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Composite Hurwitz Rings Satisfying the Ascending Chain Condition on Principal Ideals

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Abstract. Let $D \subseteq E$ be an extension of integral domains with characteristic zero, I be a nonzero proper ideal of D and let $H(D,E)$ and $H(D,I)$ (resp., $h(D,E)$ and $h(D,I)$) be composite Hurwitz series rings (resp., composite Hurwitz polynomial rings). In this paper, we show that $H(D,E)$ satisfies the ascending chain condition on principal ideals if and only if $h(D,E)$ satisfies the ascending chain condition on principal ideals, if and only if $\bigcap_{n\geq 1} a_1 \cdots a_n E = (0)$ for each infinite sequence $(a_n)_{n\geq 1}$ consisting of nonzero nonunits of D. We also prove that $H(D,I)$ satisfies the ascending chain condition on principal ideals if and only if $h(D,I)$ satisfies the ascending chain condition on principal ideals, if and only if D satisfies the ascending chain condition on principal ideals.

1. Introduction

1.1 Composite Hurwitz Rings

Let R be a commutative ring with identity and let $H(R)$ be the set of formal expressions of the type $f = \sum_{i=0}^{\infty} a_i X^i$, where $a_i \in R$. Define addition and $*$-

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product on $H(R)$ as follows: for $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in H(R)$,

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) X^i \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then $H(R)$ becomes a commutative ring with identity containing R under these two operations, *i.e.*, $H(R) = (R[\![X]\!], +, *)$. We call it the *Hurwitz series ring* over R. The *Hurwitz polynomial ring* $h(R)$ is defined as the same way, *i.e.*, $h(R)$ is the set of formal expression of the type $f = \sum_{i=0}^{n} a_i X^i$, where $a_i \in R$ with the usual addition and $*$-product. Note that $(h(R), +, *)$ is also a commutative ring with identity containing R. In order to prevent the confusion, for $n \geq 1$, we denote the nth Hurwitz power of f by $f^{(n)}$, where f is either a Hurwitz series or a Hurwitz polynomial. Also, for a Hurwitz series $f = \sum_{i=0}^{\infty} a_i X^i \in H(R)$, the *order* of f is the smallest nonnegative ingeger m such that $a_m \neq 0$ and is denoted by $\text{ord}(f)$. For an $f = \sum_{i=0}^{\infty} a_i X^i \in H(R)$, we mean by the *support* of f, denoted by $\text{supp}(f)$, the set of nonnegative integers n such that $a_n \neq 0$. Also, for an $f \in H(R)$ and a nonnegative integer n, $f(n)$ stands for the coefficient of X^n in f.

Let $D \subseteq E$ be an extension of commutative rings with identity and set $H(D, E) := \{f \in H(E) \mid f(0) \in D\}$ and $h(D, E) := \{f \in h(E) \mid f(0) \in D\}$. Then $H(D) \subseteq H(D, E) \subseteq H(E)$ and $h(D) \subseteq h(D, E) \subseteq h(E)$. The rings $H(D, E)$ and $h(D, E)$ are called the *composite Hurwitz series ring* and the *composite Hurwitz polynomial ring*, respectively. In other words, $H(D, E) = (D + XE[\![X]\!], +, *)$ and $h(D, E) = (D + XE[X], +, *)$.

Let I be a nonzero proper ideal of D and set $H(D, I) = \{f \in H(D) \mid f(n) \in I$ for all $n \geq 1\}$ (resp., $h(D, I) = \{f \in h(D) \mid f(n) \in I$ for all $n \geq 1\}$). Then $D \subsetneq H(D, I) \subsetneq H(D)$ and $D \subsetneq h(D, I) \subsetneq h(D)$.

1.2 Ascending Chain Condition on Principal Ideals

As a particular case of Noetherian rings, a ring satisfying the ascending chain condition on principal ideals has been studied by many mathematicians. Let R be a commutative ring with identity. Recall that R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. It was shown that if R is an integral domain, then R satisfies ACCP if and only if $\bigcap_{n \geq 1} a_1 \cdots a_n R = (0)$ for each infinite sequence $(a_n)_{n \geq 1}$ of nonzero nonunits of R [2, Remark 1.1]. In [2, Proposition 1.2 and Remark 1.4] (or [5, Theorem 3.4] and [6, Corollary 1.5(2)]), the authors proved that if $D \subseteq E$ is an extension of integral domains, then $D + XE[\![X]\!]$ satisfies ACCP if and only if $D + XE[X]$ satisfies ACCP, if and only if $\bigcap_{n \geq 1} a_1 \cdots a_n E = (0)$ for each infinite sequence $(a_n)_{n \geq 1}$ consisting of nonzero nonunits of D. Also, it was shown that if D is an integral domain and I is a nonzero proper ideal of D, then $D + XI[\![X]\!]$ satisfies ACCP if and only if $D + XI[X]$ satisfies ACCP, if and only if D satisfies ACCP [3, Proposition 4.6] (or [5, Theorem 3.7] and [6, Corollary 1.5(3)]).

In this article, we study when composite Hurwitz series rings $H(D, E)$ and $H(D, I)$ and composite Hurwitz polynomial rings $h(D, E)$ and $h(D, I)$ satisfy

ACCP, where $D \subseteq E$ is an extension of integral domains with characteristic zero and I is a nonzero proper ideal of D. More precisely, we show that $H(D, E)$ satisfies ACCP if and only if $h(D, E)$ satisfies ACCP, if and only if $\bigcap_{n\geq 1} a_1 \cdots a_n E = (0)$ for each infinite sequence $(a_n)_{n\geq 1}$ consisting of nonzero nonunits of D (Theorem 2.4). We also prove that $H(D, I)$ satisfies ACCP if and only if $h(D, I)$ satisfies ACCP, if and only if D satisfies ACCP (Theorem 3.4).

2. Composite Hurwitz Rings $H(D, E)$ and $h(D, E)$

In this section, we characterize when the composite Hurwitz series ring $H(D, E)$ and the composite Hurwitz polynomial ring $h(D, E)$ satisfy ACCP, where $D \subseteq E$ is an extension of integral domains with characteristic zero. We start with a simple result whose proof is straightforward.

Proposition 2.1. *Let $D \subseteq E$ be an extension of commutative rings with identity. Then the following conditions are equivalent.*

(1) *$H(D, E)$ is an integral domain.*

(2) *$h(D, E)$ is an integral domain.*

(3) *$D \subseteq E$ is an extension of integral domains with characteristic zero.*

We next characterize when a Hurwitz series (resp., Hurwitz polynomial) is a unit in the composite Hurwitz series ring $H(D, E)$ (resp., composite Hurwitz polynomial ring $h(D, E)$).

Lemma 2.2. *Let $D \subseteq E$ be an extension of commutative rings with identity. Then the following assertions hold.*

(1) *A Hurwitz series $f \in H(D, E)$ is a unit if and only if $f(0)$ is a unit in D.*

(2) *A Hurwitz polynomial $f = \sum_{i=0}^{n} a_i X^i \in h(D, E)$ is a unit if and only if a_0 is a unit in D and for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion.*

Proof. (1) If f is a unit in $H(D, E)$, then $f * g = 1$ for some $g \in H(D, E)$; so $f(0)g(0) = 1$. Hence $f(0)$ is a unit in D. Conversely, if $f(0)$ is a unit in D, then $f(0)$ is a unit in E; so $f * g = 1$ for some $g \in H(E)$ [4, Proposition 2.5]. Since $g(0) = \frac{1}{f(0)} \in D$, $g \in H(D, E)$. Thus f is a unit in $H(D, E)$.

(2) If f is a unit in $h(D, E)$, then $f * g = 1$ for some $g \in h(D, E)$; so $a_0 g(0) = 1$. Hence a_0 is a unit in D. Also, f is a unit in $h(E)$; so for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion [1, Theorem 3.1]. For the converse, assume that a_0 is a unit in D and for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion. Then f is a unit in $h(E)$ [1, Theorem 3.1]; so $f * g = 1$ for some $g \in h(E)$. Since a_0 is a unit in D, $g(0) = \frac{1}{a_0} \in D$; so $g \in h(D, E)$. Thus f is a unit in $h(D, E)$. □

To characterize when composite Hurwitz rings $H(D,E)$ and $h(D,E)$ satisfy ACCP, we need the following lemma.

Lemma 2.3. *Let $D \subseteq E$ be an extension of integral domains with characteristic zero and let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $H(D,E)$ (resp., $h(D,E)$). If $(f_n)_{n\geq 1}$ has an infinite subsequence of series with positive order (resp., polynomials with positive degree), then $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,E) = (0)$ (resp., $\bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,E) = (0)$).*

Proof. We first consider the composite Hurwitz series ring case. Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $H(D,E)$ which has an infinite subsequence of series with positive order. If $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,E)$, then for all $n \geq 1$, there exists an element $h_n \in H(D,E)$ such that $g = f_1 * \cdots * f_n * h_n$. Note that by Proposition 2.1, $H(D,E)$ is an integral domain; so $\text{ord}(g) \geq \text{ord}(f_1) + \cdots + \text{ord}(f_n)$ for all $n \geq 1$. Since $\text{ord}(f_i)$ is positive for infinitely many i, $\text{ord}(g) = \infty$. Hence $g = 0$, which indicates that $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,E) = (0)$.

We next consider the composite Hurwitz polynomial ring case. Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $h(D,E)$ which contains an infinite subsequence of polynomials with positive degree. Let $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,E)$. Then for all $n \geq 1$, we can find a suitable element $h_n \in h(D,E)$ such that $g = f_1 * \cdots * f_n * h_n$. Note that by Proposition 2.1, $h(D,E)$ is an integral domain; so $\deg(g) \geq \deg(f_1) + \cdots + \deg(f_n)$ for all $n \geq 1$. Since $\deg(f_i) \geq 1$ for infinitely many i, $\deg(g) = \infty$. Thus $g = 0$, which shows that $\bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,E) = (0)$. □

We now give the main result in this section.

Theorem 2.4. *Let $D \subseteq E$ be an extension of integral domains with characteristic zero. Then the following statements are equivalent.*

(1) *$H(D,E)$ satisfies ACCP.*

(2) *$h(D,E)$ satisfies ACCP.*

(3) *$\bigcap_{n\geq 1} a_1 \cdots a_n E = (0)$ for each infinite sequence $(a_n)_{n\geq 1}$ consisting of nonzero nonunits of D.*

(4) *$D + XE[\![X]\!]$ satisfies ACCP.*

(5) *$D + XE[X]$ satisfies ACCP.*

Proof. (1) ⇒ (2) Let $f_1 * h(D,E) \subseteq f_2 * h(D,E) \subseteq \cdots$ be an ascending chain of principal ideals of $h(D,E)$. Then for each $i \geq 1$, there exists an element $g_i \in h(D,E)$ such that $f_i = f_{i+1} * g_i$. Note that $f_1 * H(D,E) \subseteq f_2 * H(D,E) \subseteq \cdots$ is an ascending chain of principal ideals of $H(D,E)$. Since $H(D,E)$ satisfies ACCP, we can find a positive integer m such that $f_k * H(D,E) = f_m * H(D,E)$ for all $k \geq m$. Hence for all $k \geq m$, there exists a unit $h_k \in H(D,E)$ such that $f_k = f_{k+1} * h_k$. Since $H(D,E)$ is an integral domain, $h_k = g_k$ for all $k \geq m$. Note that $\{\deg(f_i)\}_{i\geq 1}$ is a monotone decreasing sequence of nonnegative integers; so it should be stationary. Let n be the smallest positive integer such that $\deg(f_i) = \deg(f_n)$ for all $i \geq n$.

Then $\deg(g_i) = 0$ for all $i \geq n$, which means that g_i is a unit in D for all $i \geq n$. Thus $f_i * h(D,E) = f_n * h(D,E)$ for all $i \geq n$.

(2) $\Rightarrow$ (3) Let $(a_n)_{n\geq 1}$ be an infinite sequence consisting of nonzero nonunits of D and let $e \in \bigcap_{n\geq 1} a_1 \cdots a_n E$. Note that by Lemma 2.2(2), $(a_n)_{n\geq 1}$ is an infinite sequence consisting of nonzero nonunits of $h(D,E)$. Since $h(D,E)$ satisfies ACCP, we have

$$\begin{aligned} eX &\in \bigcap_{n\geq 1} a_1 \cdots a_n h(D,E) \\ &= (0), \end{aligned}$$

where the equality follows from [2, Remark 1.1]. Thus $e = 0$, which indicates that $\bigcap_{n\geq 1} a_1 \cdots a_n E = (0)$.

(3) $\Rightarrow$ (1) Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $H(D,E)$. If $(f_n)_{n\geq 1}$ contains an infinite subsequence of series with positive order, then by Lemma 2.3, $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,E) = (0)$; so we may assume that for all $n \geq 1$, the order of f_n is zero. Let $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,E)$. Then for all $n \geq 1$, there exists an element $h_n \in H(D,E)$ such that $g = f_1 * \cdots * f_n * h_n$. Since $H(D,E)$ is an integral domain, we have

$$g(\mathrm{ord}(g)) = f_1(0) \cdots f_n(0) h_n(\mathrm{ord}(h_n))$$

for all $n \geq 1$. Note that by Lemma 2.2(1), $\{f_i(0)\}_{i\geq 1}$ consists of nonzero nonunits in D. Hence we obtain

$$\begin{aligned} g(\mathrm{ord}(g)) &\in \bigcap_{n\geq 1} f_1(0) \cdots f_n(0) E \\ &= (0) \end{aligned}$$

by (3). Thus $g = 0$, which implies that $H(D,E)$ satisfies ACCP [2, Remark 1.1].

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) These equivalences were shown in [2, Proposition 1.2 and Remark 1.4] (or [5, Theorem 3.4] and [6, Corollary 1.5(2)]). □

Remark 2.5. By replacing the order of a Hurwitz series with the degree of a Hurwitz polynomial, we can give a direct proof of (3) $\Rightarrow$ (2) in Theorem 2.4. For the sake of completeness, we insert it as follows. Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonunits of $h(D,E)$. If $(f_n)_{n\geq 1}$ has an infinite subsequence of polynomials with positive degree, then by Lemma 2.3, $\bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,E) = (0)$; so we may assume that f_n is a constant for all $n \geq 1$. Let $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,E)$ and a be the coefficient of the largest degree term of g. Then for all $n \geq 1$, there exists an element $h_n \in h(D,E)$ such that $g = f_1 * \cdots * f_n * h_n$; so for all $n \geq 1$, $a = f_1 \cdots f_n b_n$, where b_n is the coefficient of the largest degree term of h_n. Hence $a \in \bigcap_{n\geq 1} f_1 \cdots f_n E$. However $\bigcap_{n\geq 1} f_1 \cdots f_n E = (0)$ by (3). Thus $g = 0$, which implies that $h(D,E)$ satisfies ACCP.

3. Composite Hurwitz Rings $H(D,I)$ and $h(D,I)$

In this section, we characterize when the composite Hurwitz series ring $H(D,I)$ and the composite Hurwitz polynomial ring $h(D,I)$ satisfy ACCP, where D is an integral domain with characteristic zero and I is a nonzero proper ideal of D. To do this, we study analogues of Proposition 2.1 and Lemmas 2.2 and 2.3.

Proposition 3.1. *Let D be a commutative ring with identity and I a nonzero proper ideal of D. Then the following conditions are equivalent.*

(1) *$H(D,I)$ is an integral domain.*

(2) *$h(D,I)$ is an integral domain.*

(3) *D is both an integral domain and a torsion-free $\mathbb{Z}$-module.*

(4) *D is an integral domain with characteristic zero.*

Lemma 3.2. *Let D be a commutative ring with identity and I a nonzero proper ideal of D. Then the following assertions hold.*

(1) *A Hurwitz series $f \in H(D,I)$ is a unit if and only if $f(0)$ is a unit in D.*

(2) *A Hurwitz polynomial $f = \sum_{i=0}^{n} a_i X^i \in h(D,I)$ is a unit if and only if a_0 is a unit in D and for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion.*

Proof. (1) If f is a unit in $H(D,I)$, then there exists an element $g \in H(D,I)$ such that $f * g = 1$; so $f(0)g(0) = 1$. Hence $f(0)$ is a unit in D. Conversely, if $f(0)$ is a unit in D, then we can find a suitable element $g \in H(D)$ such that $f * g = 1$ [4, Proposition 2.5]. Now we claim that $g \in H(D,I)$. If $g \in D$, then we have nothing to prove; so we assume that $g \notin D$. Let m be the smallest positive integer in $\text{supp}(g)$. Then $0 = (f * g)(m) = f(0)g(m) + f(m)g(0)$; so $g(m) = -\frac{f(m)g(0)}{f(0)} \in I$. Let $n \in \text{supp}(g) \setminus \{0\}$ and suppose that $g(k) \in I$ for all $k \in \text{supp}(g)$ with $0 < k < n$. Note that $0 = (f*g)(n) = f(0)g(n) + \sum_{i=1}^{n} \binom{n}{i} f(i)g(n-i)$; so $g(n) = -\frac{\sum_{i=1}^{n} \binom{n}{i} f(i)g(n-i)}{f(0)} \in I$. Hence $g \in H(D,I)$, and thus f is a unit in $H(D,I)$.

(2) If f is a unit in $h(D,I)$, then $f * g = 1$ for some $g \in h(D,I)$; so $a_0 g(0) = 1$. Hence a_0 is a unit in D. Also, f is a unit in $h(D)$; so for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion [1, Theorem 3.1]. For the converse, assume that a_0 is a unit in D and for each $i = 1, \ldots, n$, a_i is nilpotent or some power of a_i is with torsion. Then f is a unit in $h(D)$ [1, Theorem 3.1]; so $f * g = 1$ for some $g \in h(D)$. Now, a simple modification of the proof of (1) shows that $g \in h(D,I)$. Thus f is a unit in $h(D,I)$. □

Lemma 3.3. *Let D be an integral domain with characteristic zero, I a nonzero proper ideal of D, and let $(f_n)_{n \geq 1}$ be an infinite sequence of nonzero nonunits* *of $H(D,I)$ (resp., $h(D,I)$). If $(f_n)_{n\geq 1}$ has an infinite subsequence of series with positive order (resp., polynomials with positive degree), then $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,I) = (0)$ (resp., $\bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,I) = (0)$).*

Proof. While the proof is similar to that of Lemma 2.3, we include it for the sake of completeness.

We first consider the composite Hurwitz series ring case. Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $H(D,I)$ which has an infinite subsequence of series with positive order. If $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,I)$, then for all $n \geq 1$, there exists an element $h_n \in H(D,I)$ such that $g = f_1 * \cdots * f_n * h_n$. Note that by Proposition 3.1, $H(D,I)$ is an integral domain; so $\text{ord}(g) \geq \text{ord}(f_1) + \cdots + \text{ord}(f_n)$ for all $n \geq 1$. Since $\text{ord}(f_i)$ is positive for infinitely many i, $\text{ord}(g) = \infty$. Hence $g = 0$, which shows that $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,I) = (0)$.

We next consider the composite Hurwitz polynomial ring case. Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $h(D,I)$ which contains an infinite subsequence of polynomials with positive degree. Let $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,I)$. Then for all $n \geq 1$, we can find a suitable element $h_n \in h(D,I)$ such that $g = f_1 * \cdots * f_n * h_n$. Note that by Proposition 3.1, $h(D,I)$ is an integral domain; so $\deg(g) \geq \deg(f_1) + \cdots + \deg(f_n)$ for all $n \geq 1$. Since f_i has the positive degree for infinitely many i, $\deg(g) = \infty$. Thus $g = 0$, which indicates that $\bigcap_{n\geq 1} f_1 * \cdots * f_n * h(D,I) = (0)$. □

We now give the main result in this section.

Theorem 3.4. *Let D be an integral domain with characteristic zero and I a nonzero proper ideal of D. Then the following statements are equivalent.*

(1) *$H(D,I)$ satisfies ACCP.*

(2) *$h(D,I)$ satisfies ACCP.*

(3) *D satisfies ACCP.*

(4) *$D + XI[\![X]\!]$ satisfies ACCP.*

(5) *$D + XI[X]$ satisfies ACCP.*

(6) *$H(D)$ satisfies ACCP.*

(7) *$h(D)$ satisfies ACCP.*

(8) *$D[\![X]\!]$ satisfies ACCP.*

(9) *$D[X]$ satisfies ACCP.*

Proof. (1) ⇒ (2) Let $f_1 * h(D,I) \subseteq f_2 * h(D,I) \subseteq \cdots$ be an ascending chain of principal ideals of $h(D,I)$. Then for each $i \geq 1$, there exists an element $g_i \in h(D,I)$ such that $f_i = f_{i+1} * g_i$. Note that $f_1 * H(D,I) \subseteq f_2 * H(D,I) \subseteq \cdots$ is an ascending chain of principal ideals of $H(D,I)$. Since $H(D,I)$ satisfies ACCP, we can find a positive integer m such that $f_k * H(D,I) = f_m * H(D,I)$ for all $k \geq m$. Hence for all $k \geq m$, there exists a unit $h_k \in H(D,I)$ such that $f_k = f_{k+1} * h_k$. Since $H(D,I)$ is an integral domain, $h_k = g_k$ for all $k \geq m$. Note that $\{\deg(f_i)\}_{i\geq 1}$ is a monotone decreasing sequence of nonnegative integers; so it should stop in finite steps. Let n be the smallest positive integer such that $\deg(f_i) = \deg(f_n)$ for all $i \geq n$. Then $\deg(g_i) = 0$ for all $i \geq n$, which means that g_i is a unit in D for all $i \geq n$. Thus $f_i * h(D,I) = f_n * h(D,I)$ for all $i \geq n$.

(2) $\Rightarrow$ (3) Let $(a_n)_{n\geq 1}$ be an infinite sequence consisting of nonzero nonunits of D and choose any $d \in \bigcap_{n\geq 1} a_1 \cdots a_n D$. Note that by Lemma 3.2(2), $(a_n)_{n\geq 1}$ is an infinite sequence consisting of nonzero nonunits of $h(D,I)$. Since $h(D,I)$ satisfies ACCP, we have

$$\begin{aligned} dX &\in \bigcap_{n\geq 1} a_1 \cdots a_n h(D,I) \\ &= (0), \end{aligned}$$

where the equality comes from [2, Remark 1.1]. Hence $d = 0$, which indicates that $\bigcap_{n\geq 1} a_1 \cdots a_n D = (0)$. Thus D satisfies ACCP.

(3) $\Rightarrow$ (1) Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $H(D,I)$. If $(f_n)_{n\geq 1}$ contains an infinite subsequence of series with positive order, then by Lemma 3.3, $\bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,I) = (0)$; so we may assume that for all $n \geq 1$, the order of f_n is zero. Let $g \in \bigcap_{n\geq 1} f_1 * \cdots * f_n * H(D,I)$. Then for all $n \geq 1$, there exists an element $h_n \in H(D,I)$ such that $g = f_1 * \cdots * f_n * h_n$. Since $H(D,I)$ is an integral domain, we have

$$g(\mathrm{ord}(g)) = f_1(0) \cdots f_n(0) h_n(\mathrm{ord}(h_n))$$

for all $n \geq 1$. Note that by Lemma 3.2(1), $\{f_i(0)\}_{i\geq 1}$ consists of nonzero nonunits in D. Since D satisfies ACCP, we obtain

$$\begin{aligned} g(\mathrm{ord}(g)) &\in \bigcap_{n\geq 1} f_1(0) \cdots f_n(0) D \\ &= (0). \end{aligned}$$

Thus $g = 0$, which implies that $H(D,I)$ satisfies ACCP.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) These equivalences appear in [3, Proposition 4.6] (or [5, Theorem 3.7] and [6, Corollary 1.5(3)]).

(3) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) These equivalences follow directly from Theorem 2.4 and [2, Remark 1.1] by applying $D = E$. □

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