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Weakly Classical Prime Submodules

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Abstract. In this paper, all rings are commutative with nonzero identity. Let M be an R-module. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. We introduce the concept of "weakly classical prime submodules" and we will show that this class of submodules enjoys many properties of weakly 2-absorbing ideals of commutative rings. A proper submodule N of M is a *weakly classical prime submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.

1. Introduction

Throughout this paper all rings are commutative with nonzero identity and all modules are considered to be unitary. Several authors have extended the notion of prime ideal to modules, see, for example [16, 19, 20]. Let M be a module over a commutative ring R. A proper submodule N of M is called *prime* if for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. Anderson and Smith [4] said that a proper ideal I of a ring R is *weakly prime* if

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whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b \in I$. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [17]. A proper submodule N of M is called *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [12, 13] (see also, [14], in which, the notion of classical prime submodules is named "weakly prime submodules"). For more information on classical prime submodules, the reader is referred to [5, 6, 15].

The annihilator of M which is denoted by $\mathrm{Ann}_R(M)$ is $(0 :_R M)$. Furthermore, for every $m \in M$, $(0 :_R m)$ is denoted by $\mathrm{Ann}_R(m)$. When $\mathrm{Ann}_R(M) = 0$, M is called a *faithful R-module*. An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R, see [18]. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. Finitely generated faithful multiplication modules are cancellation modules [24, Corollary to Theorem 9], where an R-module M is defined to be a *cancellation module* if $IM = JM$ for ideals I and J of R implies $I = J$. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $m, m' \in M$, by mm', we mean the product of Rm and Rm'. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [2]). Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by M-rad(N), is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say M-rad$(N) = M$. It is shown in [18, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$(N) = \sqrt{(N :_R M)}M$. In [22], Quartararo et al. said that a commutative ring R is a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a u-ring. Moreover, they proved that every Prüfer domain is a u-domain. Also, any ring which contains an infinite field as a subring is a u-ring, [23, Exercise 3.63].

In this paper we introduce the concept of weakly classical prime submodules and we will show that this class of submodules enjoys many properties of weakly 2-absorbing ideals of commutative rings as in [8]. We like to emphasize that our study in this paper is inspired by the work as in [4, 7, 8] and [11, Section 3]. We recall from Badawi [7] that a proper ideal of R is said to be a *2-absorbing ideal of R* if whenever $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Badawi and Darani in [8] called a proper ideal I of R a *weakly 2-absorbing ideal of R* if whenever $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. For more information about the theory of 2-absorbing ideals and its generalizations we refer to [3, 9, 10, 21]. Now we state our definition of weakly classical prime submodule. A proper submodule N of an

R-module M is called a *weakly classical prime submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$. Clearly, every classical prime submodule is a weakly classical prime submodule. Among many results in this paper, it is shown (Theorem 2.17.) that N is a weakly classical prime submodule of an R-module M if and only if for every ideals I, J of R and $m \in M$ with $0 \neq IJm \subseteq N$, either $Im \subseteq N$ or $Jm \subseteq N$. It is proved (Theorem 2.19.) that if N is a weakly classical prime submodule of an R-module M that is not classical prime, then $(N :_R M)^2 N = 0$. It is shown (Theorem 2.25.) that over a *um*-ring R, N is a weakly classical prime submodule of an R-module M if and only if for every ideals I, J of R and submodule L of M with $0 \neq IJL \subseteq N$, either $IL \subseteq N$ or $JL \subseteq N$. Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R-module where M_i is an R_i-module, for $i = 1, 2, 3$. In Theorem 2.38. it is proved that if N is a weakly classical prime submodule of M, then either $N = \{(0, 0, 0)\}$ or N is a classical prime submodule of M. Let R be a *um*-ring, M be an R-module and F be a faithfully flat R-module. It is shown (Theorem 2.39.) that N is a weakly classical prime submodule of M if and only if $F \otimes N$ is a weakly classical prime submodule of $F \otimes M$.

2. Properties of Weakly Classical Prime Submodules

First of all we give a module which has no nonzero weakly classical prime submodule.

Example 2.1. Let p be a fixed prime integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then

$$E(p) := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in \mathbb{N}_0 \right\}$$

is a nonzero submodule of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each $t \in \mathbb{N}_0$, set

$$G_t := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \right\}.$$

Notice that for each $t \in \mathbb{N}_0$, G_t is a submodule of $E(p)$ generated by $\frac{1}{p^t} + \mathbb{Z}$ for each $t \in \mathbb{N}_0$. Each proper submodule of $E(p)$ is equal to G_i for some $i \in \mathbb{N}_0$ (see, [23, Example 7.10]). However, no G_t is a weakly classical prime submodule of $E(p)$. Indeed, $\frac{1}{p^{t+2}} + \mathbb{Z} \in E(p)$. Then $0 \neq p^2 \left(\frac{1}{p^{t+2}} + \mathbb{Z} \right) = \frac{1}{p^t} + \mathbb{Z} \in G_t$ but $p \left(\frac{1}{p^{t+2}} + \mathbb{Z} \right) = \frac{1}{p^{t+1}} + \mathbb{Z} \notin G_t$.

Theorem 2.2. *Let M be an R-module and N a proper submodule of M.*

1. *If N is a weakly classical prime submodule of M, then $(N :_R m)$ is a weakly prime ideal of R for every $m \in M \backslash N$ with $Ann_R(m) = 0$.*
2. *If $(N :_R m)$ is a weakly prime ideal of R for every $m \in M \backslash N$, then N is a weakly classical prime submodule of M.*

Proof. (1) Suppose that N is a weakly classical prime submodule. Let $m \in M \backslash N$ with $\mathrm{Ann}_R(m) = 0$, and $0 \neq ab \in (N :_R m)$ for some $a, b \in R$. Then $0 \neq abm \in N$. So $am \in N$ or $bm \in N$. Hence $a \in (N :_R m)$ or $b \in (N :_R m)$.

(2) Assume that $(N :_R m)$ is a weakly prime ideal of R for every $m \in M \backslash N$. Let $0 \neq abm \in N$ for some $m \in M$ and $a, b \in R$. If $m \in N$, then we are done. So we assume that $m \notin N$. Hence $0 \neq ab \in (N :_R m)$ implies that either $a \in (N :_R m)$ or $b \in (N :_R m)$. Therefore either $am \in N$ or $bm \in N$, and so N is weakly classical prime. □

We recall that M is a torsion-free R-module if and only if for every $0 \neq m \in M$, $\mathrm{Ann}_R(m) = 0$. As a direct consequence of Theorem 2.2. the following result follows.

Corollary 2.3. *Let M be a torsion-free R-module and N a proper submodule of M. Then N is a weakly classical prime submodule of M if and only if $(N :_R m)$ is a weakly prime ideal of R for every $m \in M \backslash N$.*

Theorem 2.4. *Let $f : M \to M'$ be a homomorphism of R-modules.*

1. *Suppose that f is a monomorphism. If N' is a weakly classical prime submodule of M' with $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a weakly classical prime submodule of M.*

2. *Suppose that f is an epimorphism. If N is a weakly classical prime submodule of M containing $Ker(f)$, then $f(N)$ is a weakly classical prime submodule of M'.*

Proof. (1) Suppose that N' is a weakly classical prime submodule of M' with $f^{-1}(N') \neq M$. Let $0 \neq abm \in f^{-1}(N')$ for some $a, b \in R$ and $m \in M$. Since f is a monomorphism, $0 \neq f(abm) \in N'$. So we get $0 \neq abf(m) \in N'$. Hence $f(am) = af(m) \in N'$ or $f(bm) = bf(m) \in N'$. Thus $am \in f^{-1}(N')$ or $bm \in f^{-1}(N')$. Therefore $f^{-1}(N')$ is a weakly classical prime submodule of M.

(2) Assume that N is a weakly classical prime submodule of M. Let $a, b \in R$ and $m' \in M'$ be such that $0 \neq abm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abm) \in f(N)$. Since $Ker(f) \subseteq N$, we have $0 \neq abm \in N$. It implies that $am \in N$ or $bm \in N$. Hence $am' \in f(N)$ or $bm' \in f(N)$. Consequently $f(N)$ is a weakly classical prime submodule of M'. □

As an immediate consequence of Theorem 2.4.(2) we have the following corollary.

Corollary 2.5. *Let M be an R-module and $L \subset N$ be submodules of M. If N is a weakly classical prime submodule of M, then N/L is a weakly classical prime submodule of M/L.*

Theorem 2.6. *Let K and N be submodules of M with $K \subset N \subset M$. If K is a weakly classical prime submodule of M and N/K is a weakly classical prime submodule of M/K, then N is a weakly classical prime submodule of M.*

Proof. Let $a, b \in R$, $m \in M$ and $0 \neq abm \in N$. If $abm \in K$, then $am \in K \subset N$ or $bm \in K \subset N$ as it is needed. Thus, assume that $abm \notin K$. Then $0 \neq ab(m+K) \in N/K$, and so $a(m+K) \in N/K$ or $b(m+K) \in N/K$. It means that $am \in N$ or $bm \in N$, which completes the proof. □

For an R-module M, the set of zero-divisors of M is denoted by $Z_R(M)$.

Theorem 2.7. *Let M be an R-module, N be a submodule of M and S be a multiplicative subset of R.*

1. *If N is a weakly classical prime submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a weakly classical prime submodule of $S^{-1}M$.*

2. *If $S^{-1}N$ is a weakly classical prime submodule of $S^{-1}M$ such that $S \cap Z_R(N) = \emptyset$ and $S \cap Z_R(M/N) = \emptyset$, then N is a weakly classical prime submodule of M.*

Proof. (1) Let N be a weakly classical prime submodule of M and $(N :_R M) \cap S = \emptyset$. Suppose that $0 \neq \frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} \in S^{-1}N$ for some $a_1, a_2 \in R$, $s_1, s_2, s_3 \in S$ and $m \in M$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. If $sa_1a_2m = 0$, then $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} = \frac{sa_1a_2m}{ss_1s_2s_3} = \frac{0}{1}$, a contradiction. Since N is a weakly classical prime submodule, then we have $a_1(sm) \in N$ or $a_2(sm) \in N$. Thus $\frac{a_1}{s_1}\frac{m}{s_3} = \frac{sa_1m}{ss_1s_3} \in S^{-1}N$ or $\frac{a_2}{s_2}\frac{m}{s_3} = \frac{sa_2m}{ss_2s_3} \in S^{-1}N$. Consequently $S^{-1}N$ is a weakly classical prime submodule of $S^{-1}M$.

(2) Suppose that $S^{-1}N$ is a weakly classical prime submodule of $S^{-1}M$ and $S \cap Z_R(N) = \emptyset$ and $S \cap Z_R(M/N) = \emptyset$. Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Then $\frac{a}{1}\frac{b}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{a}{1}\frac{b}{1}\frac{m}{1} = \frac{0}{1}$, then there exists $s \in S$ such that $sabm = 0$ which contradicts $S \cap Z_R(N) = \emptyset$. Therefore $\frac{a}{1}\frac{b}{1}\frac{m}{1} \neq \frac{0}{1}$, and so either $\frac{a}{1}\frac{m}{1} \in S^{-1}N$ or $\frac{b}{1}\frac{m}{1} \in S^{-1}N$. We may assume that $\frac{a}{1}\frac{m}{1} \in S^{-1}N$. So there exists $u \in S$ such that $uam \in N$. But $S \cap Z_R(M/N) = \emptyset$, whence $am \in N$. Consequently N is a weakly classical prime submodule of M. □

Following the notion of (weakly) 2-absorbing ideals of commutative rings (as in [7] and [8]), Darani [25] generalized the concept of prime submodules (resp. weakly prime submodules) of a module over a commutative ring as following: Let N be a proper submodule of an R-module M. Then N is said to be a *2-absorbing submodule (resp. weakly 2-absorbing submodule) of* M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

Proposition 2.8. *Let N be a proper submodule of an R-module M.*

1. *If N is a weakly prime submodule of M, then N is a weakly classical prime submodule of M.*

2. *If N is a weakly classical prime submodule of M, then N is a weakly 2-absorbing submodule of M. The converse holds if in addition $(N :_R M)$ is a weakly prime ideal of R.*

Proof. (1) Assume that N is a weakly prime submodule of M. Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Therefore either $bm \in N$ or $a \in (N :_R M)$. The first case leads us to the claim. In the second case we have that $am \in N$. Consequently N is a weakly classical prime submodule.

(2) It is evident that if N is weakly classical prime, then it is weakly 2-absorbing. Assume that N is a weakly 2-absorbing submodule of M and $(N :_R M)$ is a weakly prime ideal of R. Let $0 \neq abm \in N$ for some $a, b \in R$ and $m \in M$ such that neither $am \in N$ nor $bm \in N$. Then $0 \neq ab \in (N :_R M)$ and so either $a \in (N :_R M)$ or $b \in (N :_R M)$. This contradiction shows that N is weakly classical prime. □

The following example shows that the converse of Proposition 2.8.(1) is not true.

Example 2.9. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}$ where p is a prime integer. Consider the submodule $N = \{\overline{0}\} \bigoplus \{0\} \bigoplus \mathbb{Z}$ of M. Notice that $(\overline{0}, 0, 0) \neq p(\overline{1}, 0, 1) = (\overline{0}, 0, p) \in N$, but $(\overline{1}, 0, 1) \notin N$. Also $p(\overline{1}, 1, 1) \notin N$ which shows that $p \notin (N :_{\mathbb{Z}} M)$. Therefore N is not a weakly prime submodule of M. Now, we show that N is a weakly classical prime submodule of M. Let $m, n, z, w \in \mathbb{Z}$ and $\overline{x} \in \mathbb{Z}_p$ be such that $(\overline{0}, 0, 0) \neq mn(\overline{x}, z, w) \in N$. Hence $\overline{mnx} = \overline{0}$ and $mnz = 0$. Therefore $p|mnx$ and $z = 0$. So $p|m$ or $p|nx$. If $p|m$, then $m(\overline{x}, z, w) = (\overline{mx}, 0, mw) = (\overline{0}, 0, mw) \in N$. Similarly, if $p|nx$, then $n(\overline{x}, z, w) = (\overline{nx}, 0, nw) = (\overline{0}, 0, nw) \in N$. Consequently N is a weakly classical prime submodule of M.

Proposition 2.10. *Let M be a cyclic R-module. Then a proper submodule N of M is a weakly prime submodule if and only if it is a weakly classical prime submodule.*

Proof. By Proposition 2.8.(1), the "only if" part holds. Let $M = Rm$ for some $m \in M$ and N be a weakly classical prime submodule of M. Suppose that $0 \neq rx \in N$ for some $r \in R$ and $x \in M$. Then there exists an element $s \in R$ such that $x = sm$. Therefore $0 \neq rx = rsm \in N$ and since N is a weakly classical prime submodule, $rm \in N$ or $sm \in N$. Hence $r \in (N :_R M)$ or $x \in N$. Consequently N is a weakly prime submodule. □

Example 2.11. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \bigoplus \mathbb{Z}_q \bigoplus \mathbb{Q}$ where $p,\ q$ are two distinct prime integers. One can easily see that the zero submodule of M is a weakly classical prime submodule. Notice that $pq(\overline{1}, \overline{1}, 0) = (\overline{0}, \overline{0}, 0)$, but $p(\overline{1}, \overline{1}, 0) \neq (\overline{0}, \overline{0}, 0)$ and $q(\overline{1}, \overline{1}, 0) \neq (\overline{0}, \overline{0}, 0)$. So the zero submodule of M is not classical prime. Hence the two concepts of classical prime submodules and of weakly classical prime submodules are different in general.

The following definition is an analogue of [8, Page 3] and [11, Definition 3.8].

Definition 2.12. Let N be a proper submodule of M and $a, b \in R$, $m \in M$. If N is a weakly classical prime submodule and $abm = 0$, $am \notin N$, $bm \notin N$, then (a, b, m) is called a *classical triple-zero of* N.

The following result and its proof are analogues of [11, Lemma 3.10].

Theorem 2.13. *Let N be a weakly classical prime submodule of an R-module M and suppose that $abK \subseteq N$ for some $a, b \in R$ and some submodule K of M. If (a, b, k) is not a classical triple-zero of N for every $k \in K$, then $aK \subseteq N$ or $bK \subseteq N$.*

Proof. Suppose that (a, b, k) is not a classical triple-zero of N for every $k \in K$. Assume on the contrary that $aK \not\subseteq N$ and $bK \not\subseteq N$. Then there are $k_1, k_2 \in K$ such that $ak_1 \notin N$ and $bk_2 \notin N$. If $abk_1 \neq 0$, then we have $bk_1 \in N$, because $ak_1 \notin N$ and N is a weakly classical prime submodule of M. If $abk_1 = 0$, then since $ak_1 \notin N$ and (a, b, k_1) is not a classical triple-zero of N, we conclude again that $bk_1 \in N$. By a similar argument, since (a, b, k_2) is not a classical triple-zero and $bk_2 \notin N$, then we deduce that $ak_2 \in N$. From our hypothesis, $ab(k_1 + k_2) \in N$ and $(a, b, k_1 + k_2)$ is not a classical triple-zero of N. Hence we have either $a(k_1 + k_2) \in N$ or $b(k_1 + k_2) \in N$. If $a(k_1 + k_2) = ak_1 + ak_2 \in N$, then since $ak_2 \in N$, we have $ak_1 \in N$, a contradiction. If $b(k_1 + k_2) = bk_1 + bk_2 \in N$, then since $bk_1 \in N$, we have $bk_2 \in N$, which again is a contradiction. Thus $aK \subseteq N$ or $bK \subseteq N$. □

The following definition is an analogue of [11, Definition 3.9].

Definition 2.14. Let N be a weakly classical prime submodule of an R-module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M. We say that N is a *free classical triple-zero with respect to IJK* if (a, b, k) is not a classical triple-zero of N for every $a \in I, b \in J$, and $k \in K$.

Remark 2.15. Let N be a weakly classical prime submodule of M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M such that N is a free classical triple-zero with respect to IJK. Hence if $a \in I, b \in J$, and $k \in K$, then $ak \in N$ or $bk \in N$.

The following result is an analogue of [11, Theorem 3.11].

Corollary 2.16. *Let N be a weakly classical prime submodule of an R-module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M. If N is a free classical triple-zero with respect to IJK, then $IK \subseteq N$ or $JK \subseteq N$.*

Proof. Suppose that N is a free classical triple-zero with respect to IJK. Assume that $IK \not\subseteq N$ and $JK \not\subseteq N$. Then there are $a \in I$ and $b \in J$ with $aK \not\subseteq N$ and $bK \not\subseteq N$. Since $abK \subseteq N$ and N is free classical triple-zero with respect to IJK, then Theorem 2.13. implies that $aK \subseteq N$ and $bK \subseteq N$ which is a contradiction. Consequently $IK \subseteq N$ or $JK \subseteq N$. □

Let M be an R-module and N a submodule of M. For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_M a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N.

In the next theorem we characterize weakly classical prime submodules.

Theorem 2.17. *Let M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is weakly classical prime;*

2. *For every* $a, b \in R$, $(N :_M ab) = (0 :_M ab) \cup (N :_M a) \cup (N :_M b)$*;*
3. *For every* $a \in R$ *and* $m \in M$ *with* $am \notin N$, $(N :_R am) = (0 :_R am) \cup (N :_R m)$*;*
4. *For every* $a \in R$ *and* $m \in M$ *with* $am \notin N$, $(N :_R am) = (0 :_R am)$ *or* $(N :_R am) = (N :_R m)$*;*
5. *For every* $a \in R$ *and every ideal* I *of* R *and* $m \in M$ *with* $0 \neq aIm \subseteq N$, *either* $am \in N$ *or* $Im \subseteq N$*;*
6. *For every ideal* I *of* R *and* $m \in M$ *with* $Im \nsubseteq N$, $(N :_R Im) = (0 :_R Im)$ *or* $(N :_R Im) = (N :_R m)$*;*
7. *For every ideals* I, J *of* R *and* $m \in M$ *with* $0 \neq IJm \subseteq N$, *either* $Im \subseteq N$ *or* $Jm \subseteq N$.

Proof. (1)⇒(2) Suppose that N is a weakly classical prime submodule of M. Let $m \in (N :_M ab)$. Then $abm \in N$. If $abm = 0$, then $m \in (0 :_M ab)$. Assume that $abm \neq 0$. Hence $am \in N$ or $bm \in N$. Therefore $m \in (N :_M a)$ or $m \in (N :_M b)$. Consequently, $(N :_M ab) = (0 :_M ab) \cup (N :_M a) \cup (N :_M b)$.
(2)⇒(3) Let $am \notin N$ for some $a \in R$ and $m \in M$. Assume that $x \in (N :_R am)$. Then $axm \in N$, and so $m \in (N :_M ax)$. Since $am \notin N$, then $m \notin (N :_M a)$. Thus by part (2), $m \in (0 :_M ax)$ or $m \in (N :_M x)$, whence $x \in (0 :_R am)$ or $x \in (N :_R m)$. Therefore $(N :_R am) = (0 :_R am) \cup (N :_R m)$.
(3)⇒(4) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.
(4)⇒(5) Let for some $a \in R$, an ideal I of R and $m \in M$, $0 \neq aIm \subseteq N$. Hence $I \subseteq (N :_R am)$ and $I \nsubseteq (0 :_R am)$. If $am \in N$, then we are done. So, assume that $am \notin N$. Therefore by part (4) we have that $I \subseteq (N :_R m)$, i.e., $Im \subseteq N$.
(5)⇒(6)⇒(7) Have proofs similar to that of the previous implications.
(7)⇒(1) Is trivial. □

An analogue of [8, Theorem 2.3] is the following result.

Theorem 2.18. *Let* N *be a weakly classical prime submodule of* M *and suppose that* (a, b, m) *is a classical triple-zero of* N *for some* $a, b \in R$ *and* $m \in M$. *Then*

1. $abN = 0$.
2. $a(N :_R M)m = 0$.
3. $b(N :_R M)m = 0$.
4. $(N :_R M)^2 m = 0$.
5. $a(N :_R M)N = 0$.
6. $b(N :_R M)N = 0$.

Proof. (1) Suppose that $abN \neq 0$. Then there exists $n \in N$ with $abn \neq 0$. Hence $0 \neq ab(m+n) = abn \in N$, so we conclude that $a(m+n) \in N$ or $b(m+n) \in N$. Thus $am \in N$ or $bm \in N$, which contradicts the assumption that (a, b, m) is classical triple-zero. Thus $abN = 0$.

(2) Let $axm \neq 0$ for some $x \in (N :_R M)$. Then $a(b+x)m \neq 0$, because $abm = 0$. Since $xm \in N$, $a(b+x)m \in N$. Then $am \in N$ or $(b+x)m \in N$. Hence $am \in N$ or $bm \in N$, which contradicts our hypothesis.

(3) The proof is similar to part (2).

(4) Assume that $x_1x_2m \neq 0$ for some $x_1, x_2 \in (N :_R M)$. Then by parts (2) and (3), $(a + x_1)(b + x_2)m = x_1x_2m \neq 0$. Clearly $(a + x_1)(b + x_2)m \in N$. Then $(a + x_1)m \in N$ or $(b + x_2)m \in N$. Therefore $am \in N$ or $bm \in N$ which is a contradiction. Consequently $(N :_R M)^2m = 0$.

(5) Let $axn \neq 0$ for some $x \in (N :_R M)$ and $n \in N$. Therefore by parts (1) and (2) we conclude that $0 \neq a(b + x)(m + n) = axn \in N$. So $a(m + n) \in N$ or $(b + x)(m + n) \in N$. Hence $am \in N$ or $bm \in N$. This contradiction shows that $a(N :_R M)N = 0$.

(6) Similart to part (5). □

A submodule N of an R-module M is called a nilpotent submodule if $(N :_R M)^kN = 0$ for some positive integer k (see [1]), and we say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M.

Theorem 2.19. *If N is a weakly classical prime submodule of an R-module M that is not classical prime, then $(N :_R M)^2N = 0$ and so N is nilpotent.*

Proof. Suppose that N is a weakly classical prime submodule of M that is not classical prime. Then there exists a classical triple-zero (a, b, m) of N for some $a, b \in R$ and $m \in M$. Assume that $(N :_R M)^2N \neq 0$. Hence there are $x_1, x_2 \in (N :_R M)$ and $n \in N$ such that $x_1x_2n \neq 0$. By Theorem 2.18. $0 \neq (a + x_1)(b + x_2)(m + n) = x_1x_2n \in N$. So $(a + x_1)(m + n) \in N$ or $(b + x_1)(m + n) \in N$. Therefore $am \in N$ or $bm \in N$, a contradiction. □

Remark 2.20. Let M be a multiplication R-module and K, L be submodules of M. Then there are ideals I, J of R such that $K = IM$ and $L = JM$. Thus $KL = IJM = IL$. In particular $KM = IM = K$. Also, for any $m \in M$ we define $Km := KRm$. Hence $Km = IRm = Im$.

The following corollary is an analogue of [8, Theorem 2.4].

Corollary 2.21. *If N is a weakly classical prime submodule of a multiplication R-module M that is not classical prime, then $N^3 = 0$.*

Proof. Since M is multiplication, then $N = (N :_R M)M$. Therefore by Theorem 2.19. and Remark 2.20. $N^3 = (N :_R M)^2N = 0$. □

Assume that $\text{Nil}(M)$ is the set of nilpotent elements of M. If M is faithful, then $\text{Nil}(M)$ is a submodule of M and if M is faithful multiplication, then $\text{Nil}(M) = \text{Nil}(R)M = \bigcap Q$ $(= M\text{-rad}(\{0\}))$, where the intersection runs over all prime submodules of M, [1, Theorem 6].

Theorem 2.22. *Let N be a weakly classical prime submodule of M. If N is not classical prime, then*

1. $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$.
2. *If M is multiplication, then M-rad(N)=M-rad$(\{0\})$. If in addition M is faithful, then M-rad$(N) = \mathrm{Nil}(M)$.*

Proof. (1) Assume that N is not classical prime. By Theorem 2.19. $(N :_R M)^2N = 0$. Then

$$\begin{aligned}(N :_R M)^3 &= (N :_R M)^2(N :_R M)\\ &\subseteq ((N :_R M)^2N :_R M)\\ &= (0 :_R M),\end{aligned}$$

and so $(N :_R M) \subseteq \sqrt{(0 :_R M)}$. Hence, we have $\sqrt{(N :_R M)} = \sqrt{(0 :_R M)} = \sqrt{\mathrm{Ann}_R(M)}$.

(2) By part (1), M-rad$(N) = \sqrt{(N :_R M)}M = \sqrt{(0 :_R M)}M = M$-rad$(\{0\}) = \mathrm{Nil}(M)$. □

Corollary 2.23. *Let R be a ring and I be a proper ideal of R.*

1. *${}_RI$ is a weakly classical prime submodule of ${}_RR$ if and only if I is a weakly prime ideal of R.*
2. *Every proper ideal of R is weakly prime if and only if for every R-module M and every proper submodule N of M, N is a weakly classical prime submodule of M.*

Proof. (1) Let ${}_RI$ be a weakly classical prime submodule of ${}_RR$. Then by Theorem 2.2.(1), $(I :_R 1) = I$ is a weakly prime ideal of R. For the converse, notice that ${}_RI$ is a weakly prime submodule of ${}_RR$ if and only if I is a weakly prime ideal of R. Now, apply part (1) of Proposition 2.8.

(2) Assume that every proper ideal of R is weakly prime. Let N be a proper submodule of an R-module M. Since for every $m \in M\backslash N$, $(N :_R m)$ is a proper ideal of R, then it is a weakly prime ideal of R. Hence by Theorem 2.2.(2), N is a weakly classical prime submodule of M. We have the converse immediately by part (1). □

Regarding Remark 2.20. we have the next proposition.

Proposition 2.24. *Let M be a multiplication R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is a weakly classical prime submodule of M;*
2. *If $0 \neq N_1N_2m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$, then either $N_1m \subseteq N$ or $N_2m \subseteq N$.*

Proof. (1)⇒(2) Let $0 \neq N_1N_2m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$. Since M is multiplication, there are ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Therefore $0 \neq N_1N_2m = I_1I_2m \subseteq N$, and so either $I_1m \subseteq N$ or $I_2m \subseteq N$. Hence $N_1m \subseteq N$ or $N_2m \subseteq N$.
(2)⇒(1) Suppose that $0 \neq I_1I_2m \subseteq N$ for some ideals I_1, I_2 of R and some $m \in M$. It is sufficient to set $N_1 := I_1M$ and $N_2 := I_2M$ in part (2). □

Theorem 2.25. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is weakly classical prime;*
2. *For every $a, b \in R$, $(N :_M ab) = (0 :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$;*
3. *For every $a, b \in R$ and every submodule L of M, $0 \neq abL \subseteq N$ implies that $aL \subseteq N$ or $bL \subseteq N$;*
4. *For every $a \in R$ and every submodule L of M with $aL \nsubseteq N$, $(N :_R aL) = (0 :_R aL)$ or $(N :_R aL) = (N :_R L)$;*
5. *For every $a \in R$, every ideal I of R and every submodule L of M, $0 \neq aIL \subseteq N$ implies that $aL \subseteq N$ or $IL \subseteq N$;*
6. *For every ideal I of R and every submodule L of M with $IL \nsubseteq N$, $(N :_R IL) = (0 :_R IL)$ or $(N :_R IL) = (N :_R L)$;*
7. *For every ideals I, J of R and every submodule L of M, $0 \neq IJL \subseteq N$ implies that $IL \subseteq N$ or $JL \subseteq N$.*

Proof. Similar to that of Theorem 2.17. □

Remark 2.26. The zero submodule of the $\mathbb{Z}$-module $\mathbb{Z}_4$, is a weakly classical prime submodule (weakly prime ideal) of $\mathbb{Z}_4$, but $(0 :_{\mathbb{Z}} \mathbb{Z}_4) = 4\mathbb{Z}$ is not a weakly prime ideal of $\mathbb{Z}$.

Proposition 2.27. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. If N is a weakly classical prime submodule of M, then $(N :_R L)$ is a weakly prime ideal of R for every faithful submodule L of M that is not contained in N.*

Proof. Assume that N is a weakly classical prime submodule of M and L is a faithful submodule of M such that $L \nsubseteq N$. Let $0 \neq ab \in (N :_R L)$ for some $a, b \in R$. Then $0 \neq abL \subseteq N$, because L is faithful. Hence Theorem 2.25. implies that $aL \subseteq N$ or $bL \subseteq N$, i.e., $a \in (N :_R L)$ or $b \in (N :_R L)$. Consequently $(N :_R L)$ is a weakly prime ideal of R. □

Proposition 2.28. *Let M be an R-module and N be a weakly classical prime submodule of M. Then*

1. *For every* $a, b \in R$ *and* $m \in M$, $(N :_R abm) = (0 :_R abm) \cup (N :_R am) \cup (N :_R bm)$*;*
2. *If* R *is a u-ring, then for every* $a, b \in R$ *and* $m \in M$, $(N :_R abm) = (0 :_R abm)$ *or* $(N :_R abm) = (N :_R am)$ *or* $(N :_R abm) = (N :_R bm)$*.*

Proof. (1) Let $a, b \in R$ and $m \in M$. Suppose that $r \in (N :_R abm)$. Then $ab(rm) \in N$. If $ab(rm) = 0$, then $r \in (0 :_R abm)$. Therefore we assume that $ab(rm) \neq 0$. So, either $a(rm) \in N$ or $b(rm) \in N$. Thus, either $r \in (N :_R am)$ or $r \in (N :_R bm)$. Consequently $(N :_R abm) = (0 :_R abm) \cup (N :_R am) \cup (N :_R bm)$.

(2) Apply part (1). □

Theorem 2.29. *Let* R *be a um-ring,* M *be a faithful multiplication* R*-module and* N *be a proper submodule of* M*. The following conditions are equivalent:*

1. N *is a weakly classical prime submodule of* M*;*
2. *If* $0 \neq N_1N_2N_3 \subseteq N$ *for some submodules* N_1, N_2, N_3 *of* M*, then either* $N_1N_3 \subseteq N$ *or* $N_2N_3 \subseteq N$*;*
3. *If* $0 \neq N_1N_2 \subseteq N$ *for some submodules* N_1, N_2 *of* M*, then either* $N_1 \subseteq N$ *or* $N_2 \subseteq N$*;*
4. N *is a weakly prime submodule of* M*;*
5. $(N :_R M)$ *is a weakly prime ideal of* R*.*

Proof. (1)⇒(2) Let $0 \neq N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M. Since M is multiplication, there are ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Therefore $0 \neq I_1I_2N_3 \subseteq N$, and so by Theorem 2.25. $I_1N_3 \subseteq N$ or $I_2N_3 \subseteq N$. Thus, either $N_1N_3 \subseteq N$ or $N_2N_3 \subseteq N$.
(2)⇒(3) Is easy.
(3)⇒(4) Suppose that $0 \neq IK \subseteq N$ for some ideal I of R and some submodule K of M. It is sufficient to set $N_1 := IM$ and $N_2 = K$ in part (3).
(4)⇒(1) By part (1) of Proposition 2.8.
(1)⇒(5) By Proposition 2.27.
(5)⇒(4) Let $0 \neq IK \subseteq N$ for some ideal I of R and some submodule K of M. Since M is multiplication, then there is an ideal J of R such that $K = JM$. Hence $0 \neq IJ \subseteq (N :_R M)$ which implies that either $I \subseteq (N :_R M)$ or $J \subseteq (N :_R M)$. If $I \subseteq (N :_R M)$, then we are done. So, suppose that $J \subseteq (N :_R M)$. Thus $K = JM \subseteq N$. □

Proposition 2.30. *Let* R *be a um-ring. Let* M *be a finitely generated faithful multiplication* R*-module and* N *a submodule of* M*. Then the following conditions are equivalent:*

1. N *is a weakly classical prime submodule;*
2. $(N :_R M)$ *is a weakly prime ideal of* R*;*

3. $N = IM$ *for some weakly prime ideal* I *of* R.

Proof. (1) $\Leftrightarrow$ (2) . By Theorem 2.29.
(2) $\Rightarrow$ (3) Since $(N :_R M)$ is a weakly prime ideal and $N = (N :_R M)\,M$, then condition (3) holds.
(3) $\Rightarrow$ (2) Suppose that $N = IM$ for some weakly prime ideal I of R. Since M is a multiplication module, we have $N = (N : M)\,M$. Therefore $N = IM = (N : M)\,M$ and so $I = (N : M)$, because by [24, Corollary to Theorem 9] M is cancellation. □

Theorem 2.31. *Let* M_1, M_2 *be* R*-modules and* N_1 *be a proper submodule of* M_1*. Then the following conditions are equivalent:*

1. $N = N_1 \times M_2$ *is a weakly classical prime submodule of* $M = M_1 \times M_2$*;*
2. N_1 *is a weakly classical prime submodule of* M_1 *and for each* $r, s \in R$ *and* $m_1 \in M_1$ *we have*

$$rsm_1 = 0,\ rm_1 \notin N_1,\ sm_1 \notin N_1 \Rightarrow rs \in Ann_R(M_2).$$

Proof. (1)$\Rightarrow$(2) Suppose that $N = N_1 \times M_2$ is a weakly classical prime submodule of $M = M_1 \times M_2$. Let $r, s \in R$ and $m_1 \in M_1$ be such that $0 \neq rsm_1 \in N_1$. Then $(0, 0) \neq rs(m_1, 0) \in N$. Thus $r(m_1, 0) \in N$ or $s(m_1, 0) \in N$, and so $rm_1 \in N_1$ or $sm_1 \in N_1$. Consequently N_1 is a weakly classical prime submodule of M_1. Now, assume that $rsm_1 = 0$ for some $r, s \in R$ and $m_1 \in M_1$ such that $rm_1 \notin N_1$ and $sm_1 \notin N_1$. Suppose that $rs \notin \mathrm{Ann}_R(M_2)$. Therefore there exists $m_2 \in M_2$ such that $rsm_2 \neq 0$. Hence $(0, 0) \neq rs(m_1, m_2) \in N$, and so $r(m_1, m_2) \in N$ or $s(m_1, m_2) \in N$. Thus $rm_1 \in N_1$ or $sm_1 \in N_1$ which is a contradiction. Consequently $rs \in \mathrm{Ann}_R(M_2)$.
(2)$\Rightarrow$(1) Let $r, s \in R$ and $(m_1, m_2) \in M = M_1 \times M_2$ be such that $(0, 0) \neq rs(m_1, m_2) \in N = N_1 \times M_2$. First assume that $rsm_1 \neq 0$. Then by part (2), $rm_1 \in N_1$ or $sm_1 \in N_1$. So $r(m_1, m_2) \in N$ or $s(m_1, m_2) \in N$, and thus we are done. If $rsm_1 = 0$, then $rsm_2 \neq 0$. Therefore $rs \notin \mathrm{Ann}_R(M_2)$, and so part (2) implies that either $rm_1 \in N_1$ or $sm_1 \in N_1$. Again we have that $r(m_1, m_2) \in N$ or $s(m_1, m_2) \in N$ which shows N is a weakly classical prime submodule of M. □

The following two propositions have easy verifications.

Proposition 2.32. *Let* M_1, M_2 *be* R*-modules and* N_1 *be a proper submodule of* M_1*. Then* $N = N_1 \times M_2$ *is a classical prime submodule of* $M = M_1 \times M_2$ *if and only if* N_1 *is a classical prime submodule of* M_1*.*

Proposition 2.33. *Let* M_1, M_2 *be* R*-modules and* N_1, N_2 *be proper submodules of* M_1, M_2*, respectively. If* $N = N_1 \times N_2$ *is a weakly classical prime (resp. classical prime) submodule of* $M = M_1 \times M_2$*, then* N_1 *is a weakly classical prime (resp. classical prime) submodule of* M_1 *and* N_2 *is a weakly classical prime (resp. classical prime) submodule of* M_2*.*

Example 2.34. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$ and $N = p\mathbb{Z} \times q\mathbb{Z}$ where p, q are two distinct prime integers. Since $p\mathbb{Z}$, $q\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $p\mathbb{Z}$, $q\mathbb{Z}$ are weakly classical prime $\mathbb{Z}$-submodules of $\mathbb{Z}$. Notice that $(0,0) \neq pq(1,1) = (pq, pq) \in N$, but neither $p(1,1) \in N$ nor $q(1,1) \in N$. So N is not a weakly classical prime submodule of M. This example shows that the converse of Proposition 2.33. is not true.

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.35. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times M_2$ is a proper submodule of M. Then the following conditions are equivalent:*

1. *N_1 is a classical prime submodule of M_1;*
2. *N is a classical prime submodule of M;*
3. *N is a weakly classical prime submodule of M.*

Proof. (1)⇒(2) Let $(a_1, a_2)(b_1, b_2)(m_1, m_2) \in N$ for some $(a_1, a_2), (b_1, b_2) \in R$ and $(m_1, m_2) \in M$. Then $a_1 b_1 m_1 \in N_1$ so either $a_1 m_1 \in N_1$ or $b_1 m_1 \in N_1$ which shows that either $(a_1, a_2)(m_1, m_2) \in N$ or $(b_1, b_2)(m_1, m_2) \in N$. Consequently N is a classical prime submodule of M.
(2)⇒(3) It is clear that every classical prime submodule is a weakly classical prime submodule.
(3)⇒(1) Let $abm \in N_1$ for some $a, b \in R_1$ and $m \in M_1$. We may assume that $0 \neq m' \in M_2$. Therefore $0 \neq (a,1)(b,1)(m,m') \in N$. So either $(a,1)(m,m') \in N$ or $(b,1)(m,m') \in N$. Therefore $am \in N_1$ or $bm \in N_1$. Hence N_1 is a classical prime submodule of M_1. □

Proposition 2.36. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that N_1, N_2 are proper submodules of M_1, M_2, respectively. If $N = N_1 \times N_2$ is a weakly classical prime submodule of M, then N_1 is a weakly prime submodule of M_1 and N_2 is a weakly prime submodule of M_2.*

Proof. Suppose that $N = N_1 \times N_2$ is a weakly classical prime submodule of M. By hypothesis, there exist $x \in M_1 \backslash N_1$ and $y \in M_2 \backslash N_2$. First we show that N_1 is a weakly prime submodule of M_1. Let $0 \neq am_1 \in N_1$ for some $a \in R_1$ and $m_1 \in M_1$. Then $0 \neq (1,0)(a,1)(m_1, y) \in N_1 \times N_2 = N$. Notice that if $(a,1)(m_1,y) \in N_1 \times N_2 = N$, then $y \in N_2$ which is a contradiction. So we get $(1,0)(m_1,y) \in N_1 \times N_2 = N$. Thus $m_1 \in N_1$. Hence N_1 is a weakly prime submodule of M_1. A similar argument shows that N_2 is a weakly prime submodule of M_2. □

The following example shows that the converse of Proposition 2.36. is not true in general.

Example 2.37. Let $R = M = \mathbb{Z} \times \mathbb{Z}$ and $N = p\mathbb{Z} \times q\mathbb{Z}$ where $p,\ q$ are two distinct prime integers. Since $p\mathbb{Z},\ q\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $p\mathbb{Z},\ q\mathbb{Z}$ are weakly prime (weakly classical prime) $\mathbb{Z}$-submodules of $\mathbb{Z}$. Notice that $(0,0) \neq (p,1)(1,q)(1,1) = (p,q) \in N$, but neither $(p,1)(1,1) \in N$ nor $(1,q)(1,1) \in N$. So N is not a weakly classical prime submodule of M.

Theorem 2.38. *Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R-module where M_1 is an R_1-module, M_2 is an R_2-module and M_3 is an R_3-module. If N is a weakly classical prime submodule of M, then either $N = \{(0,0,0)\}$ or N is a classical prime submodule of M.*

Proof. Since $\{(0,0,0)\}$ is a weakly classical prime submodule in any module, we may assume that $N = N_1 \times N_2 \times N_3 \neq \{(0,0,0)\}$. We assume that N is not a classical prime submodule of M and reach a contradiction. Without loss of generality we may assume that $N_1 \neq 0$ and so there is $0 \neq n \in N_1$. We claim that $N_2 = M_2$ or $N_3 = M_3$. Suppose that there are $m_2 \in M_2 \backslash N_2$ and $m_3 \in M_3 \backslash N_3$. Get $r \in (N_2 :_{R_2} M_2)$ and $s \in (N_3 :_{R_3} M_3)$. Since $(0,0,0) \neq (1,r,1)(1,1,s)(n,m_2,m_3) = (n,rm_2,sm_3) \in N$, then $(1,r,1)(n,m_2,m_3) = (n,rm_2,m_3) \in N$ or $(1,1,s)(n,m_2,m_3) = (n,m_2,sm_3) \in N$. Therefore either $m_3 \in N_3$ or $m_2 \in N_2$, a contradiction. Hence $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. Let $N = N_1 \times M_2 \times N_3$. Then $(0,1,0) \in (N :_R M)$. Clearly $(0,1,0)^2 N \neq \{(0,0,0)\}$. So $(N :_R M)^2 N \neq \{(0,0,0)\}$ which is a contradiction, by Theorem 2.19. In the case when $N = N_1 \times N_2 \times M_3$ we have that $(0,0,1) \in (N :_R M)$ and similar to the previous case we reach a contradiction. □

Theorem 2.39. *Let R be a um-ring and M be an R-module.*

1. *If F is a flat R-module and N is a weakly classical prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly classical prime submodule of $F \otimes M$.*
2. *Suppose that F is a faithfully flat R-module. Then N is a weakly classical prime submodule of M if and only if $F \otimes N$ is a weakly classical prime submodule of $F \otimes M$.*

Proof. (1) Let $a,b \in R$. Then by Theorem 2.25. either $(N :_M ab) = (0 :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. Assume that $(N :_M ab) = (0 :_M ab)$. Then by [6, Lemma 3.2],

$$(F \otimes N :_{F\otimes M} ab) = F \otimes (N :_M ab) = F \otimes (0 :_M ab)$$

$$= (F \otimes 0 :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)\,.$$

Now, suppose that $(N :_M ab) = (N :_M a)$. Again by [6, Lemma 3.2],

$$(F \otimes N :_{F\otimes M} ab) = F \otimes (N :_M ab) = F \otimes (N :_M a)$$

$$= (F \otimes N :_{F\otimes M} a)\,.$$

Similarly, we can show that if $(N :_M ab) = (N :_M b)$, then $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} b)$. Consequently by Theorem 2.25. we deduce that $F \otimes N$ is a weakly classical prime submodule of $F \otimes M$.

(2) Let N be a weakly classical prime submodule of M and assume that $F \otimes N = F \otimes M$. Then $0 \to F \otimes N \overset{\subseteq}{\to} F \otimes M \to 0$ is an exact sequence. Since F is a faithfully flat module, $0 \to N \overset{\subseteq}{\to} M \to 0$ is an exact sequence. So $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a weakly classical prime submodule by (1). Now for the converse, let $F \otimes N$ be a weakly classical prime submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b \in R$. Then $(F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$ or $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} a)$ or $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} b)$ by Theorem 2.25. Suppose that $(F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$. Hence

$$F \otimes (N :_M ab) = (F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$$
$$= (F \otimes 0 :_{F\otimes M} ab) = F \otimes (0 :_M ab)\,.$$

Thus $0 \to F \otimes (0 :_M ab) \overset{\subseteq}{\to} F \otimes (N :_M ab) \to 0$ is an exact sequence. Since F is a faithfully flat module, $0 \to (0 :_M ab) \overset{\subseteq}{\to} (N :_M ab) \to 0$ is an exact sequence which implies that $(N :_M ab) = (0 :_M ab)$. With a similar argument we can deduce that if $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} a)$ or $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} b)$, then $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. Consequently N is a weakly classical prime submodule of M by Theorem 2.25. □

Corollary 2.40. *Let R be a um-ring, M be an R-module and X be an indeterminate. If N is a weakly classical prime submodule of M, then $N[X]$ is a weakly classical prime submodule of $M[X]$.*

Proof. Assume that N is a weakly classical prime submodule of M. Notice that $R[X]$ is a flat R-module. Then by Theorem 2.39. $R[X] \otimes N \simeq N[X]$ is a weakly classical prime submodule of $R[X] \otimes M \simeq M[X]$. □

References

[1] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra, **36**(2008), 4620–4642.

[2] R. Ameri, *On the prime submodules of multiplication modules*, Inter. J. Math. Math. Sci., **27**(2003), 1715–1724.

[3] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, **39**(2011), 1646–1672.

[4] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math., **29**(2003), 831–840.

[5] A. Azizi, *On prime and weakly prime submodules*, Vietnam J. Math., **36(3)**(2008), 315–325.

[6] A. Azizi, *Weakly prime submodules and prime submodules*, Glasgow Math. J., **48**(2006), 343–346.

[7] A. Badawi, On 2-absorbing ideals of commutative rings, Bull.Austral. Math. Soc., **75**(2007), 417–429.

[8] A. Badawi and A. Y. Darani, On weakly 2-absorbing ideals of commutative rings, Houston J. Math., **39(2)**(2013), 441–452.

[9] A. Badawi, Ü. Tekir, E. A. Uğurlu, G. Ulucak and E. Y. Çelikel, *Generalizations of 2-absorbing primary ideals of commutative rings*, Turkish J. Math., **40**(2016), 703–717.

[10] A. Badawi, Ü. Tekir and E. Yetkin, *On* 2*-absorbing primary ideals of commutative rings*, Bull. Korean Math. Soc., **51(4)**(2014), 1163–1173.

[11] A. Badawi, Ü. Tekir and E. Yetkin, *On weakly* 2*-absorbing primary ideals of commutative rings*, J. Korean Math. Soc., **52(1)**(2015), 97–111.

[12] M. Behboodi, *A generalization of Bears lower nilradical for modules*, J. Algebra Appl., **6(2)**(2007), 337-353.

[13] M. Behboodi, *On weakly prime radical of modules and semi-compatible modules*, Acta Math. Hungar., **113(3)**(2006), 239-250.

[14] M. Behboodi and H. Koohy, *Weakly prime modules*, Vietnam J. Math., **32(2)**(2004), 185-195.

[15] M. Behboodi and S. H. Shojaee, *On chains of classical prime submodules and dimension theory of modules*, Bull. Iranian Math. Soc., **36(1)**(2010), 149–166.

[16] J. Dauns, *Prime modules*, J. Reine Angew. Math., **298**(1978), 156–181.

[17] S. Ebrahimi Atani and F. Farzalipour, *On weakly prime submodules*, Tamk. J. Math., **38(3)**(2007), 247–252.

[18] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, **16**(1988), 755–779.

[19] C.-P. Lu, *Prime submodules of modules*, Comm. Math. Univ. Sancti Pauli, **33**(1984), 61-69.

[20] R. L. McCasland and M. E. Moore, *Prime submodules*, Comm. Algebra, **20**(1992), 1803–1817.

[21] H. Mostafanasab and A. Y. Darani, *On ϕ-n-absorbing primary ideals of commutative rings*, J. Korean Math. Soc., **53(3)**(2016), 549–582.

[22] P. Quartararo and H. S. Butts, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc., **52**(1975), 91-96.

[23] R.Y. Sharp, *Steps in commutative algebra*, Second edition, Cambridge University Press, Cambridge, 2000.

[24] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50**(1988), 223–235.

[25] A. Yousefian Darani and F. Soheilnia, *On* 2*-absorbing and weakly* 2*-absorbing submodules*, Thai J. Math., **9**(2011), 577-584.