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$h(x)$-B-Tribonacci and $h(x)$-B-Tri Lucas Polynomials

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Abstract. In this paper we introduce $h(x)$-B-Tribonacci and $h(x)$-B-Tri Lucas polynomials. We also obtain the identities for these polynomials.

1. Introduction

Fibonacci and Lucas polynomials studied in [4] and [5] are the natural extension of Fibonacci and Lucas sequences respectively. Many interesting identities relating to h(x)-Fibonacci polynomials are studied in [3]. The B- Tribonacci sequence which is an extension of generalized Fibonacci sequence is introduced in [1]. In [2] we have discussed the identities relating to bivariate B-Tribonacci and B-Lucas polynomials. In this paper we introduce $h(x)$-B-Tribonacci and $h(x)$-B-Tri Lucas polynomials and study various identities involving these polynomials.

Definition 1.1. Let $h(x)$ be a non-zero polynomial with real coefficients. The $h(x)$-B-Tribonacci polynomials $({}^tB)_{h,n}(x), n \in \mathbb{N} \cup \{0\}$ are defined by

$$({}^tB)_{h,n+2}(x) = h^2(x)\ ({}^tB)_{h,n+1}(x) + 2h(x)\ ({}^tB)_{h,n}(x) + \ ({}^tB)_{h,n-1}(x), \tag{1.1}$$

$$\forall n \geq 1, \text{with } \ ({}^tB)_{h,0}(x) = 0\ ,\ ({}^tB)_{h,1}(x) = 0 \text{ and } ({}^tB)_{h,2}(x) = 1,$$

where the coefficients on the right hand side are the terms of the binomial expansion of $\big(h(x)+1\big)^2$ and $({}^tB)_{h,n}(x)$ is the n^{th} polynomial of (1.1). In particular if $h(x) = 1$,

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then (1.1) reduces to B- Tribonacci sequence defined in [1] with $a = 1$ and $b = 1$, namely,

$$(1.2) \qquad ({}^tB)_{1,n+2}(x) = ({}^tB)_{1,n+1}(x) + 2\,({}^tB)_{1,n}(x) + ({}^tB)_{1,n-1}(x), \forall n \geq 1,$$

$$\text{with } ({}^tB)_{1,0}(x) = 0\ ,\ ({}^tB)_{1,1}(x) = 0 \text{ and } ({}^tB)_{1,2}(x) = 1.$$

First few terms of (1.2) are $({}^tB)_{1,0}(x) = 0$, $({}^tB)_{1,1}(x) = 0$, $({}^tB)_{1,2}(x) = 1$, $({}^tB)_{1,3}(x) = 1$, $({}^tB)_{1,4}(x) = 3$, $({}^tB)_{1,5}(x) = 6$, $({}^tB)_{1,6}(x) = 13$, $({}^tB)_{1,7}(x) = 28$ and $({}^tB)_{1,8}(x) = 60$.

Table 1 shows the coefficients of $h(x)$-B-Tribonacci polynomials, $({}^tB)_{h,n}(x)$, arranged in ascending order and also the sequence $({}^tB)_{1,n}$.

n	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8	h^9	h^{10}	h^{11}	h^{12}	$({}^tB)_{1,n}$
0	0													0
1	0													0
2	1													1
3	0	0	1											1
4	0	2	0	0	1									3
5	1	0	0	4	0	0	1							6
6	0	0	6	0	0	6	0	0	1					13
7	0	4	0	0	15	0	0	8	0	0	1			28
8	1	0	0	20	0	0	28	0	0	10	0	0	1	60

Table 1: Showing the coefficients of $({}^tB)_{h,n}(x)$ and the terms of $({}^tB)_{1,n}$.

Comparing the Table 1 with the Pascal type triangle, the sum of the n^{th} row is the n^{th} term of the sequence $({}^tB)_{1,n}$. In Table 1, for n ≥ 2, sum of the elements in the anti diagonal of corresponding (2n-3)x(2n-3) matrix is $2^{2(n-2)}$.

2. Identities for the n^{th} term $({}^tB)_{h,n}(x)$, of $h(x)$-B-Tribonacci Polynomials

In this section we discuss some identities of $h(x)$-B- Tribonacci polynomials which can be proved by usual method. For simplicity we use $({}^tB)_{h,n}(x) = ({}^tB)_{h,n}$ and $h(x) = h$.

(1) Combinatorial formula: The n^{th} term $({}^tB)_{h,n}$ of (1.1) is given by

$$(2.1) \qquad ({}^tB)_{h,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}, \text{for all } n \geq 2.$$

(2) Binet type Formula: The n^{th} term of (1.1) is given by

$$({}^tB)_{h,n} = \frac{(\alpha-\beta)\gamma^n - (\alpha-\gamma)\beta^n + (\beta-\gamma)\alpha^n}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}. \tag{2.2}$$

where α, β and γ are the distinct roots of the characteristics equation corresponding to (1.1).

(3) Generating function: The generating function for $h(x)$-B-Tribonacci polynomials (1.1) is given by

$$({}^tG_{(B)})_h(z) = \frac{1}{1-z(h+z)^2} \tag{2.3}$$

(4) Sum of the first $(n+1)$ terms: The sum of the first $n+1$ terms of (1.1) is

$$\sum_{r=0}^{n} ({}^tB)_{h,r} = \frac{({}^tB)_{h,n+2} + \big(1-h^2\big)({}^tB)_{h,n+1} + ({}^tB)_{h,n} - 1}{h^2+2h}, \tag{2.4}$$

provided $h \neq -2, 0$.

Next two theorems are related to the recurrence properties of $h(x)$-B-Tribonacci polynomials.

Theorem 2.1.

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\, ({}^tB)_{h,n+i}\, h^i = ({}^tB)_{h,n+3s}, \forall s \geq 0. \tag{2.5}$$

Proof. For $s = 0$, the result is true. For s $= 1$,

L. H. S. of (2.5) $= \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} ({}^tB)_{h,n+i}\, h^i$

$$= ({}^tB)_{h,n} + 2h\, ({}^tB)_{h,n+1} + h^2\, ({}^tB)_{h,n+2} = ({}^tB)_{h,n+3} = \text{R.H.S.}$$

Therefore (2.5) is true for $s = 1$. Assume that the result holds for all $s \leq m$.

Consider, $\sum_{i=0}^{2m+2} \frac{(2m+2)^{\underline{i}}}{i!}\, ({}^tB)_{h,n+i}\, h^i$

$= \sum_{i=0}^{2m+2} \left(\frac{(2m)^{\underline{i-2}}}{(i-2)!} + 2\,\frac{(2m)^{\underline{i-1}}}{(i-1)!} + \frac{(2m)^{\underline{i}}}{i!}\right) ({}^tB)_{h,n+i}\, h^i$

$= \sum_{i=-2}^{2m} \frac{(2m)^{\underline{i}}}{i!} ({}^tB)_{h,n+i+2}\, h^{i+2} + 2\, \sum_{i=-1}^{2m} \frac{(2m)^{\underline{i}}}{i!} ({}^tB)_{h,n+i+1}\, h^{i+1}$
$+ \sum_{i=0}^{2m} \frac{(2m)^{\underline{i}}}{i!}\, ({}^tB)_{h,n+i}\, h^i$

$= \sum_{i=0}^{2m} \frac{(2m)^{\underline{i}}}{i!}\ h^i\ \Big(h^2\ ({}^tB)_{h,n+i+2} + 2h\ ({}^tB)_{h,n+i+1} + ({}^tB)_{h,n+i}\Big)$

$= h^2({}^tB)_{h,n+3m+2} + 2h({}^tB)_{h,n+3m+1} + ({}^tB)_{h,n+3m}$

$= ({}^tB)_{h,n+3m+3}$.

Hence the result is true for $s = m+1$. Therefore by Mathematical induction on s, the theorem is proved. □

Theorem 2.2 For all $s \geq 1$,

(2.6)

$$\sum_{i=0}^{s-1}\Big(2\ h^{2s-1-2i}({}^tB)_{h,n+1+i} + h^{2s-2-2i}({}^tB)_{h,n+i}\Big) = ({}^tB)_{h,n+2+s}\ -\ h^{2s}({}^tB)_{h,n+2}.$$

Proof. We use induction on s. Equation (1.1) gives,

$$2h\ ({}^tB)_{h,n+1}\ +\ ({}^tB)_{h,n} = ({}^tB)_{h,n+3} - h^2\ ({}^tB)_{h,n+2}$$

Hence (2.6) holds for $s = 1$. Now let the result be true for $s \leq m$. We prove it for $s = m+1$.

Consider,

$\sum_{i=0}^{m}\Big(2\ h^{2m+1-2i}({}^tB)_{h,n+i+1} + h^{2m-2i}({}^tB)_{h,n+i}\Big)$

$= \sum_{i=0}^{m-1}\Big(2h^{2m+1-2i}({}^tB)_{h,n+i+1} + h^{2m-2i}({}^tB)_{h,n+i}\Big)$

$+ \Big(2h({}^tB)_{h,n+m+1} + ({}^tB)_{h,n+m}\Big)$

$= h^2\Big(\ \sum_{i=0}^{m-1}\big(2h^{2m-1-2i}({}^tB)_{h,n+i+1} + h^{2m-2-2i}({}^tB)_{h,n+i}\big)\Big)$

$+\Big(2h({}^tB)_{h,n+m+1} + ({}^tB)_{h,n+m}\Big)$

$= h^2\Big(({}^tB)_{h,n+m+2} - h^{2m}({}^tB)_{h,n+2}\Big) + 2\ h({}^tB)_{h,n+m+1} + ({}^tB)_{h,n+m}$

$= h^2({}^tB)_{h,n+m+2} - h^{2m+2}({}^tB)_{h,n+2} + 2\ h({}^tB)_{h,n+m+1} + ({}^tB)_{h,n+m}$

$= ({}^tB)_{h,n+m+3} - h^{2m+2}({}^tB)_{h,n+2}$.

Hence the theorem is proved. □

Theorem 2.3 The derivative of $({}^tB)_{h,n}$ with respect to x is given by

(2.7) $$({}^tB)'_{h,n} = 2\sum_{i=0}^{n}\Big(h\ ({}^tB)_{h,n+1-i} + ({}^tB)_{h,n-i}\Big)({}^tB)_{h,i}.$$

Proof. Consider,

$$({}^tG_{(B)})_h(z) = \frac{1}{1 - z\,(h+z)^2}$$

Thus,

$$\sum_{n=0}^{\infty}({}^tB)_{h,n}z^{n-2} = \frac{1}{1 - z(h+z)^2} \tag{2.8}$$

Differentiating (2.8) both sides with respect to x we get,

$\sum_{n=0}^{\infty}({}^tB)'_{h,n}z^{n-2}h' = \frac{2hz}{[1-z(h+z)^2]^2}\,h' + \frac{2z^2}{[1-z(h+z)^2]^2}\,h'$

$= 2hh'z\big[\sum_{n=0}^{\infty}({}^tB)_{h,n}z^{n-2}\big]^2 + 2h'z^2\big[\sum_{n=0}^{\infty}({}^tB)_{h,n}z^{n-2}\big]^2$

$= 2hh'z^{-3}\big[\sum_{n=0}^{\infty}({}^tB)_{h,n}z^{n}\big]^2 + 2h'z^{-2}\big[\sum_{n=0}^{\infty}({}^tB)_{h,n}z^{n}\big]^2$

Therefore,

$\sum_{n=0}^{\infty}({}^tB)'_{h,n}z^{n+1} = 2h\sum_{n=0}^{\infty}\big(\sum_{i=0}^{n}({}^tB)_{h,i}({}^tB)_{h,n-i}\big)z^n$
$+2z\sum_{n=0}^{\infty}\big(\sum_{i=0}^{n}({}^tB)_{h,i}({}^tB)_{h,n-i}\big)z^n$

Comparing the coefficients of z^{n+1},

$({}^tB)'_{h,n} = 2\sum_{i=0}^{n}\Big(h\;({}^tB)_{h,n+1-i} + ({}^tB)_{h,n-i}\Big)({}^tB)_{h,i}.$

□

3. $h(x)$-B-Tri Lucas Polynomials

In this section we define $h(x)$-B-Tri Lucas polynomials and prove some identities related to these polynomials.

Definition 3.1. Let $h(x)$ be a non zero polynomial with real coefficients. The $h(x)$-B-Tri Lucas polynomials $({}^tL)_{h,n}(x), n \in \mathbb{N}\cup\{0\}$ are defined by

$$({}^tL)_{h,n+2}(x) = h^2(x)\;({}^tL)_{h,n+1}(x) + 2h(x)\;({}^tL)_{h,n} + \;({}^tL)_{h,n-1}(x), \tag{3.1}$$

for all $n \geq 1$, with $({}^tL)_{h,0}(x) = 0$, $({}^tL)_{h,1}(x) = 2$, and $({}^tL)_{h,2}(x) = h^2(x)$,

where the coefficients on the right hand side are the terms of the binomial expansion of $(h(x)+1)^2$ and $({}^tL)_{h,n}(x)$ is the n^{th} polynomial. In particular if $h(x) = 1$, then (3.1) reduces to B- Tri Lucas sequence defined by

$$({}^tL)_{1,n+2}(x) = \;({}^tL)_{1,n+1}(x) + 2\;({}^tL)_{1,n}(x) + \;({}^tL)_{1,n-1}(x), \forall n \geq 1, \tag{3.2}$$

with $({}^tL)_{1,0}(x) = 0$, $({}^tL)_{1,1}(x) = 2$ and $({}^tL)_{1,2}(x) = 1$.

First few terms of (3.2) are $({}^tL)_{1,0}(x) = 0, ({}^tL)_{1,1}(x) = 2, ({}^tL)_{1,2}(x) = 1, ({}^tL)_{1,3}(x) = 5, ({}^tL)_{1,4}(x) = 9, ({}^tL)_{1,5}(x) = 20, ({}^tL)_{1,6}(x) = 43$ and $({}^tL)_{1,7}(x) = 92$.

Table 2 shows the coefficients of $h(x)$-B-Tri Lucas polynomials $({}^tL)_{h,n}(x)$ arranged in ascending order and also the sequence $({}^tL)_{1,n}$.

n	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8	h^9	h^{10}	h^{11}	h^{12}	$({}^tL)_{1,n}$
0	0													0
1	2													2
2	0	0	1											1
3	0	4	0	0	1									5
4	2	0	0	6	0	0	1							9
5	0	0	11	0	0	8	0	0	1					20
6	0	8	0	0	24	0	0	10	0	0	1			43
7	2	0	0	36	0	0	41	0	0	12	0	0	1	92

Table 2: Showing the coefficients of $({}^tL)_{h,n}(x)$ and the terms of $({}^tL)_{1,n}$.

Comparing Table 2 with the Pascal type triangle, the sum of the n^{th} row is the term $({}^tL)_{1,n}$. In Table 2, for n $\geq$ 2, sum of the elements in the anti diagonal of corresponding (2n-1)x(2n-1) matrix is 7 $\left(2^{2(n-2)}\right)$.

We state below the identities related to the n^{th} term $({}^tL)_{h,n}(x)$, of $h(x)$-B-Tri Lucas polynomials. For simplicity we use $({}^tL)_{h,n}(x) = ({}^tL)_{h,n}$ and $h(x) = h$.

(1) Combinatorial formula: The n^{th} term $({}^tL)_{h,n}$ of (3.1) is given by

(3.3) $$({}^tL)_{h,n} = \sum_{r=0}^{\left[\frac{2n-2}{3}\right]} \left(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \right) h^{2n-2-3r},$$

$\forall n \geq 2$.

(2) Binet type formula: The n^{th} term of (3.1) is given by

(3.4) $$({}^tL)_{h,n} = \frac{(\alpha-\beta)\gamma^n(2\gamma-h^2) - (\alpha-\gamma)\beta^n(2\beta-h^2) + (\beta-\gamma)\alpha^n(2\alpha-h^2)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}.$$

where α, β and γ are the distinct roots of the characteristics equation corresponding to (3.1).

(3) Generating function: The generating function for $h(x)$-B-Tri Lucas polynomials (3.1) is given by

$$({}^tG_{(L)})_h(z) = \frac{2 - h^2 z}{1 - z\big(h+z\big)^2} \tag{3.5}$$

(4) Sum of the first n+1 terms: The sum of the first $n+1$ terms of (3.1) is

$$\sum_{r=0}^{n} ({}^tL)_{h,r} = \frac{({}^tL)_{h,n+2} + \big(1 - h^2\big)({}^tL)_{h,n+1} + ({}^tL)_{h,n} + ({}^tL)_{h,2} - ({}^tL)_{h,1}}{h^2 + 2h}, \tag{3.6}$$

provided $h \neq -2, 0$.

We have the following theorems on recurrence properties of $h(x)$-B-Tri Lucas polynomials.

Theorem 3.2.

$$({}^tL)_{h,n+1} = ({}^tB)_{h,n+2} + 2h\ ({}^tB)_{h,n} + ({}^tB)_{h,n-1}, \text{ for all } n \geq 1. \tag{3.7}$$

Proof. By induction on n. Note that (3.7) holds for $n = 1$. Now assume that it holds for $n \leq m-1$ and consider,

$$\begin{aligned}
({}^tL)_{h,m+1} &= h^2\ ({}^tL)_{h,m} + 2h\ ({}^tL)_{h,m-1} + ({}^tL)_{h,m-2} \\
&= h^2\Big(\ ({}^tB)_{h,m+1} + 2h\ ({}^tB)_{h,m-1} + ({}^tB)_{h,m-2}\Big) \\
&\quad +2h\ \Big(\ ({}^tB)_{h,m} + 2h\ ({}^tB)_{h,m-2} + ({}^tB)_{h,m-3}\Big) \\
&\quad +\Big(\ ({}^tB)_{h,m-1} + 2h\ ({}^tB)_{h,m-3} + ({}^tB)_{h,m-4}\Big) \\
&= ({}^tB)_{h,m+2} + 2h\ ({}^tB)_{h,m} + ({}^tB)_{h,m-1}.
\end{aligned}$$

Hence the theorem is proved. □

Following corollary can be deduced from equations (1.1) and (3.7).

Corollary 3.3

$$({}^tL)_{h,n} = 2\ ({}^tB)_{h,n+1} - h^2\ ({}^tB)_{h,n}, \text{ for all } n \geq 0. \tag{3.8}$$

Theorem 3.4

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \, ({}^{t}L)_{h,n+i} \, h^{i} = ({}^{t}L)_{h,n+3s}. \tag{3.9}$$

Proof. Since $({}^{t}L)_{h,n} = 2\,({}^{t}B)_{h,n+1} - h^{2}({}^{t}B)_{h,n}$,

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \, ({}^{t}L)_{h,n+i} \, h^{i} = \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \Big(2\,({}^{t}B)_{h,n+1+i} - h^{2}({}^{t}B)_{h,n+i}\Big) \, h^{i}$$

$$= 2\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \, ({}^{t}B)_{h,n+1+i} \, h^{i} - h^{2}\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \, ({}^{t}B)_{h,n+i} \, h^{i}$$

$$= 2\,({}^{t}B)_{h,n+1+3s} - h^{2}\,({}^{t}B)_{h,n+3s}, \text{ from equation (2.5).}$$

$$= ({}^{t}L)_{h,n+3s}.$$ □

Using the procedure similar to the one used to prove Theorem 2.2, we get the following result.

Theorem 3.5. *For all $s \geq 1$,*

$$\sum_{i=0}^{s-1} \Big(2\,h^{2s-1-2i}({}^{t}L)_{h,n+1+i} + h^{2s-2-2i}({}^{t}L)_{h,n+i}\Big) = ({}^{t}L)_{h,n+2+s} - h^{2s}({}^{t}L)_{h,n+2}. \tag{3.10}$$

To prove the next theorem we use equation (2.7).

Theorem 3.6. *The derivative of $({}^{t}L)_{h,n}$ with respect to x is given by*

$$({}^{t}L)'_{h,n} = \sum_{i=0}^{n} \Big(2h\,({}^{t}L)_{h,n+1-i} + 2\,({}^{t}L)_{h,n-i}\Big)({}^{t}B)_{h,i} - 2h\,({}^{t}B)_{h,n}. \tag{3.11}$$

Proof. Consider,

$$({}^{t}L)_{h,n} = 2\,({}^{t}B)_{h,n+1} - h^{2}({}^{t}B)_{h,n} \tag{3.12}$$

Differentiating (3.12) both sides with respect to x, we get

$$({}^{t}L)'_{h,n} = 2\,({}^{t}B)'_{h,n+1} - h^{2}({}^{t}B)'_{h,n} - 2h\,({}^{t}B)_{h,n}$$

$$= 2\,\sum_{i=0}^{n+1} \Big(2h\,({}^{t}B)_{h,n+2-i} + 2\,({}^{t}B)_{h,n+1-i}\Big)({}^{t}B)_{h,i}$$

$$-h^{2}\,\sum_{i=0}^{n} \Big(2h\,({}^{t}B)_{h,n+1-i} + 2\,({}^{t}B)_{h,n-i}\Big)({}^{t}B)_{h,i} - 2h\,({}^{t}B)_{h,n}$$

$$= \sum_{i=0}^{n} \Big(2h\,\big(2({}^{t}B)_{h,n+2-i} - h^{2}\,({}^{t}B)_{h,n+1-i}\big)$$

$$+2\left(2\ ({}^tB)_{h,n+1-i} - h^2({}^tB)_{h,n-i}\right)({}^tB)_{h,i}\Big) - 2h\ ({}^tB)_{h,n}$$

$$= \sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,n+1-i} + 2\ ({}^tL)_{h,n-i}\Big)({}^tB)_{h,i} - 2h\ ({}^tB)_{h,n}. \qquad \square$$

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