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Certain Inequalities Involving Pathway Fractional Integral Operators

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Abstract. Belarbi and Dahmani [3], recently, using the Riemann-Liouville fractional integral, presented some interesting integral inequalities for the Chebyshev functional in the case of two synchronous functions. Subsequently, Dahmani *et al.* [5] and Sulaiman [17], provided some fractional integral inequalities. Here, motivated essentially by Belarbi and Dahmani's work [3], we aim at establishing certain (presumably) new inequalities associated with pathway fractional integral operators by using synchronous functions which are involved in the Chebychev functional. Relevant connections of the results presented here with those involving Riemann-Liouville fractional integrals are also pointed out.

1. Introduction and Preliminaries

We begin by recalling the well-known celebrated functional introduced by Chebyshev [4] and defined by

(1.1)
$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

where $f(x)$ and $g(x)$ are two integrable functions which are synchronous on $[a,\, b]$,

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i.e.,

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0, \tag{1.2}$$

for any $x,\, y \in [a, b]$.

The functional (1.1) has attracted many researchers' attention due to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [1, 3, 5, 7, 9, 14, 17, 6]; for a very recent work, see also [2]).

Recently, Nair [13] introduced and investigated a new fractional integral operator through the idea of pathway model given by Mathai [10](and further studied by Mathai and Haubold [11, 12]). Belarbi and Dahmani [3], recently, using the Riemann-Liouville fractional integral, presented some interesting integral inequalities for the Chebyshev functional in the case of two synchronous functions. Subsequently, Dahmani *et al.* [5] and Sulaiman [17], provided some fractional integral inequalities. Here, motivated essentially by Belarbi and Dahmani's work [3], we aim at establishing certain (presumably) new inequalities associated with pathway fractional integral operators by using synchronous functions which are involved in the Chebychev functional. Relevant connections of the results presented here with those involving Riemann-Liouville fractional integrals are also indicated.

For our purpose, we need the following definitions and some properties.

Definition 1.1. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space C_μ^n $(n,\, \mu \in \mathbb{R})$, if there exists a real number $p > \mu$ such that $f^{(n)}(t) = t^p\, \phi(t)$, where $\phi(t) \in C(0,\, \infty)$. Here, for the case $n = 1$, we use a simpler notation $C_\mu^1 = C_\mu$.

Definition 1.2. Let $f(x) \in L(a, b)$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator is defined and represented as follows (see [13, p. 239]):

$$\left(P_{0^+}^{(\eta,\alpha,a)} f\right)(t) = t^\eta \int_0^{\left[\frac{t}{a(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}} f(\tau)\, d\tau. \tag{1.3}$$

Let $[a,\, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals $I_{a+}^\eta\, f$ and $I_{b-}^\eta\, f$ of order $\eta \in \mathbb{C}$ $(\Re(\eta) > 0)$ are defined by

$$\left(I_{a+}^\eta\, f\right)(x) := \frac{1}{\Gamma(\eta)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\eta}} \quad (x > a;\ \Re(\eta) > 0) \tag{1.4}$$

and

$$\left(I_{b-}^\eta\, f\right)(x) := \frac{1}{\Gamma(\eta)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\eta}} \quad (x < b;\ \Re(\eta) > 0), \tag{1.5}$$

respectively, where $f \in C_\mu$ $(\mu \geq -1)$ (see, *e.g.*, [8, p. 69]) and $\Gamma(\eta)$ is the familiar Gamma function ((see, *e.g.*, [15, Section 1.1] and [16, Section 1.1]).

Remark 1.3. The special case of the pathway fractional integration operator $\left(P_{0^+}^{(\eta,\alpha,a)} f\right)(t)$ in (1.3) when $\alpha = 0$, $a = 1$, and $\eta \to \eta - 1$ reduces immediately to the left-sided Riemann-Liouville fractional integrals as follows:

$$\left(P_{0^+}^{(\eta-1,0,1)} f\right)(t) = \int_0^t (t-\tau)^{\eta-1} f(\tau)\, d\tau = \Gamma(\eta) \left(I_{0+}^{\eta} f\right)(t) \quad (\Re(\eta) > 0). \tag{1.6}$$

Further one of the Erdélyi-Kober type fractional integrals (see [8, p. 105, Eq. (2.6.1)]) defined by

$$\begin{gathered}\left(I_{a+;\sigma,\alpha}^{\eta} f\right)(t) := \frac{\sigma\, t^{-\sigma(\eta+\alpha)}}{\Gamma(\eta)} \int_a^t \frac{\tau^{\sigma\alpha+\sigma-1} f(\tau)\, d\tau}{\left(t^{\sigma} - \tau^{\sigma}\right)^{1-\eta}} \\ (0 \leqq a < t < b \leqq \infty;\ \Re(\eta) > 0;\ \sigma > 0;\ \alpha \in \mathbb{C})\end{gathered} \tag{1.7}$$

appears to be closely related to the pathway fractional integration operator (1.3). It is found that one integral cannot contain the other one as a purely special case. Yet it is easy to see that some special cases of the two integrals have, for example, the following relationship:

$$\left(P_{0^+}^{(\eta-1,0,1)} f\right)(t) = \Gamma(\eta)\, t^{\eta} \left(I_{0+;1,0}^{\eta} f\right)(t). \tag{1.8}$$

Setting $f(t) = t^{\beta-1}$ in (1.3), we obtain the following formula (see [13, Eq. (12)]):

$$\begin{gathered}P_{0^+}^{(\eta,\alpha)} \left\{t^{\beta-1}\right\} = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\, \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha} + \beta + 1\right)} \\ (\alpha < 1;\ \Re(\eta) > 0;\ \Re(\beta) > 0)\,.\end{gathered} \tag{1.9}$$

Indeed, setting $f(t) = t^{\beta-1}$ in (1.3) and then changing $u = \frac{a(1-\alpha)\tau}{t}$, some algebra gives us that

$$P_{0^+}^{(\eta,\alpha)} \left\{t^{\beta-1}\right\} = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\, B\left(\frac{\eta}{1-\alpha} + 1,\, \beta\right),$$

where $B(\alpha, \beta)$ is the well-known Beta function which is closely related to the Gamma function as follows:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha + \beta)}, \tag{1.10}$$

where α and β are complex numbers which are neither 0 nor negative integers (see, *e.g.*, [15, pp. 9-11] and [16, pp. 7-10]).

2. Main Results

We establish Chebyshev type integral inequalities for the synchronous functions involving the pathway fractional integral operator (1.3).

Theorem 2.1. *Let f and g be two synchronous functions on $[0,\infty)$, Then the following inequality holds true*:

$$P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\} \geq \frac{a(1-\alpha)+a\eta}{t^{\eta+1}} P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{g(t)\} \tag{2.1}$$

for all $a>0$, $\alpha<1$, $t>0$, and $\eta>0$.

Proof. Let f and g be two synchronous functions on $[0,\infty)$. Then, for all $\tau,\ \rho \in (0,\ t)$ with $t>0$, we have

$$(f(\tau)-f(\rho))\,(g(\tau)-g(\rho)) \geq 0, \tag{2.2}$$

or, equivalently,

$$f(\tau)g(\tau)+f(\rho)g(\rho) \geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{2.3}$$

Now, multiplying both sides of (2.3) by $t^{\eta}\left[1-\frac{a(1-\alpha)\tau}{t}\right]^{\frac{\eta}{1-\alpha}}$ and integrating with respect to τ from 0 to $\frac{t}{a(1-\alpha)}$, in view of (1.3), we get

$$P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\}+f(\rho)g(\rho)P_{0^+}^{(\eta,\alpha,a)}\{1\} \geq g(\rho)P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}+f(\rho)P_{0^+}^{(\eta,\alpha,a)}\{g(t)\}. \tag{2.4}$$

Again, multiplying both sides of (2.4) by $t^{\eta}[1-\frac{a(1-\alpha)\rho}{t}]^{\frac{\eta}{1-\alpha}}$ and integrating each side of the resulting inequality with respect to ρ from 0 to $\frac{t}{a(1-\alpha)}$ and applying (1.3), we finally use (1.9), after some simplifications, to prove the desired inequality (2.1). □

Theorem 2.2. *Let f and g be two synchronous functions on $[0,\infty)$. Then, for all $a>0$, $\alpha<1$, $t>0$, $\eta>0$ and $\zeta>0$, we have*

$$\begin{aligned} &\frac{t^{\zeta+1}}{a(1-\alpha)+a\zeta} P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\} + \frac{t^{\eta+1}}{a(1-\alpha)+a\eta} P_{0^+}^{(\zeta,\alpha,a)}\{f(t)g(t)\} \\ &\qquad \geq P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\, P_{0^+}^{(\zeta,\alpha,a)}\{g(t)\} + P_{0^+}^{(\zeta,\alpha,a)}\{f(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{g(t)\}. \end{aligned} \tag{2.5}$$

Proof. Multiplying both sides of (2.4) by

$$t^{\zeta}\left[1-\frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\zeta}{1-\alpha}} \qquad (a>0;\ \alpha<1;\ t>0;\ \zeta>0)$$

and integrating the resulting inequality with respect to ρ from 0 to $\frac{t}{a(1-\alpha)}$ and applying (1.3), we obtain

$$\begin{aligned} &P_{0^+}^{(\zeta,\alpha,a)}\{1\}\, P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\} + P_{0^+}^{(\zeta,\alpha,a)}\{f(t)g(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{1\} \\ &\qquad \geq P_{0^+}^{(\zeta,\alpha,a)}\{g(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{f(t)\} + P_{0^+}^{(\zeta,\alpha,a)}\{f(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{g(t)\}. \end{aligned} \tag{2.6}$$

Finally, applying (1.9) to (2.6) yields the desired result (2.5). □

Remark 2.3. It may be noted that the inequalities (2.1) and (2.5) are reversed if the functions are asynchronous on $[0, \infty)$, i.e.,

$$(f(x) - f(y))\,(g(x) - g(y)) \leq 0, \tag{2.7}$$

for any $x,\, y \in [0, \infty)$.

It is also observed that the inequality in (2.5) when $\zeta = \eta$ reduces immediately to that in (2.1).

Theorem 2.4. *Let $n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\}$ and $\{f_j\}_{j=1}^{n}$ be a sequence of positive increasing functions on $[0,\, \infty)$. Then we have*

$$P_{0^+}^{(\eta,\alpha,a)}\left\{\prod_{j=1}^{n} f_j(t)\right\} \geq \left[\frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\right]^{n-1} \prod_{j=1}^{n} P_{0^+}^{(\eta,\alpha,a)}\left\{f_j(t)\right\}, \tag{2.8}$$

for all $a > 0$, $\alpha < 1$, $t > 0$, and $\eta > 0$.

Proof. We proceed to prove (2.8) by mathematical induction on n. For $n = 1$, the inequality (2.8) obviously holds. For $n = 2$, the inequality (2.8) immediately follows from (2.1). So we assume that the inequality in (2.8) holds true for some positive integer $k \in \mathbb{N} \setminus \{1\}$, i.e.,

$$P_{0^+}^{(\eta,\alpha,a)}\left\{\prod_{j=1}^{k} f_j(t)\right\} \geq \left[\frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\right]^{k-1} \prod_{j=1}^{k} P_{0^+}^{(\eta,\alpha,a)}\left\{f_j(t)\right\}, \tag{2.9}$$

under the given conditions of Theorem 3. It is observed that, since $\{f_j\}_{j=1}^{k}$ are a sequence of increasing functions, so is $\prod_{j=1}^{k} f_j(t)$ on $(0,\, \infty)$. Now, we can apply the inequality in (2.1) to the functions

$$g(t) := \prod_{j=1}^{k} f_j(t) \quad \text{and} \quad f(t) := f_{k+1}(t)$$

to get the following inequality:

$$P_{0^+}^{(\eta,\alpha,a)}\left\{\prod_{j=1}^{k+1} f_j(t)\right\} \geq \frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\, P_{0^+}^{(\eta,\alpha,a)}\left\{\prod_{j=1}^{k} f_j(t)\right\} \cdot P_{0^+}^{(\eta,\alpha,a)}\left\{f_{k+1}(t)\right\}, \tag{2.10}$$

under the given conditions of Theorem 3. Then, if we apply (2.9) to the right-hand side of (2.10), we have

$$P_{0^+}^{(\eta,\alpha,a)}\left\{\prod_{j=1}^{k+1} f_j(t)\right\} \geq \left[\frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\right]^{k} \prod_{j=1}^{k+1} P_{0^+}^{(\eta,\alpha,a)}\left\{f_j(t)\right\}.$$

Hence, by the principle of mathematical induction, the inequality (2.8) holds true for any $n \in \mathbb{N}$. $\square$

Here we consider some other variations of the fractional integral inequalities.

Theorem 2.5. *Let f and g be two functions defined on $[0, \infty)$ such that f is increasing, g is differentiable and g' is bounded below on $[0, \infty)$. Then we have*

$$\begin{aligned} P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\} \geq \frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\; P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\; P_{0^+}^{(\eta,\alpha,a)}\{g(t)\} \\ - \frac{m\,t}{2a(1-\alpha)+a\eta} P_{0^+}^{(\eta,\alpha,a)}\{f(t)\} + m\; P_{0^+}^{(\eta,\alpha,a)}\{tf(t)\}, \end{aligned} \tag{2.11}$$

for all $a > 0$, $t > 0$, $\alpha < 0$, $\eta > 0$, and $m := \inf_{t\in[0,\,\infty)} g'(t)$.

Proof. Consider the function $h(t) := g(t) - mt$. Then we observe that h is differentiable and increasing on $[0, \infty)$. We can, therefore, use Theorem 1 to get

$$\begin{aligned} P_{0^+}^{(\eta,\alpha,a)}\{f(t)(g(t)-mt)\} &\geq \frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\; P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\; P_{0^+}^{(\eta,\alpha,a)}\{g(t)-mt\} \\ &= \frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\; P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\; P_{0^+}^{(\eta,\alpha,a)}\{g(t)\} \\ &\quad - \frac{a(1-\alpha)+a\eta}{t^{\eta+1}} \cdot m \cdot P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\, P_{0^+}^{(\eta,\alpha,a)}\{t\}. \end{aligned}$$

Finally, using the special case of (1.9) when $\beta = 2$:

$$P_{0^+}^{(\eta,\alpha,a)}\{t\} = \frac{t^{\eta+2}}{a^2\,(2-2\alpha+\eta)(1-\alpha+\eta)} \tag{2.12}$$

for the last term in the last resulting inequality, after a little simplification, we are led to the inequality (2.11). $\square$

Theorem 2.6. *Let f and g be two functions defined on $[0, \infty)$ such that f is increasing, g is differentiable and bounded above on $[0, \infty)$. Then we have*

$$\begin{aligned} P_{0^+}^{(\eta,\alpha,a)}\{f(t)g(t)\} \leq \frac{a(1-\alpha)+a\eta}{t^{\eta+1}}\; P_{0^+}^{(\eta,\alpha,a)}\{f(t)\}\; P_{0^+}^{(\eta,\alpha,a)}\{g(t)\} \\ - \frac{M\;t}{2a(1-\alpha)+a\eta} P_{0^+}^{(\eta,\alpha,a)}\{f(t)\} + M\; P_{0^+}^{(\eta,\alpha,a)}\{tf(t)\}, \end{aligned} \tag{2.13}$$

for all $a > 0$, $t > 0$, $\alpha < 0$, $\eta > 0$ and $M := \sup_{t\in[0,\,\infty)} g'(t)$.

Proof. Here consider $h(t) := M\,t - g(t)$. We see that $h'(t) \geq 0$ and h is increasing on $[0, \infty)$. Then, a similar argument as in the proof of Theorem 4 will establish the inequality (2.13). So the detailed algebra is left to the interested reader. $\square$

Concluding Remarks. In view of (1.6), since the pathway fractional integral operator (1.3) reduces to a Riemann-Liouville type fractional integral operator (1.4), we find that Theorems 1 to 5 may yield those known results due to Belarbi and Dahmani [3]. We also note that the results derived here are of a general character and can give numerous special inequalities, which are (potentially) useful in various applications, in particular, to an establishment of uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations.

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