\nabla_W A)Y, X) \\ & = -2c(\eta(Y)u(X) + \eta(X)u(Y)) - du(U,X)(\eta(Y) + \tau w(Y)) \\ & \quad - du(U,Y)(\eta(X) + \tau w(X)) + \mu^2((X\tau)w(Y) + (Y\tau)w(X)) \\ & \quad - (U\tau)(u(X)w(Y) + u(Y)w(X)) \\ & \quad - \{(\lambda-\alpha)\eta(AX) - 2\mu w(AX)\}u(Y) \\ & \quad - \{(\lambda-\alpha)\eta(AY) - 2\mu w(AY)\}u(X) \\ & \quad + \mu^2(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)) \\ & \quad + \tau\mu^2(g(\nabla_X W, Y) + g(\nabla_Y W, X)) \\ & \quad - \frac{1}{\alpha}u(X)\Big\{(2c\mu - U\mu)w(Y) - (U\alpha + 2c(\lambda-\alpha))\eta(Y) \\ & \qquad - \{\mu^2 + 2(\lambda-\alpha)g(AW,W)\}\eta(AY) + 2\mu\left(\lambda + \frac{c}{\alpha}\right)w(AY)\Big\} \\ & \quad - \frac{1}{\alpha}u(Y)\Big\{(2c\mu - U\mu)w(X) - \{U\alpha + 2c(\lambda-\alpha)\}\eta(X) \\ & \qquad - \{\mu^2 + 2(\lambda-\alpha)g(AW,W)\}\eta(AX) + 2\mu\left(\lambda + \frac{c}{\alpha}\right)w(AX)\Big\}. \end{aligned}$$

Combining this to (4.12), we have

$$
\begin{aligned}
&-2\mu(W\alpha)g(AY,X)\\
&+2\eta(AY)\Big\{-\Big(\alpha+\frac{c}{\alpha}\Big)u(AX)-\frac{c}{\alpha}(\alpha+\lambda)u(X)+\mu X\mu\Big\}\\
&+2\Big\{-\Big(\alpha+\frac{c}{\alpha}\Big)u(AY)-\frac{c}{\alpha}(\alpha+\lambda)u(Y)+\mu(Y\mu)\Big\}\eta(AX)\\
&-\frac{2c\mu}{\alpha}\{u(AX)w(Y)+u(AY)w(X)\\
&\quad+(\lambda-\alpha)(w(X)u(Y)+w(Y)u(X))\}\\
&=-2\alpha c(\eta(Y)u(X)+\eta(X)u(Y))-\alpha du(U,X)(\eta(Y)+\tau w(Y))\\
&\quad-\alpha du(U,Y)(\eta(X)+\tau w(X))+\alpha\mu^2((X\tau)w(Y)+(Y\tau)w(X))\\
&\quad-\alpha(U\tau)(u(X)w(Y)+u(Y)w(X))-\mu^2(\eta(AX)u(Y)+\eta(AY)u(X))\\
&\quad+2\alpha\mu(w(AY)u(X)+w(AX)u(Y))+\alpha\mu^2(g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X))\\
&\quad+\mu^3(g(\nabla_XW,Y)+g(\nabla_YW,X))\\
&\quad-u(X)\Big\{(2c\mu-U\mu)w(Y)-(U\alpha+2c(\lambda-\alpha))\eta(Y)\\
&\qquad-(\mu^2+2(\lambda-\alpha)g(AW,W))\eta(AY)+2\mu\Big(\lambda+\frac{c}{\alpha}\Big)w(AY)\Big\}\\
&\quad-u(Y)\Big\{(2c\mu-U\mu)w(X)-(U\alpha+2c(\lambda-\alpha))\eta(X)\\
&\qquad-(\mu^2+2(\lambda-\alpha)g(AW,W))\eta(AX)+2\mu\Big(\lambda+\frac{c}{\alpha}\Big)w(AX)\Big\}.
\end{aligned}
\tag{4.27}
$$

If we put $Y=W$ in (4.27) and take account of (2.1), (3.5) and (4.19), then we find

$$
\begin{aligned}
&-2\mu(W\alpha)w(AX)+\mu^2(X\mu)\\
&+2\mu(W\mu)\eta(AX)-\frac{2c\mu}{\alpha}\{u(AX)+(\lambda-\alpha)u(X)\}\\
&=-\mu du(U,X)-\alpha du(U,W)(\eta(X)+\tau w(X))\\
&\quad+\alpha\mu^2((X\tau)+(W\tau)w(X))\\
&\quad-\mu^2\{(W\alpha)\eta(X)+(W\mu)w(X)\}\\
&\quad+\Big(U\mu-\alpha(U\tau)-\frac{2c}{\alpha}\mu g(AW,W)\Big)u(X),
\end{aligned}
$$

or, using (4.25) and (4.26)

$$
\begin{aligned}
&2\mu(W\alpha)AW-2c\mu U+\{\mu(\lambda-\alpha)\xi\alpha-3\mu^2W\alpha-\alpha\mu^2(\xi\tau)\}\xi\\
&-\{\mu^2(W\mu)+\tau\mu^2(W\alpha)+2\mu^3(\xi\tau)\}W+\mu^2\nabla\mu-\mu(\lambda-\alpha)\nabla\alpha\\
&-2\mu(2\lambda-\alpha)AU-\mu\Big\{\frac{U\alpha}{\alpha}+2(\lambda-\alpha)g(AW,W)-2(\lambda-\alpha)^2\Big\}U\\
&+\alpha\mu^2((W\tau)W+\nabla\tau)+\Big\{U\mu-\alpha(U\tau)-\frac{2c\mu}{\alpha}g(AW,W)\Big\}U=0.
\end{aligned}
$$

By the way, since $\alpha\tau = \mu$, we find

$$\alpha\mu\nabla\tau = \mu\nabla\mu - (\lambda - \alpha)\nabla\alpha. \tag{4.28}$$

Using this, above equation is reduced to

$$\begin{aligned}
&\mu\nabla\mu - (\lambda - \alpha)\nabla\alpha \\
&= (2\lambda - \alpha)AU + \left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W) - (\lambda - \alpha)^2 + c\right\} U \\
&\quad -(W\alpha)AW + \{2\mu(W\alpha) - (\lambda - \alpha)\xi\alpha\}\xi + (\lambda - \alpha)(2W\alpha - \tau(\xi\alpha))W.
\end{aligned} \tag{4.29}$$

If we take the inner product (4.29) with W, then we get

$$\mu(W\mu) = \{3(\lambda - \alpha) - g(AW, W)\}W\alpha - \tau(\lambda - \alpha)\xi\alpha. \tag{4.30}$$

Also, taking the inner product (4.29) with U and making use of (4.7), we obtain

$$\frac{U\mu}{\mu} - \frac{U\alpha}{\alpha} = \left(3\lambda - 2\alpha + \frac{c}{\alpha}\right) g(AW, W) + (\lambda - \alpha)(2\alpha - 3\lambda) + c. \tag{4.31}$$

On the other hand, replacing Y by W in (4.23) and using (4.3) and (4.14), we find

$$\begin{aligned}
&g(\nabla_X U, W) + g(\phi\nabla\mu, X) - \frac{\lambda - \alpha}{\mu} g(\phi\nabla\alpha, X) \\
&-(\xi\tau)u(X) + 2(\lambda - \alpha)w(AX) \\
&+\left\{\frac{U\alpha}{\alpha} + (\lambda - \alpha)(5\alpha - 6\lambda + 4g(AW, W))\right\} w(X) \\
&+\left\{\frac{U\alpha}{\mu} + \mu(4\alpha - 5\lambda + 3g(AW, W))\right\} \eta(X) = 0.
\end{aligned}$$

By the way, applying (4.29) by ϕ and making use of (2.6), (3.3) and (3.5), we get

$$\begin{aligned}
&\mu\phi\nabla\mu - (\lambda - \alpha)\phi\nabla\alpha \\
&= -\frac{1}{\mu}(W\alpha)AU + \mu(\xi\tau)U + \mu^2(2\lambda - \alpha)\xi - \mu(2\lambda - \alpha)AW \\
&\quad - \mu\left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W) - (\lambda - \alpha)(3\lambda - 2\alpha) + c\right\} W.
\end{aligned}$$

Substituting this into the last equation, we find

$$
\begin{aligned}
&g(\nabla_X U, W)\\
&= \frac{W\alpha}{\mu^2}u(AX) + \alpha w(AX)\\
&\quad + \Big\{3(\lambda-\alpha)^2 + \frac{c}{\alpha}g(AW,W) + c\\
&\qquad - \frac{U\alpha}{\alpha} - 3(\lambda-\alpha)g(AW,W)\Big\}w(X)\\
&\quad + \Big\{3\mu(\lambda-\alpha-g(AW,W)) - \frac{U\alpha}{\mu}\Big\}\eta(X).
\end{aligned} \tag{4.32}
$$

On the other hand, (4.12) turns out, using (2.4), to be

$$
\begin{aligned}
&\alpha(\nabla_X A)W\\
&= \frac{c\alpha}{\mu}(\eta(X)U + 2u(X)\xi) - (W\alpha)AX\\
&\quad + \frac{1}{\mu}\eta(AX)\Big\{\mu\nabla\mu - \Big(\alpha+\frac{c}{\alpha}\Big)AU - \frac{c}{\alpha}(\lambda+\alpha)U\Big\}\\
&\quad + \frac{1}{\mu}\Big\{\mu(X\mu) - \Big(\alpha+\frac{c}{\alpha}\Big)u(AX) - \frac{c}{\alpha}(\lambda+\alpha)u(X)\Big\}A\xi\\
&\quad - \frac{c}{\alpha}\{w(X)AU + u(AX)W + (\lambda-\alpha)(u(X)W + w(X)U)\}.
\end{aligned}
$$

If we apply by ϕ to this and make use of (3.3), then we find

$$
\begin{aligned}
&-\alpha\phi(\nabla_X A)W = (W\alpha)\phi AX + c\alpha\eta(X)W - (X\mu)U\\
&+\frac{1}{\mu}\eta(AX)\Big\{\Big(\alpha+\frac{c}{\alpha}\Big)\{(\lambda-\alpha)A\xi - \mu AW\} - \frac{c}{\alpha}\mu(\lambda+\alpha)W - \mu\phi\nabla\mu\Big\}\\
&+\frac{1}{\mu}\Big\{\Big(\alpha+\frac{c}{\alpha}\Big)u(AX) + \frac{2c\lambda}{\alpha}u(X)\Big\}U + \frac{c}{\alpha}w(X)(\mu^2\xi - \mu AW).
\end{aligned} \tag{4.33}
$$

Now, if we put $Z = W$ in (3.12), then we find

$$
\begin{aligned}
&2g(\phi(\nabla_Y A)W, X)\\
&= 2\{(w(A^2Y) - cw(Y))\eta(X) - w(AY)\eta(AX)\}\\
&\quad + du(W,X)(\eta(Y) + \tau w(Y)) + \tau du(Y,X)\\
&\quad + (W\tau)(w(Y)u(X) + w(X)u(Y))\\
&\quad + (g(\nabla_W U, Y) + g(\nabla_Y U, W))(\eta(X) + \tau w(X))\\
&\quad + \frac{2}{\mu}\{u(AX) + (\lambda-\alpha)u(X)\}u(Y)\\
&\quad + (Y\tau)u(X) - (X\tau)u(Y) + \tau(u(Y)g(\nabla_W W, X)\\
&\quad + u(X)g(\nabla_W W, Y)).
\end{aligned}
$$

Using (2.1), (2.10), (3.5), (3.8) and (4.33), we can write the above equation as

$$\begin{aligned}
&\mu du(X,Y)\\
&= (W\alpha)g((\phi A + A\phi)X,Y) + \frac{2c}{\alpha}\mu(w(X)w(AY) - w(Y)w(AX))\\
&\quad + \eta(AX)g(\phi\nabla\mu, Y) - \eta(AY)g(\phi\nabla\mu, X)\\
&\quad + \frac{2c}{\mu\alpha}(u(X)u(AY) - u(Y)u(AX)) - (X\mu)u(Y) + (Y\mu)u(X)\\
&\quad + \alpha((X\tau)u(Y) - (Y\tau)u(X))\\
&\quad + g(\nabla_Y U, W)\eta(AX) - g(\nabla_X U, W)\eta(AY)\\
&\quad + \{2c\alpha - 2c\lambda - \mu^2(\alpha + g(AW,W))\}(\eta(X)w(Y) - \eta(Y)w(X)),
\end{aligned}$$

which together with (4.28) and (4.32) yields

$$\begin{aligned}
&\mu du(X,Y)\\
&= (W\alpha)g((\phi A + A\phi)X,Y) + \frac{2c\mu}{\alpha}\{w(X)w(AY) - w(Y)w(AX)\}\\
&\quad + \frac{W\alpha}{\mu^2}\{\eta(AX)u(AY) - \eta(AY)u(AX)\}\\
&\quad + \eta(AX)g(\phi\nabla\mu, Y) - \eta(AY)g(\phi\nabla\mu, X)\\
&\quad + \alpha\{\eta(AX)w(AY) - \eta(AY)w(AX)\}\\
&\quad + \frac{2c}{\mu\alpha}(u(X)u(AY) - u(Y)u(AX)) + \frac{\mu}{\alpha}((X\alpha)u(Y) - (Y\alpha)u(X))\\
&\quad + \{(\mu^2 + c)g(AW,W) + \alpha\mu^2 - c\alpha + 2c\lambda\}(\eta(X)w(Y) - \eta(Y)w(X)).
\end{aligned} \tag{4.34}$$

Putting $X = \phi e_i$ and $Y = e_i$ in this and summing up for $i = 1, 2, \cdots, n$, we obtain

$$\mu \sum_{i=0}^{2n} du(\phi e_i, e_i) = (h - \alpha - g(AW,W))W\alpha - \mu(W\mu),$$

where we have used (2.6)–(2.8), (3.5) and (4.7). Taking the trace of (2.12), we obtain

$$\sum_{i=0}^{2n} g(\phi\nabla_{e_i} U, e_i) = \xi\alpha - \xi h.$$

Thus, it follows that

$$\mu(\xi h - \xi\alpha) = \mu(W\mu) + (g(AW,W) + \alpha - h)W\alpha, \tag{4.35}$$

which together with (4.16) gives

$$\mu^2(\mathrm{div}W) = (g(AW,W) + \alpha - h)W\alpha. \tag{4.36}$$

We notice here that

Remark 4.1. If $AU = \rho U$ for some function ρ on Ω, then $AW \in \mathrm{span}\{\xi, W\}$ on Ω, where $\mathrm{span}\{\xi, W\}$ is a linear subspace spanned by ξ and W.

In fact, because of the hypothesis $AU = \rho U$, (3.5) reformed as

$$\mu\phi AW = (\rho + \lambda - \alpha)U,$$

which implies that $AW = \mu\xi + (\rho + \lambda - \alpha)W \in \mathrm{span}\{\xi, W\}$.

In the previous paper [6], it is proved that

Lemma 4.2. *Let M be a real hypersurface of $M_n(c), c \neq 0$ which satisfies $R_\xi \phi = \phi R_\xi$ and $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. If $AW \in \mathrm{span}\{\xi, W\}$, then $\Omega = \emptyset$.*

The sketch of proof. Since $AW \in \mathrm{span}\{\xi, W\}$, (3.5) becomes

$$AU = (g(AW, W) + \alpha - \lambda)U. \tag{4.37}$$

From (4.2) we also have

$$g(AW, W)(\alpha AU + cU) = 0.$$

Now, suppose that $g(AW, W) \neq 0$ on Ω. Then we have $\alpha AU + cU = 0$ on this subset, which together with (4.37) gives

$$\mu^2 = \alpha g(AW, W) + c.$$

Because of (2.6), (2.16) and this fact, we verified that $R_\xi A\xi = 0$ on the subset, which together with (3.1) and the fact $\alpha AU + cU = 0$ implies that

$$R_\xi(\phi A - A\phi) = 0.$$

Owing to Theorem 1.1, we conclude that $A\xi = \alpha\xi$, a contradiction. Therefore we see that $g(AW, W) = 0$ on Ω. So we have

$$AW = \mu\xi. \tag{4.38}$$

Hence (4.37) reformed as

$$AU = (\alpha - \lambda)U. \tag{4.39}$$

Differentiating (4.38) covariantly, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi,$$

which together with (2.4), (2.10) and (4.39) gives

$$(\nabla_W A)W = 2(\lambda - \alpha)U. \tag{4.40}$$

Using this, (4.38) and (4.39), we can write (4.13) as

$$\mu(X\mu) = \mu(W\alpha)\eta(X) + (\mu^2 + 2c)u(X). \tag{4.41}$$

Differentiating this covariantly and taking account of (2.1) and (2.4), we find

$$\begin{aligned} &Y(\mu(W\alpha))\eta(X) - X(\mu(W\alpha))\eta(Y) \\ &+(\mu(W\alpha))g((\phi A + A\phi)Y, X) \\ &+2\mu(W\alpha)(\eta(Y)u(X) - \eta(X)u(Y)) \\ &+(\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0, \end{aligned} \tag{4.42}$$

which together with (2.8), (2.10), (4.34), (4.38), (4.39) and (4.40) implies that $W\alpha = 0$. Thus, (4.41) and (4.42) becomes respectively to

$$\mu\nabla\mu = (\mu^2 + 2c)U, \tag{4.43}$$

$$(\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0. \tag{4.44}$$

Using (4.38), (4.39) and these, we see from (4.34) that $(\mu^2 + 2c)(\mu^2 + c) = 0$, which shows that $\mu^2 + 2c = 0$ on Ω. Therefore, (4.29) can be written as

$$\mu^2\nabla\alpha = \{\mu^2(3\lambda - 2\alpha) - c\alpha\}U,$$

where we have used (4.38), (4.39) and the face that μ is constant on Ω.

As in the same method as those used from (4.41) to derive (4.44), we can deduce from the last equation that

$$\{\mu^2(3\lambda - \alpha) - c\alpha\}(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0.$$

If $g(\nabla_Y U, X) - g(\nabla_X U, Y) = 0$, then similarly as above we have a contradiction. Hence we see that $\mu^2(3\lambda - 2\alpha) - c\alpha = 0$, which together with $\mu^2 + 2c = 0$ yields $2\lambda - \alpha = 0$. that is, $2\mu^2 + \alpha^2 = 0$, a contradiction. This completes the proof. □

5. Constant Mean Curvature

We will continue our discussions under the same hypotheses as those stated in Section 4. Further we assume that mean curvature of the hypersurface M in $M_n(c), c \neq 0$ is constant. Then h is constant. So (4.35) becomes

$$\mu(\xi\alpha + W\mu) = (h - \alpha - g(AW, W))W\alpha. \tag{5.1}$$

Taking the trace of (4.12), we also have $h(W\alpha) - 2g(A\xi, \nabla\mu) = 0$, which together with (2.6) and (4.11) gives

$$2\mu(W\mu) = (h - 2\alpha)W\alpha. \tag{5.2}$$

Combining the last two equations, we obtain

$$\mu(\xi\alpha) = \left(\frac{h}{2} - g(AW, W)\right)W\alpha. \tag{5.3}$$

From (3.2) we have

$$2\mu(\nabla\mu) = \alpha(\nabla\lambda) + (\lambda - 2\alpha)\nabla\alpha, \tag{5.4}$$

which tells us that $2\mu(W\alpha) = \alpha(\xi\lambda) + (\lambda - 2\alpha)\xi\alpha$, which connected to (5.3) implies that

$$\alpha\mu(\xi\lambda) = \left\{2\alpha\lambda - 2\alpha^2 - \frac{h}{2}\lambda + (\lambda - 2\alpha)g(AW, W)\right\} W\alpha. \tag{5.5}$$

Combining (4.30) to (5.2) and (5.3), we get

$$\left\{(2\alpha - \lambda)g(AW, W) - 3\alpha\lambda + 2\alpha^2 + \frac{h\lambda}{\alpha}\right\} W\alpha = 0. \tag{5.6}$$

On the other hand, putting $X = Y = e_i$ in (3.12) and summing up for $i = 0, 1, \cdots, 2n$ and using (2.1) and (2.4), we find

$$\begin{aligned}
&\phi\nabla\alpha + \{2c(n-1) - \mathrm{Tr}A^2\}\eta(Z) + h\eta(AZ) \\
&= g(\nabla_\xi U, Z) - g(\nabla_Z U, \xi) + \tau(g(\nabla_W U, Z) + g(\nabla_U W, Z)) \\
&\quad + (\mathrm{div}U)(\eta(Z) + \tau w(Z)) + g((\phi A + A\phi)U, Z) \\
&\quad + (W\tau)u(Z) + (U\tau)w(Z) + \tau(\mathrm{div}W)u(Z),
\end{aligned}$$

where we have used h is constant, or using (2.10), (3.3), (3.7) and (4.24)

$$\begin{aligned}
&\phi\nabla\alpha + \frac{\mu}{\alpha}(\nabla_W U + \nabla_U W) - 4\mu AW + (W\tau + \tau(\mathrm{div}W))\,U \\
&+ (U\tau + \tau(\mathrm{div}U) + \mu(4\lambda - 3\alpha - h))\,W \\
&+(\lambda - \alpha)(\lambda + 3\alpha)\xi = 0.
\end{aligned} \tag{5.7}$$

By the way, combining (4.20) to (4.33) and taking account of (4.32), we find

$$\begin{aligned}
\nabla_U W =&\frac{1}{\mu}\{\mu\phi\nabla\mu - (\lambda - \alpha)\phi\nabla\alpha\} - (\xi\tau)U + 2\left(2\lambda - \alpha + \frac{c}{\alpha}\right) AW \\
&+ \left\{(\lambda - \alpha)(2\alpha - 3\lambda) + c - \left(\lambda + \frac{c}{\alpha}\right) g(AW, W)\right\} W \\
&+ \mu\left\{g(AW, W) + 3\alpha - 5\lambda - \frac{2c}{\alpha}\right\}\xi.
\end{aligned}$$

Substituting this and (4.14) into (5.7), we obtain

$$\begin{aligned}
&\alpha\phi\nabla\alpha + 2\mu\phi\nabla\mu - (\lambda - \alpha)\phi\nabla\alpha \\
&= 4\mu\left(\alpha - \lambda - \frac{c}{\alpha}\right) AW + (\mu(\xi\tau) - \alpha(W\tau) - \mu(\mathrm{div}W))\,U \\
&\quad - 4(\lambda - \alpha)(\mu^2 + c)\xi - \tau f W
\end{aligned}$$

for some function f on Ω. Because of (4.28), (4.36), (5.2) and (5.3), it is verified that

$$\mu\left(\mu(\xi\tau) - \alpha(W\tau) - \mu(\mathrm{div}W)\right) = (h-\lambda)W\alpha.$$

From this and (5.4), above equation can be written as

$$\alpha^2\phi\nabla\lambda = -4\mu(\mu^2+c)(AW-\mu\xi) + \frac{\alpha}{\mu}(h-\lambda)(W\alpha)U - \mu f W, \tag{5.8}$$

which together with (3.5) implies that

$$\alpha^2(\nabla\lambda - (\xi\lambda)\xi) = 4(\mu^2+c)\{AU + (\lambda-\alpha)U\} + \alpha(h-\lambda)(W\alpha)W + fU. \tag{5.9}$$

Differentiation (4.4) with respect to W gives

$$\begin{aligned}\alpha W(g(A^2W,W)) = \ & (\mu^2 - g(A^2W,W))W\alpha \\ & + 2\mu(\alpha + g(AW,W))W\mu + (\mu^2-c)W(g(AW,W)).\end{aligned} \tag{5.10}$$

By the definition of $g(AW,W)$, differentiation $g(AW,W)$ with respect to X gives

$$\alpha X(g(AW,W)) = \alpha g((\nabla_W A)W,X) + 2\alpha g(\nabla_X W, AW),$$

which together with (4.13) yields

$$\begin{aligned}\alpha X(g(AW,W)) = \ & -(W\alpha)w(AX) + (W\mu)\eta(AX) - \left(\alpha + \frac{2c}{\alpha}\right)u(AX) \\ & -\frac{2c}{\alpha}\lambda u(X) + \mu(X\mu) + 2\alpha g(\nabla_X W, AW).\end{aligned} \tag{5.11}$$

Replacing X by W in this and making use of (4.19), we find

$$\begin{aligned}W(g(AW,W)) = \ & -\left(2 + \frac{1}{\alpha}g(AW,W)\right)W\alpha + \frac{2}{\mu}g(AW,\nabla\mu) \\ & + 2\mu\left(\frac{1}{\alpha} - \frac{g(AW,W)}{\mu^2}\right)W\mu,\end{aligned}$$

which together with (5.10) implies that

$$\begin{aligned}\frac{1}{2}\alpha W(g(A^2W,W)) = \ & \left\{\left(\alpha - \lambda + \frac{c}{\alpha}\right)g(AW,W) + c - \mu^2\right\}W\alpha \\ & + \mu\left(\lambda - \frac{c}{\alpha} + \frac{c}{\mu^2}g(AW,W)\right)W\mu \\ & + \left(\mu - \frac{c}{\mu}\right)g(AW,\nabla\mu).\end{aligned} \tag{5.12}$$

On the other hand, by the definition $g(A^2W,W)$, we find

$$\frac{1}{2}\alpha X(g(A^2W,W)) = \alpha g((\nabla_X A)W, AW) + \alpha g(A^2W, \nabla_X W)$$

for any vector field X. Putting $X = W$ in this and making use of (4.3), (4.12) and (4.19), we obtain

$$\begin{aligned}&\frac{1}{2}\alpha W(g(A^2W, W))\\&= (c - g(A^2W, W))W\alpha\\&\quad + \mu\left(\alpha + g(AW, W) + \frac{c}{\mu^2}g(AW, W)\right)W\mu\\&\quad + \left(\mu - \frac{c}{\mu}\right)g(AW, \nabla\mu),\end{aligned}$$

which connected to (5.12) yields

$$\begin{aligned}&\left\{g(A^2W, W) - \left(\lambda - \alpha - \frac{c}{\alpha}\right)g(AW, W) - \mu^2\right\}W\alpha\\&+ \mu\left(\lambda - \alpha - \frac{c}{\alpha} - g(AW, W)\right)W\mu = 0.\end{aligned}$$

Because of (4.5), it follows that

$$(\mu^2 - c - \alpha g(AW, W))W\mu = 0.$$

Now, if we assume that $W\mu \neq 0$ on Ω, then we get $\alpha g(AW, W) = \mu^2 - c$ on this subset, which together with (4.5) gives

$$g(A^2W, W) = \mu^2 + (g(AW, W))^2$$

on the subset. Using these facts, it is verified that

$$||AW - \mu\xi - g(AW, W)W|| = 0$$

on the set. Consequently we have $AW = \mu\xi + g(AW, W)W$ on the subset. It is contradictory because of Lemma 4.2. Therefore we see that $W\mu = 0$ on Ω. Thus, (5.2) becomes $(h - 2\alpha)W\alpha = 0$ and hence $W\alpha = 0$ because of h is constant. Therefore it is clear that $\xi\alpha = 0$ and $\xi\lambda = 0$ by virtue of (5.2) and (5.5).

Summing up, we conclude that

Lemma 5.1. $\xi\alpha = W\alpha = \xi\mu = W\mu = \xi\lambda = W\lambda = 0$ *on* Ω *provided that the mean curvature of* M *is constant.*

According to Lemma 5.1, (4.29) and (5.9) reduced respectively to

$$\begin{aligned}&\mu\nabla\mu - (\lambda - \alpha)\nabla\alpha\\&= (2\lambda - \alpha)AU + \left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right)g(AW, W) - (\lambda - \alpha)^2 + c\right\}U,\end{aligned} \tag{5.13}$$

$$\alpha^2\nabla\lambda = 4(\mu^2 + c)AU + f_1U, \tag{5.14}$$

where we have put $f_1 = f + 4(\lambda - \alpha)(\mu^2 + c)$. Combining (5.13) and (5.14) to (5.4), we find

$$\alpha\lambda\nabla\alpha + 2(\alpha^2 - 2c)AU = f_2U, \tag{5.15}$$

where we have put

$$f_2 = f_1 - 2(\mu^2 + c)g(AW, W) + 2\alpha(\lambda - \alpha)^2 - 2c\alpha.$$

From (5.15) we see that $g(AW, \nabla\alpha) = 0$ and hence $g(AW, \nabla\mu) = 0$ by virtue of (5.13). Replacing X by ξ in (5.11) and taking account of $g(AW, \nabla\alpha) = 0$ and (4.18), it is verified that

$$\xi(g(AW, W)) = 0, \tag{5.16}$$

where we have used Lemma 5.1. Similarly we see from (4.19) and (5.11) that

$$W(g(AW, W)) = 0. \tag{5.17}$$

Applying (5.15) by ϕ and using (2.6), (2.8) and (3.3), we find

$$\alpha\lambda\phi\nabla\alpha = 2\mu(\alpha^2 - 2c)AW - 2\alpha(\lambda - \alpha)(\alpha^2 - 2c)A\xi - \mu f_2W.$$

From this and (3.7), we get

$$\begin{aligned}\alpha\lambda\nabla_\xi U = {} & \mu(2\alpha^2 - 3\alpha\lambda - 4c)AW \\ & +\{3\alpha\lambda^2 - 4\alpha^2\lambda + 2\alpha^3 + 4c(\lambda - \alpha)\}A\xi \\ & -(\alpha\lambda)^2\xi - \mu f_2W.\end{aligned} \tag{5.18}$$

On the other hand, if we combine (5.13) to (5.14), then we have

$$\alpha^2(2\lambda - \alpha)X\lambda - 4\mu(\mu^2 + c)X\mu + 4(\lambda - \alpha)(\mu^2 + c)X\alpha = f_3u(X) \tag{5.19}$$

for any vector field X, where we have put

$$f_3 = (2\lambda - \alpha)f_1 - 4(\mu^2 + c)\left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right)g(AW, W) - (\lambda - \alpha)^2 + c\right\}. \tag{5.20}$$

Differentiating (5.19) covariantly with respect to a vector field Y, and taking the skew-symmetric part, we find

$$\begin{aligned}& f_4((X\alpha)(Y\lambda) - (Y\alpha)(X\lambda)) \\ & + (Yf_3)u(X) - (Xf_3)u(Y) + f_3(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0\end{aligned}$$

for some function f_4. Replacing X by ξ in this and using (2.11) and Lemma 5.1, we obtain

$$(\xi f_3)u(Y) + f_3(\mu w(AY) + g(\nabla_\xi U, Y)) = 0,$$

which tells us that $\xi f_3 = 0$ by virtue of Lemma 5.1. Therefore, above equation implies that $f_3(\mu^2 + 2c)AW \in \mathrm{span}\{\xi, W\}$ by virtue of (5.18). Owing to Lemma 5.1, it is clear that $(\mu^2 + 2c)f_3 = 0$.

We are now going to prove $f_3 = 0$ on Ω. If not, then we have $\mu^2 + 2c = 0$ on this subset and hence μ is constant. So (5.13) reformed as

$$2c\nabla\alpha = (\alpha^2 - 4c)AU + c(2\lambda - \alpha - g(AW, W))U.$$

From (5.15) we also have

$$(\alpha^2 - 2c)(\nabla\alpha + 2AU) = f_2 U.$$

Combining the last two equations, it follows that $(\alpha^2 - 2c)AU = xU$ for some function x. Owing to Lemma 4.2 and Remark 4.1, we see that $\alpha^2 - 2c = 0$, a contradiction because of $\mu^2 + 2c = 0$. Accordingly we prove that $f_3 = 0$ and consequently it is seen that

$$(2\lambda - \alpha)f_1 = 4(\mu^2 + c)\left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W) - (\lambda - \alpha)^2 + c\right\}, \tag{5.21}$$

because of (5.20). Using $f_3 = 0$ and (5.4), we can write (5.19) as

$$\alpha(\alpha^2 - 2c)\nabla\lambda + 2\lambda(\mu^2 + c)\nabla\alpha = 0. \tag{5.22}$$

By the definition of f_2 and (5.21), it is verified that

$$\alpha(\alpha - 2\lambda)f_2 = 2(\alpha^2 - 2c)\left\{\left(\mu^2 + c\right) g(AW, W) - \alpha(\lambda - \alpha)^2 + c\alpha\right\}. \tag{5.23}$$

Finally we prepare the following lemma for later use.

Lemma 5.2. *Let* $\mathrm{span}\{\xi, W\}$ *be the linear subspace spanned by* ξ *and* W*. Then there exists* $P \in \mathrm{span}\{\xi, W\}$ *such that*

$$\begin{aligned} g(AW, \nabla_X U) =& \frac{c}{\alpha} w(A^2X) - \left\{\mu^2 + \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W)\right\} w(AX) \\ &+ g(P, X) \end{aligned}$$

provided that h *is constant.*

Proof. Putting $Y = AW$ in (4.34) and using (3.5), (4.3), (5.15) and Lemma 5.1, we find

$$\begin{aligned} &\mu du(X, AW) \\ &= \frac{2c}{\alpha}\mu\{g(A^2W, W)w(X) - g(AW, W)w(AX)\} \\ &\quad + \eta(AX)g(\phi\nabla\mu, AW) - \mu(\alpha + g(AW, W))g(\phi\nabla\mu, X) \\ &\quad + \alpha\{g(A^2W, W)\eta(AX) - \mu(\alpha + g(AW, W))w(AX)\} \\ &\quad + \{(\mu^2 + c)g(AW, W) + \alpha\mu^2 - c\alpha + 2c\lambda\}(g(AW, W)\eta(X) - \mu w(X)), \end{aligned}$$

which enables us to obtain

$$\begin{aligned}&g(AW,\nabla_X U)-g(\nabla_{AW}U,X)\\&=-\alpha\left(\alpha+g(AW,W)+\frac{2c}{\alpha^2}g(AW,W)\right)w(AX)\\&\quad-(\alpha+g(AW,W))g(\phi\nabla\mu,X)+g(P_1,X)\end{aligned}$$

for some $P_1\in\operatorname{span}\{\xi,W\}$. If we replace X by AW in (4.23) and make use of (3.5), (4.3), (5.15) and Lemma 5.1, then we get

$$\begin{aligned}&g(\nabla_X U,AW)+g(\nabla_{AW}U,X)\\&=2cw(AX)+2\alpha w(A^2X)-2w(A^3X)\\&\quad+\left(\mu+\frac{\mu}{\alpha}g(AW,W)\right)\{(3\lambda-2\alpha)\eta(AX)-2\mu w(AX)\\&\quad-\alpha\lambda\eta(X)+g(\phi\nabla\alpha,X)\}\\&\quad+\Big\{\mu(3\lambda-2\alpha)(\alpha+g(AW,W))-2\mu g(AW,W)-\alpha\lambda\mu\\&\quad-\frac{1}{\mu}g(AU+(\lambda-\alpha)U,\nabla\alpha)\Big\}(\eta(X)+\tau w(X))-2c\mu\eta(X),\end{aligned}$$

which shows that

$$\begin{aligned}&g(\nabla_X U,AW)+g(\nabla_{AW}U,X)\\&=-2w(A^3X)+2\alpha w(A^2X)+2cw(AX)\\&\quad-2(\lambda-\alpha)(\alpha+g(AW,W))w(AX)\\&\quad+\frac{\mu}{\alpha}(\alpha+g(AW,W))g(\phi\nabla\alpha,X)+g(P_2,X)\end{aligned}$$

for some $P_2\in\operatorname{span}\{\xi,W\}$. Adding to the last two equations, we obtain

$$\begin{aligned}2g(AW,\nabla_X U)=&-2w(A^3X)+2\alpha w(A^2X)+2cw(AX)\\&-2(\lambda-\alpha)(\alpha+g(AW,W))w(AX)\\&-\alpha\left(\alpha+g(AW,W)+\frac{2c}{\alpha^2}g(AW,W)\right)w(AX)\\&-(\alpha+g(AW,W))\left(\phi\nabla\mu-\frac{\mu}{\alpha}\phi\nabla\alpha\right)\\&+g(P_3,X)\end{aligned}$$

for some $P_3\in\operatorname{span}\{\xi,W\}$.

By the way, applying (5.13) by ϕ, and using (2.8) and (3.3), we find

$$\phi\nabla\mu-\frac{\mu}{\alpha}\phi\nabla\alpha=(2\lambda-\alpha)\{-AW+\mu\xi+(\lambda-\alpha)W\}-\varepsilon W, \tag{5.24}$$

where we have put

$$\varepsilon = \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W) - (\lambda - \alpha)^2 + c. \tag{5.25}$$

Because of (4.3), we have

$$\begin{aligned} A^3W = &- \frac{c}{\alpha} A^2W + (\lambda - \alpha)(\alpha + g(AW, W))AW \\ &+ \mu \left(\alpha + \frac{c}{\alpha} + g(AW, W)\right) A\xi. \end{aligned}$$

Combining the last three equations, we obtain

$$\begin{aligned} &g(AW, \nabla_X U) \\ &= \frac{c}{\alpha} w(A^2X) - \left\{\mu^2 + \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W)\right\} w(AX) + g(P_4, X) \end{aligned}$$

for some $P_4 \in \operatorname{span}\{\xi, W\}$. The completes the proof. □

6. Proof of the Main Theorem

We will continue our discussions under the same hypotheses as those stated in Section 5. Because of (5.13) and (5.25), we have

$$\mu(X\mu) - (\lambda - \alpha)X\alpha = (2\lambda - \alpha)u(AX) + \varepsilon u(X) \tag{6.1}$$

for any vector field X. Differentiating this covariantly with respect to a vector field Y and taking the skew-symmetric part, we find

$$\begin{aligned} &(X\lambda)(Y\alpha) - (Y\lambda)(X\alpha) \\ &= (2Y\lambda - Y\alpha)u(AX) - (2X\lambda - X\alpha)u(AY) \\ &\quad + c\mu(2\lambda - \alpha)(\eta(Y)w(X) - \eta(X)w(Y)) \\ &\quad + (2\lambda - \alpha)(g(A\nabla_Y U, X) - g(A\nabla_X U, Y)) \\ &\quad + (Y\varepsilon)u(X) - (X\varepsilon)u(Y) + \varepsilon(g(\nabla_Y U, X) - g(\nabla_X U, Y)), \end{aligned} \tag{6.2}$$

where we have used (2.4) and (2.8). From (5.16), (5.25) and Lemma 5.1, we have $\xi\varepsilon = 0$. If we put $Y = \xi$ in (6.2) and make use of (2.6) and $\xi\varepsilon = 0$, we find

$$\begin{aligned} &c\mu(2\lambda - \alpha)w(X) - (2\lambda - \alpha)(g(\alpha\xi + \mu W, \nabla_X U) - g(\nabla_\xi U, AX)) \\ &- \varepsilon(g(\nabla_X U, \xi) + g(\nabla_\xi U, X)) = 0, \end{aligned}$$

where we have used Lemma 5.1, or using (2.11) and (4.32),

$$(2\lambda - \alpha)A\nabla_\xi U + \varepsilon\nabla_\xi U + \mu\varepsilon AW \in \operatorname{span}\{\xi, W\}. \tag{6.3}$$

From (5.17), (5.24) and Lemma 5.1, we have $W\varepsilon = 0$. Putting $Y = W$ in (6.2) and using Lemma 5.1 and $W\varepsilon = 0$, we obtain

$$\begin{aligned}(6.4)\qquad &(2\lambda-\alpha)(g(\nabla_X U, AW) - g(A\nabla_W U, X) + c\mu\eta(X))\\ &+\varepsilon(g(\nabla_X U, W) - g(\nabla_W U, X)) = 0.\end{aligned}$$

By the way, putting $Y = W$ in (4.34), we have

$$\begin{aligned}&g(\nabla_X U, W) - g(\nabla_W U, X)\\ &= -\left(\alpha + \frac{2c}{\alpha}\right) w(AX) - g(\phi\nabla\mu, X) + g(P_5, X)\end{aligned}$$

for some $P_5 \in \operatorname{span}\{\xi, W\}$, which together with Lemma 5.2 and (6.3) implies that

$$\begin{aligned}&(2\lambda-\alpha)\left\{\frac{c}{\alpha}A^2W - \left(\mu^2 + \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W)\right) AW - A\nabla_W U\right\}\\ &- \varepsilon\left\{\left(\alpha + \frac{2c}{\alpha}\right) AW + \phi\nabla\mu\right\} \in \operatorname{span}\{\xi, W\}.\end{aligned}$$

It follows from this and (4.3) and (4.14) that

$$\begin{aligned}&(2\lambda-\alpha)A\phi\nabla\mu + \varepsilon\phi\nabla\mu\\ &+ (2\lambda-\alpha)\left\{\frac{c}{\alpha}A^2W + \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W)AW\right\}\\ &+ \varepsilon\left(\alpha + \frac{2c}{\alpha}\right) AW \in \operatorname{span}\{\xi, W\}.\end{aligned}$$

On the other hand, applying (5.23) by A, we find

$$A\phi\nabla\mu = \frac{\mu}{\alpha}A\phi\nabla\alpha - (2\lambda-\alpha)\{A^2W - \mu A\xi - (\lambda-\alpha)AW\} - \varepsilon AW.$$

Substituting this into the last equation, we obtain

$$\begin{aligned}(6.5)\qquad &\frac{\mu}{\alpha}(2\lambda-\alpha)A\phi\nabla\mu + \varepsilon\phi\nabla\mu + (2\lambda-\alpha)\frac{c}{\alpha}A^2W\\ &+\Big\{(2\lambda-\alpha)^2\left(\lambda - \alpha + \frac{c}{\alpha}\right)\\ &+(2\lambda-\alpha)((\lambda-\alpha)^2 - c) + \varepsilon\left(\alpha + \frac{2c}{\alpha}\right)\Big\}AW \in \operatorname{span}\{\xi, W\}.\end{aligned}$$

In the next place we have from (3.7)

$$(6.6)\qquad A\nabla_\xi U = \mu(3\lambda - 2\alpha)AW - 3\mu A^2W + 2\mu^2 A\xi + A\phi\nabla\alpha,$$

which together with (6.3) implies that

$$\begin{aligned}&\frac{2\lambda-\alpha}{\mu}A\phi\nabla\alpha + \frac{\varepsilon}{\mu}\phi\nabla\alpha - 2\varepsilon AW\\ &- (2\lambda-\alpha)\{3A^2W + (2\alpha - 3\lambda)AW\} \in \operatorname{span}\{\xi, W\}.\end{aligned}$$

Combining this to (6.5), we obtain

$$\begin{aligned}&\varepsilon\left(\phi\nabla\mu-\frac{\mu}{\alpha}\phi\nabla\alpha\right)+\varepsilon\left(2\lambda-\alpha+\frac{2c}{\alpha}\right)AW\\&+(\lambda-\alpha)(2\lambda-\alpha)(3A^2W+(2\alpha-3\lambda)AW)\\&+(2\lambda-\alpha)\left\{(2\lambda-\alpha)\left(\lambda-\alpha+\frac{c}{\alpha}\right)+(\lambda-\alpha)^2-c\right\}AW\\&+\frac{c}{\alpha}(2\lambda-\alpha)A^2W\in\operatorname{span}\{\xi,W\},\end{aligned}$$

which together with (4.3) and (5.24) implies that

$$\{2\varepsilon\alpha-(2\lambda-\alpha)(\mu^2+c)\}AW\in\operatorname{span}\{\xi,W\},$$

Because of Lemma 4.2, it follows that $2\alpha\varepsilon=(2\lambda-\alpha)(\mu^2+c)$. Thus, it is, using (5.23) and (5.24), verified that

$$\alpha f_2+(\alpha^2-2c)(\mu^2+c)=0 \tag{6.7}$$

by virtue of $2\lambda-\alpha\neq 0$ on Ω.

Using same method as that used to derive (6.2) from (6.1), we can deduce from (5.15) that

$$2(\alpha^2-2c)A\nabla_\xi U-f_2(\nabla_\xi U+\mu AW)\in\operatorname{span}\{\xi,W\}, \tag{6.8}$$

where we have used (2.4), (2.11), (4.3), (4.32), (5.23) and Lemma 5.1.

By the way we see from (5.18) that

$$\begin{aligned}\alpha\lambda A\nabla_\xi U-\mu\Big\{&3\alpha\lambda^2-4\alpha^2\lambda+2\alpha^3\\&+c(7\lambda-6\alpha)+\frac{4c^2}{\alpha}-f_2\Big\}AW\in\operatorname{span}\{\xi,W\},\end{aligned}$$

because of (4.3). Substituting this into (6.8) and taking account of (5.18), we obtain

$$\begin{aligned}&\big\{(\alpha^2-2c)(3\alpha^2\lambda^2-4\alpha^3\lambda+2\alpha^4+c(7\alpha\lambda-6\alpha^2)+4c^2)\\&-\alpha f_2(2\alpha^2-\alpha\lambda-4c)\big\}AW\in\operatorname{span}\{\xi,W\}.\end{aligned}$$

According to Lemma 4.2, we verify that

$$(\alpha^2-2c)\{3\alpha^2\lambda^2-4\alpha^3\lambda+2\alpha^4+c(7\alpha\lambda-6\alpha^2)+4c^2\}=\alpha f_2(2\alpha^2-\alpha\lambda-4c), \tag{6.9}$$

which together with (6.7) yields $(\alpha^2-2c)(\alpha^2-2c-2\alpha\lambda)=0$.

If $\alpha^2-2c\neq 0$, then we have $\alpha^2-2\alpha\lambda-2c=0$ on this subset, which shows that $\alpha\nabla\lambda=(\alpha-\lambda)\nabla\alpha$. Thus, (5.22) reformed as $\lambda\nabla\alpha=0$ and hence $\nabla\alpha=0$ on the subset. Accordingly, using Remark 4.1 and Lemma 4.2, it is contradictory.

Hence it is verified that $\alpha^2 - 2c = 0$ on Ω, which tells us that α is constant. So (3.2) becomes $\mu^2 = \alpha\lambda - 2c$, which implies that $2\mu\nabla\mu = \alpha\nabla\lambda$. Thus, (6.1) reformed as

$$\frac{\alpha}{2}\nabla\lambda = (2\lambda - \alpha)AU + \varepsilon U,$$

which together with $2\alpha\varepsilon = (2\lambda - \alpha)(\mu^2 + c)$ and $\alpha^2 = 2c$ implies that

$$\frac{2c}{2\lambda - \alpha}\nabla\lambda = 2\alpha AU + (\mu^2 + c)U,$$

because $2\lambda - \alpha \neq 0$ on Ω. Therefore, it is clear that

$$\frac{2c}{2\lambda - \alpha}\nabla\lambda = 2\alpha AU + (\alpha\lambda - c)U.$$

Using the same method as that used to derive (5.3) from (5.4), we can deduce from this that

$$2\alpha A\nabla_\xi U + (\alpha\lambda - c)\nabla_\xi U + \mu(\alpha\lambda - c)AW \in \operatorname{span}\{\xi, W\},$$

which together with (4.3), (5.18), (6.6) and the fact that $\alpha^2 = 2c$ implies that

$$\alpha\lambda AW \in \operatorname{span}\{\xi, W\},$$

Because of Lemma 4.2, we see that $\alpha\lambda = 0$. This is not compatible with (3.2). It is contradictory. Hence, we conclude that $\Omega = \emptyset$, that is, $A\xi = \alpha\xi$ on M. Consequently we verify that $R_\xi S = SR_\xi$ because of (2.18). Therefore from Theorem 1.2 ([10]) M is homogeneous real hypersurfaces of Type A.

Let M be of Type A. Then M always satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ and mean curvature is constant. From (2.16), it is easy to see that $\phi R_\xi = R_\xi\phi$.

Consequently we conclude that

Theorem 6.1. *Let M be a real hypersurface with constant mean curvature of a complex space form $M_n(c), c \neq 0, n \geq 3$ which satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. Then M holds $\phi R_\xi = R_\xi\phi$ if and only if $A\xi = 0$ or M is locally congruent to one of following:*

(I) *In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(II) *In cases $M_n(c) = H_n\mathbb{C}$,*

- (A_0) *a horosphere;*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

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