

ANALYTIC EXTENSIONS OF M -HYPONORMAL OPERATORS

SALAH MECHERI AND FEI ZUO

ABSTRACT. In this paper, we introduce the class of analytic extensions of M -hyponormal operators and we study various properties of this class. We also use a special Sobolev space to show that every analytic extension of an M -hyponormal operator T is subscalar of order $2k + 2$. Finally we obtain that an analytic extension of an M -hyponormal operator satisfies Weyl's theorem.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H . If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and range space of T . As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved interesting problems for hyponormal operators (e.g., the invariant subspace problem), one of recent trends in operator theory is studying natural extensions of hyponormal operators. So we introduce some of these non-hyponormal operators. An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$, and T is said to be M -hyponormal if there exists a real positive number M such that

$$M^2(T - \lambda)^*(T - \lambda) \geq (T - \lambda)(T - \lambda)^* \text{ for all } \lambda \in \mathbb{C}.$$

It is an easy extension of hyponormal operators, evidently,

$$T \text{ is hyponormal} \Rightarrow T \text{ is } M\text{-hyponormal}.$$

The following facts follow from the above definition and some well known facts about M -hyponormal operators.

- (i) If T is M -hyponormal, then so is $T - \lambda$ for each $\lambda \in \mathbb{C}$.
- (ii) If T is M -hyponormal and $\mathcal{M} \subseteq H$ is a closed T -invariant subspace, then $T|_{\mathcal{M}}$ is M -hyponormal.
- (iii) If T is M -hyponormal, then $N(T - \lambda) \subseteq N(T - \lambda)^*$ for every $\lambda \in \mathbb{C}$.

Received November 7, 2014; Revised July 16, 2015.

2010 *Mathematics Subject Classification*. Primary 47B20, 47A15.

Key words and phrases. M -hyponormal operator, Bishop's property (β), subscalar operator, Weyl's theorem.

(iv) Every quasinilpotent M -hyponormal operator is a zero operator.

(v) T is M -hyponormal if and only if there exists a positive real number M such that $M\|(T - \lambda)x\| \geq \|(T - \lambda)^*x\|$ for all $x \in H$ and for all $\lambda \in \mathbb{C}$. We give the following example to indicate that there exists an M -hyponormal operator which is not hyponormal.

Example 1.1. Consider the unilateral weighted shift operator as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \dots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on $H = l_2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^\infty$ is the canonical orthogonal basis for l_2 . It is well known that W_α is hyponormal if and only if α is monotonically increasing. Also, if α is eventually increasing, then W_α is M -hyponormal (see [14]). Hence, if we take the weights α such that $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 2, \alpha_3 = \alpha_4 = \dots$, then W_α is an M -hyponormal operator, but it is not hyponormal.

In [24] the author introduce the class of k -quasi- M -hyponormal operators defined as follows:

Definition 1.1. An operator T is said to be a k -quasi- M -hyponormal operator if there exists a real positive number M such that

$$T^{*k}(M^2(T - \lambda)^*(T - \lambda))T^k \geq T^{*k}(T - \lambda)(T - \lambda)^*T^k \text{ for all } \lambda \in \mathbb{C},$$

where k is a positive integer.

It is clear that

$$\text{hyponormal} \Rightarrow M\text{-Hyponormal} \Rightarrow k\text{-quasi-}M\text{-hyponormal}.$$

In order to generalize these classes we introduce the class of analytic extensions of M -hyponormal operators defined as follows:

Definition 1.2. An operator $T \in B(H_1 \oplus H_2)$ is said to be an analytic extension of an M -hyponormal operator, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is an M -hyponormal operator and T_3 is analytic of order k , where k is a positive integer.

An operator $T \in B(H)$ is said to be analytic if there exists a nonconstant analytic function F on a neighborhood of $\sigma(T)$ such that $F(T) = 0$. We say that an operator $T \in B(H)$ is algebraic if there is a nonconstant polynomial p such that $p(T) = 0$. In particular, if $T^k = 0$ for some positive integer k , then T is called nilpotent. If an operator $T \in B(H)$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T)$. Since F cannot have infinitely many zeros in D , we write $F(z) = G(z)p(z)$ where G is a function that is analytic and does not vanish on D and p is a nonconstant polynomial with zeros in D . By Riesz-Dunford calculus, $G(T)$ is invertible and then $p(T) = 0$, which means that T is algebraic (see [6]). When p has degree k , we say that T is analytic with order k .

A bounded linear operator T on H is called scalar of order m if it possesses a spectral distribution of order m , i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \rightarrow B(H)$$

such that $\Phi(z) = T$, where z stands for the identity function on \mathbb{C} , and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on \mathbb{C} , continuously differentiable of order m , $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

In 1984, Putinar [27] (or [28]) showed that every hyponormal operator is subscalar of order 2. In 1987, his theorem was used to show that hyponormal operator with thick spectra has a nontrivial invariant subspace, which was a result due to Brown (see [5]). In this paper, we study various properties of an analytic extension of an M -hyponormal operator. We show that every analytic extension of an M -hyponormal operator T is subscalar of order $2k+2$. Finally, we study Weyl's theorem of analytic extensions of M -hyponormal operators, we obtain if T is an analytic extension of an M -hyponormal operator, then Weyl's theorem holds for $f(T)$, where f is any analytic function on some neighborhood of $\sigma(T)$.

2. Preliminaries

Let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, $\sigma(T)$ and $\text{iso}\sigma(T)$ for the spectrum and the isolated spectrum point of T , respectively. An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimension null space and its range of finite co-dimension. The index of a Fredholm operator $T \in B(H)$ is given by $i(T) = \alpha(T) - \beta(T)$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $w(T)$ of $T \in B(H)$ are defined by $\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$ and $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$, respectively (see [15, 16]). Let $\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ denote the set of isolated eigenvalues of finite multiplicity. We say that Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$.

Let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of \mathbb{C} . We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \rightarrow H$, such that

$$\|f\|_{2,U} := \left(\int_U \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, H)$ which are analytic functions in U (i.e., $\bar{\partial}f = 0$) is defined by $A^2(U, H) := L^2(U, H) \cap O(U, H)$, where $O(U, H)$ denotes the Fréchet space of H -valued analytic functions on U with respect to uniform topology. Note that $A^2(U, H)$ is a Hilbert space.

Let us define now a special Sobolev type space. Let U be again a bounded open subset of \mathbb{C} and m be a fixed nonnegative integer. The vector valued

Sobolev space $W^m(U, H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\bar{\partial}f, \dots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

$$\|f\|_{W^m}^2 := \sum_{i=0}^m \|\bar{\partial}^i f\|_{2,U}^2.$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$.

Let U be a (connected) bounded open subset of \mathbb{C} and let m be a non-negative integer. The linear operator M_f of multiplication by f on $W^m(U, H)$ is continuous and it has a spectral distribution of order m , defined by the functional calculus

$$\Phi_M : C_0^m(\mathbb{C}) \rightarrow B(W^m(U, H)), \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m .

An operator $T \in B(H)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbrev. SVEP at λ_0), if for every open neighborhood G of λ_0 , the only analytic function $f : G \rightarrow H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in G$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0 , with values in H , which verifies $(T - z)f(z) = x$, and it is called the local resolvent set of T at x . We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x , and define the local spectral subspace of T , $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of \mathbb{C} . An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of \mathbb{C} and every sequence $f_n : G \rightarrow H$ of H -valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G , $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G . An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of \mathbb{C} . An operator $T \in B(H)$ is said to have property (δ) if for every open covering (U, V) of \mathbb{C} , we have $H = H_T(\bar{U}) + H_T(\bar{V})$. It is well known that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

In particular, the single valued extension property of operators was first introduced by N. Dunford to investigate the class of spectral operators which is another important generalization of normal operators (see [9]). In the local spectral theory, for given an operator T on a complex Banach space \mathcal{X} and a vector $x \in \mathcal{X}$, one is often interested in the existence and the uniqueness of analytic solution $f(\cdot) : U \rightarrow \mathcal{X}$ of the local resolvent equation

$$(T - \lambda)f(\lambda) = x$$

on suitable open subset U of \mathbb{C} . Obviously, if T has SVEP, then the existence of analytic solution to any local resolvent equation (related to T) implies the

uniqueness of its analytic solution. SVEP is possessed by many important classes of operators such as hyponormal operators and decomposable operators. The interested reader is referred to [8, 19, 21, 22, 23] for more details. To underline the significance of Bishop's property (β); we mention the important connections to sheaf theory and the spectral theory of several commuting operators from the monograph by Eschmeier and Putinar [12]. There are also interesting applications to invariant subspaces [12], harmonic analysis [11], and the theory of automatic continuity [18].

For many reasons the most satisfactory generalization to the general Banach space setting of the normal operators on a Hilbert space is the concept of decomposable operator. These operators possess a spectral theorem and a rich lattice structure for which it is possible to develop what is called a local spectral theory, i.e., a local analysis of their spectra. Decomposability may be defined in several ways, for instance by means of the concept of the local spectral subspace.

Definition 2.1. An operator $T \in B(H)$ is said to be decomposable if T has both Dunford property (C) and property (δ).

Every scalar operator is decomposable. Standard examples of decomposable operators are normal operators on Hilbert spaces and operators which have totally disconnected spectra.

Two important subspaces in local spectral theory, as well as in Fredholm theory, are $H_T(\{\lambda\})$ associated with the singleton set $\{\lambda\}$ and $H_T(\{\mathbb{C} \setminus \lambda\})$. We have

a) $H_T(\{\lambda\})$ coincides with the quasi-nilpotent part $H_0(\lambda I - T)$ of $\lambda I - T$, defined as

$$H_0(\lambda I - T) := \{x \in H : \lim_{n \rightarrow \infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0\}.$$

b) $H_T(\{\mathbb{C} \setminus \lambda\})$ coincides with the analytic core $\mathcal{K}(\lambda I - T)$ defined as the set of $x \in H$ such that there exist $c > 0$ and a sequence (x_n) in H for which

$$(\lambda I - T)x_1 = x, (\lambda I - T)x_{n+1} = x_n$$

and

$$\|x_n\| \leq c^n \|x\| \text{ for all } n \in \mathbb{N}.$$

$H_0(\lambda I - T)$ and $\mathcal{K}(\lambda I - T)$ are in general not closed. Moreover, $H_0(\lambda I - T)$ contains the kernels $N(\lambda I - T)^n$ for all $n \in \mathbb{N}$. We also have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda.$$

Definition 2.2. An operator $T \in B(H)$ is said to have property (Q) if $H_T(\{\lambda\}) = H_0(\lambda I - T)$ is closed for all $\lambda \in \mathbb{C}$.

We have

$$\text{Bishop's property}(\beta) \Rightarrow \text{Dunford property}(\mathcal{C}) \Rightarrow \text{property}(\mathcal{Q}) \Rightarrow \text{SVEP}.$$

Although the property (Q) seems to be rather strong, the class of operators having property (Q) is considerably large. A first example of operators with this property is given by the convolution operators of the group algebra $L_1(G)$, G is an abelian locally compact group.

Definition 2.3. An operator $T \in B(H)$ is said to belong to the class $H(p)$ if there exists a natural number $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in \mathbb{C}.$$

Clearly, every operator T which belongs to $H(p)$ has property (Q). Every convolution operator of the group algebra $L^1(G)$ is $H(1)$.

Theorem 2.1 ([26]). *Every subscalar operator $T \in B(H)$ is $H(p)$.*

Classical examples of subscalar operators are hyponormal operators. In this paper we will show that other important classes of operators are $H(p)$.

Definition 2.4. An operator $T \in B(H)$ is said to be polaroid if every $\lambda \in \text{iso}\sigma(T)$ is a pole of the resolvent of T .

Note that

$$T \text{ is polaroid} \Leftrightarrow T^* \text{ is polaroid.}$$

The condition of being polaroid may be characterized by means of the quasi-nilpotent part:

Theorem 2.2 ([3]). *An operator $T \in B(H)$ is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in \text{iso}\sigma(T).$$

Corollary 2.1. *Every $H(p)$ operator is polaroid.*

Since a subscalar operator is $H(p)$ we have:

Corollary 2.2. *Every subscalar operator is polaroid.*

3. Analytic extension of M -hyponormal operator

Lemma 3.1 (See [27, Proposition 2.1]). *For a bounded open disk D in the complex plane \mathbb{C} , there is a positive number C_D such that for an arbitrary operator $T \in B(H)$ and $f \in W^2(D, H)$ we have*

$$\|(I - P)f\|_{2,D} \leq C_D(\|(T - z)^*\bar{\partial}f\|_{2,D} + \|(T - z)^*\bar{\partial}^2f\|_{2,D}),$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

Corollary 3.1. *Let D and C_D be as in Lemma 3.1. If $T \in B(H)$ is an M -hyponormal operator and $f \in W^2(D, H)$, then for all $z \in \mathbb{C}$, there exists a positive number M such that*

$$\|(I - P)f\|_{2,D} \leq MC_D(\|(T - z)\bar{\partial}f\|_{2,D} + \|(T - z)\bar{\partial}^2f\|_{2,D}),$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

Proof. This follows from Lemma 3.1 and the definition of M -hyponormal operator. \square

Lemma 3.2. *Let $T \in B(H)$ be an M -hyponormal operator and let D be a bounded disk in \mathbb{C} . If $\{f_n\}$ is a sequence in $W^m(D, H)$ ($m > 2$) such that*

$$\lim_{n \rightarrow \infty} \|(z - T)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \dots, m$, then $\lim_{n \rightarrow \infty} \|\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, 2, \dots, m - 2$ where D_0 is a disk strictly contained in D .

Proof. Since T is an M -hyponormal operator, it follows from Corollary 3.1 that there exist positive numbers C_D and M such that

$$(1) \quad \|(I - P)\bar{\partial}^i f_n\|_{2,D} \leq MC_D (\|(T - z)\bar{\partial}^{i+1} f_n\|_{2,D} + \|(T - z)\bar{\partial}^{i+2} f_n\|_{2,D})$$

for $i = 0, 1, 2, \dots, m - 2$. From (1), we have

$$(2) \quad \lim_{n \rightarrow \infty} \|(I - P)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 0, 1, 2, \dots, m - 2$. So, it holds that

$$\lim_{n \rightarrow \infty} \|(T - z)P\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \dots, m - 2$. Since T has Bishop's property (β) [24], we have

$$(3) \quad \lim_{n \rightarrow \infty} \|P\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, 2, \dots, m - 2$, where D_0 denotes a disk with $\overline{D_0} \subset D$. From (2) and (3), we get that

$$\lim_{n \rightarrow \infty} \|\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, 2, \dots, m - 2$. \square

Lemma 3.3. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is an M -hyponormal operator and T_3 is analytic with order k and let D be a bounded disk in \mathbb{C} containing $\sigma(T)$. Define the map*

$$V : H_1 \oplus H_2 \rightarrow M(D)$$

by

$$Vh = 1 \otimes h + \overline{(T - z)W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)} (= \widetilde{1 \otimes h}),$$

where

$$\begin{aligned} &M(D) \\ &:= W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2) / \overline{(T - z)W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)} \end{aligned}$$

and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h . Then V is one-to-one and has closed range.

Proof. Let $h_n = h_n^1 \oplus h_n^2 \in H_1 \oplus H_2$ and

$$f_n = f_n^1 \oplus f_n^2 \in W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)$$

be sequences such that

$$(4) \quad \lim_{n \rightarrow \infty} \|(z - T)f_n + 1 \otimes h_n\|_{W^{2k+2}} = 0.$$

Then from (4) we have the following equations:

$$(5) \quad \begin{cases} \lim_{n \rightarrow \infty} \|(z - T_1)f_n^1 + T_2 f_n^2 + 1 \otimes h_n^1\|_{W^{2k+2}} = 0, \\ \lim_{n \rightarrow \infty} \|(z - T_3)f_n^2 + 1 \otimes h_n^2\|_{W^{2k+2}} = 0. \end{cases}$$

By the definition of the norm of Sobolev space, (5) implies

$$(6) \quad \begin{cases} \lim_{n \rightarrow \infty} \|(z - T_1)\bar{\partial}^i f_n^1 + T_2 \bar{\partial}^i f_n^2\|_{2,D} = 0, \\ \lim_{n \rightarrow \infty} \|(z - T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0, \end{cases}$$

for $i = 1, 2, \dots, 2k + 2$. Since T_3 is analytic with order k , there exists a nonconstant analytic function F on a neighborhood of $\sigma(T_3)$ such that $F(T_3) = 0$. As remarked in section one, write $F(z) = G(z)p(z)$ where G is a function that is analytic and does not vanish on a neighborhood of $\sigma(T_3)$ and $p(z) = (z - z_1)(z - z_2) \cdots (z - z_k)$ is a polynomial of order k . Set $q_j(z) = (z - z_{j+1}) \cdots (z - z_k)$ for $j = 0, 1, 2, \dots, k - 1$ and $q_k(z) = 1$.

Claim. It holds for every $j = 0, 1, 2, \dots, k$ that

$$\lim_{n \rightarrow \infty} \|q_j(T_3)\bar{\partial}^i f_n^2\|_{2,D_j} = 0$$

for $i = 1, 2, \dots, 2k - 2j + 2$, where $\sigma(T) \subsetneq D_k \subsetneq \cdots \subsetneq D_2 \subsetneq D_1 \subsetneq D$.

To prove the claim, we will use the induction on j . Since $0 = F(T_3) = G(T_3)p(T_3)$ and $G(T_3)$ is invertible, it follows that $q_0(T_3) = p(T_3) = 0$, and so the claim holds when $j = 0$. Suppose that the claim is true for some $j = r$ where $0 \leq r < k$. That is,

$$(7) \quad \lim_{n \rightarrow \infty} \|q_r(T_3)\bar{\partial}^i f_n^2\|_{2,D_r} = 0$$

for $i = 1, 2, \dots, 2k - 2r + 2$, where $\sigma(T) \subsetneq D_r \subsetneq \cdots \subsetneq D_2 \subsetneq D_1 \subsetneq D$. By the second equation of (6) and (7), we get that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|q_{r+1}(T_3)(T_3 - z)\bar{\partial}^i f_n^2\|_{2,D_r} \\ &= \lim_{n \rightarrow \infty} \|q_{r+1}(T_3)(T_3 - z_{r+1} + z_{r+1} - z)\bar{\partial}^i f_n^2\|_{2,D_r} \\ &= \lim_{n \rightarrow \infty} \|(z_{r+1} - z)q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D_r} \end{aligned}$$

for $i = 1, 2, \dots, 2k - 2r + 2$. Since $z_{r+1}I$ is hyponormal, by applying Lemma 3.2 we obtain that

$$(8) \quad \lim_{n \rightarrow \infty} \|q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2, D_{r+1}} = 0$$

for $i = 1, 2, \dots, 2k - 2r$, where $\sigma(T) \subsetneq D_{r+1} \subsetneq D_r$. Hence we complete the proof of our claim.

From the claim with $j = k$, we have

$$(9) \quad \lim_{n \rightarrow \infty} \|\bar{\partial}^i f_n^2\|_{2, D_k} = 0$$

for $i = 1, 2$, which implies by Lemma 3.1 that

$$(10) \quad \lim_{n \rightarrow \infty} \|(I - P_2)f_n^2\|_{2, D_k} = 0,$$

where P_2 denotes the orthogonal projection of $L^2(D_k, H_2)$ onto $A^2(D_k, H_2)$. By combining (9) with the first equation of (6), we obtain that

$$(11) \quad \lim_{n \rightarrow \infty} \|(z - T_1)\bar{\partial}^i f_n^1\|_{2, D_k} = 0$$

for $i = 1, 2$. It follows from Corollary 3.1 that

$$(12) \quad \lim_{n \rightarrow \infty} \|(I - P_1)f_n^1\|_{2, D_{k,1}} = 0.$$

Set $Pf_n := \begin{pmatrix} P_1 f_n^1 \\ P_2 f_n^2 \end{pmatrix}$. Combining (10) and (12) with (5), we have

$$\lim_{n \rightarrow \infty} \|(z - T)Pf_n + 1 \otimes h_n\|_{2, D_{k,1}} = 0.$$

Let Γ be a curve in $D_{k,1}$ surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n \rightarrow \infty} \|Pf_n(z) + (z - T)^{-1}(1 \otimes h_n)(z)\| = 0$$

uniformly. Hence, by Riesz functional calculus,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz + h_n \right\| = 0.$$

But $\frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz = 0$ by Cauchy's theorem. Hence, $\lim_{n \rightarrow \infty} h_n = 0$, and so V is one-to-one and has closed range. \square

Now we are ready to prove that every analytic extension of an M -hyponormal operator has a scalar extension.

Theorem 3.1. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then T is subscalar of order $2k + 2$.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ be an operator matrix on $H_1 \oplus H_2$ where T_1 is an M -hyponormal operator and T_3 is analytic with order k and let D be a bounded disk in \mathbb{C} containing $\sigma(T)$. As in Lemma 3.3, if we define the map $V : H_1 \oplus H_2 \rightarrow M(D)$ by

$$Vh = 1 \otimes h + \overline{(T - z)W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)} (= \widetilde{1 \otimes h}),$$

then V is one-to-one and has closed range. The class of a vector f or an operator S on $M(D)$ will be denoted by \widetilde{f} , \widetilde{S} respectively. Let M be the operator of multiplication by z on $W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)$. Then M is a scalar operator of order $2k + 2$ and has a spectral distribution Φ . Since the range of $T - z$ is invariant under M , \widetilde{M} can be well-defined. Moreover, consider the spectral distribution $\Phi : C_0^{2k+2}(\mathbb{C}) \rightarrow B(W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2))$ defined by the following relation: for $\varphi \in C_0^{2k+2}(\mathbb{C})$ and $f \in W^{2k+2}(D, H_1) \oplus W^{2k+2}(D, H_2)$, $\Phi(\varphi)f = \varphi f$. Then the spectral distribution Φ of M commutes with $T - z$, and so \widetilde{M} is still a scalar operator of order $2k + 2$ with $\widetilde{\Phi}$ as a spectral distribution. Since

$$VT h = 1 \otimes \widetilde{Th} = z \otimes \widetilde{h} = \widetilde{M}(1 \otimes \widetilde{h}) = \widetilde{M}V h$$

for all $h \in H_1 \oplus H_2$, $VT = \widetilde{M}V$. In particular, $R(V)$ is invariant under \widetilde{M} . Since $R(V)$ is closed, it is a closed invariant subspace of the scalar operator \widetilde{M} . Since T is similar to the restriction $\widetilde{M}|_{R(V)}$ and \widetilde{M} is scalar of order $2k + 2$, T is a subscalar operator of order $2k + 2$. \square

Corollary 3.2. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then T has property (β) , Dunford's property (C) , and SVEP.*

Proof. From section two, it suffices to prove that T has property (β) . Since property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.1 to the case of a scalar operator. Since every scalar operator has property (β) (see [27]), T has property (β) . \square

Lemma 3.4. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is an M -hyponormal operator and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$. Then $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ and $\sigma(T_3)$ is a subset of $\{z \in \mathbb{C} : p(z) = 0\}$ where $F(z) = G(z)p(z)$, G is analytic and does not vanish on D , and p is a polynomial.*

Proof. Since $p(T_3) = 0$, choose a minimal polynomial q such that $q(T_3) = 0$ and $q(z)$ is a factor of $p(z)$. Then $\{z \in \mathbb{C} : q(z) = 0\}$ is nonempty and is contained in $\{z \in \mathbb{C} : p(z) = 0\}$. First we will show that $\sigma(T_3) = \sigma_p(T_3) = \{z \in \mathbb{C} : q(z) = 0\}$. Since $q(T_3) = 0$, we have $q(\sigma(T_3)) = \sigma(q(T_3)) = \{0\}$ by the spectral mapping theorem. This means that $\sigma(T_3) \subseteq \{z \in \mathbb{C} : q(z) = 0\}$. Moreover if we assume that z_1, \dots, z_k are all the roots of $q(z) = 0$, not necessarily distinct, then $(T_3 - z_1)(T_3 - z_2) \cdots (T_3 - z_k)x = 0$ for all $x \in H_2$. By the minimality of the degree of q , we can select a vector $x_0 \in H_2$ such that $(T_3 - z_2) \cdots (T_3 - z_k)x_0 \neq 0$, and so $z_1 \in \sigma_p(T_3)$. Similarly, $z_i \in \sigma_p(T_3)$ for all $i = 1, 2, \dots, k$. Hence $\sigma(T_3) = \sigma_p(T_3) = \{z \in \mathbb{C} : q(z) = 0\}$. Since $\{z \in \mathbb{C} : q(z) = 0\}$ is a finite set, $\sigma(T_1) \cap \sigma(T_3)$ is also finite, which implies that $\sigma(T_1) \cap \sigma(T_3)$ has no interior point. By using [13], we get $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$. \square

By using Theorem 2.1 and Theorem 3.1, we deduce the following corollaries.

Corollary 3.3. *Every analytic extension of an M -hyponormal operator is $H(p)$.*

Since $H(p)$ operators are polaroid, we have:

Corollary 3.4. *Every analytic extension of an M -hyponormal operator is polaroid.*

Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T . Next we show that every analytic extension of an M -hyponormal operator is isoloid.

Corollary 3.5. *Every analytic extension of an M -hyponormal operator is isoloid.*

Proposition 3.1. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is an M -hyponormal operator and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$. Then the following statements hold*

- (i) $H_T(E) \supseteq H_{T_1}(E) \oplus \{0\}$ for every subset E of \mathbb{C} .
- (ii) $\sigma_{T_3}(x_2) \subset \sigma_T(x_1 \oplus x_2)$ and $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$ where $x_1 \oplus x_2 \in H_1 \oplus H_2$.
- (iii) $R_{T_1}(F) \oplus 0 \subset H_T(F)$ where $R_{T_1}(F) := \{y \in H_1 : \sigma_{T_1}(y) \subset F\}$ for any subset $F \subset \mathbb{C}$.

Proof. (i) Let E be any subset of \mathbb{C} and let $x_1 \in H_{T_1}(E)$. Since T has SVEP by Corollary 3.2, there exists an H -valued analytic function f_1 on $\mathbb{C} \setminus E$ such that $(T_1 - z)f_1(z) \equiv x_1$ on $\mathbb{C} \setminus E$. Hence $(T - z)(f_1(z) \oplus 0) \equiv x_1 \oplus 0$ on $\mathbb{C} \setminus E$, and so $x_1 \oplus 0 \in H_T(E)$.

(ii) Let $x_1 \oplus x_2 \in H_1 \oplus H_2$. If $z_0 \in \rho_T(x_1 \oplus x_2)$, then there exists an H -valued analytic function defined on a neighborhood U of z_0 such that $(T - \lambda)f(\lambda) = x_1 \oplus x_2$ for all $\lambda \in U$. We can write $f = f_1 \oplus f_2$ where $f_1 \in O(U, H_1)$ and $f_2 \in O(U, H_2)$. Then we get

$$\begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $(T_3 - \lambda)f_2(\lambda) \equiv x_2$. Hence $z_0 \in \rho_{T_3}(x_2)$.

On the other hand, if $z_0 \in \rho_T(x_1 \oplus 0)$, then there exists an H -valued analytic function defined on a neighborhood U of z_0 such that $(T - \lambda)g(\lambda) = x_1 \oplus 0$ for all $\lambda \in U$. We can write $g = g_1 \oplus g_2$ where $g_1 \in O(U, H_1)$ and $g_2 \in O(U, H_2)$. Then we get

$$\begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus $(T_1 - \lambda)g_1(\lambda) + T_2g_2(\lambda) \equiv x_1$ and $(T_3 - \lambda)g_2(\lambda) \equiv 0$. Since T_3 is algebraic of order k , it has SVEP, which implies that $g_2(\lambda) \equiv 0$. Thus $(T_1 - \lambda)g_1(\lambda) \equiv x_1$, and so $z_0 \in \rho_{T_1}(x_1)$. Conversely, let $z_0 \in \rho_{T_1}(x_1)$. Then there exists a function

$g_1 \in O(U, H_1)$ for some neighborhood U of z_0 such that $(T_1 - \lambda)g_1(\lambda) \equiv x_1$. Then $(T - \lambda)(g_1(\lambda \oplus 0)) \equiv x_1 \oplus 0$. Hence $z_0 \in \rho_T(x_1 \oplus 0)$.

(iii) If $x_1 \in R_{T_1}(F)$, then $\sigma_{T_1}(x_1) \subset F$. Since $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$ by (ii), $\sigma_T(x_1 \oplus 0) \subset F$. Thus $x_1 \oplus 0 \in H_T(F)$, and hence $R_{T_1}(F) \oplus 0 \subset H_T(F)$. \square

It was known by Oberai [25] that the mapping $T \rightarrow w(T)$ is upper semi-continuous, but not continuous at T . However if $T_n \rightarrow T$ with $T_n T = T T_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} w(T_n) = w(T).$$

It was known that $w(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$, then

$$w(f(T)) \subset f(w(T)).$$

This last inclusion may be proper (see Berberian [4, Example 3.3]). If T is normal, then $\sigma_e(T)$ and $w(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $w(T)$ satisfies the spectral mapping theorem for analytic functions. Next we show that the spectral mapping theorem for the Weyl spectrum and Weyl's theorem hold for analytic extension of M -hyponormal operator, more generally for $f(T)$ where f is any analytic function on some neighborhood of $\sigma(T)$.

Lemma 3.5. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then $f(w(T)) = w(f(T))$ for any analytic function f on some neighborhood of $\sigma(T)$.*

Proof. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(H_1 \oplus H_2)$$

be an analytic extension of an M -hyponormal operator. If f is analytic on some neighborhood of $\sigma(T)$, then $w(f(T_1)) = f(w(T_1))$ by [7]. Since T_3 is algebraic, $w(f(T_3)) = f(w(T_3))$ and $w(T_1) \cap w(T_3)$ is finite and so has no interior points. Since $w(T_1) \cap w(T_3)$ is finite,

$$w(f(T_1)) \cap w(f(T_3)) = f(w(T_1)) \cap f(w(T_3))$$

also has no interior points. It results from [20] that

$$\begin{aligned} w(f(T)) &= w(f(T_1)) \cup w(f(T_3)) = f(w(T_1)) \cup f(w(T_3)) \\ &= f(w(T_1) \cup w(T_3)) = f(w(T)). \end{aligned} \quad \square$$

Theorem 3.2. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then Weyl's theorem holds for $f(T)$ where f is any analytic function on some neighborhood of $\sigma(T)$.*

Proof. We use the fact that if T is polaroid and T has SVEP, then T satisfies Weyl's theorem in [2, Theorem 3.3]. Suppose that T is an analytic extension of an M -hyponormal operator. By Corollary 3.2 and Corollary 3.4 we have that

T satisfies Weyl's theorem. We show next that Weyl's theorem holds for $f(T)$. Since T is polaroid and has SVEP, then $f(T)$ is polaroid by [2, Lemma 3.11] and has SVEP by [1, Theorem 2.40]. Consequently, Weyl's theorem holds for $f(T)$. \square

Lemma 3.6. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then Weyl's theorem holds for $T + F$ for any finite rank operator F commuting with T .*

Proof. Since T is isoloid and Weyl's theorem holds for T . The result follows by [17, Theorem 3.3]. \square

Theorem 3.3. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of an M -hyponormal operator. Then for any function f analytic on a neighborhood of $\sigma(T)$ and any finite rank operator F commuting with T , Weyl's theorem holds for $f(T) + F$.*

Proof. Since T is isoloid, $f(T)$ is isoloid for any function f analytic on a neighborhood of $\sigma(T)$ [17]. Since $f(T)$ obeys Weyl's theorem for any function f analytic on a neighborhood of $\sigma(T)$ by Theorem 3.2 and $f(T)$ is isoloid, the result holds by Lemma 3.6. \square

Acknowledgement. We wish to thank the referees for careful reading and valuable comments for the origin draft. This research is supported by Taibah university research center and by National Natural Science Foundation of China (No. 11401180, 11201126, 11201127).

References

- [1] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer Academic Publishers, London, 2004.
- [2] P. Aiena, E. Aponte, and E. Balzan, *Weyl type theorems for left and right polaroid operators*, Integral Equations Operator Theory **66** (2010), no. 1, 1–20.
- [3] P. Aiena, M. Chō, and M. González, *Polaroid type operators under quasi-affinities*, J. Math. Anal. Appl. **371** (2010), no. 2, 485–495.
- [4] S. K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J. **20** (1970), 529–544.
- [5] S. Brown, *Hyponormal operators with thick spectra have invariant subspaces*, Ann. of Math. **125** (1987), no. 1, 93–103.
- [6] X. H. Cao, *Analytically class A operators and Weyl's theorem*, J. Math. Anal. Appl. **320** (2006), no. 2, 795–803.
- [7] M. Chō and Y. M. Han, *Riesz idempotent and algebraically M -hyponormal operators*, Integral Equations Operator Theory **53** (2005), no. 3, 311–320.
- [8] H. R. Dowsow, *Spectral Theory of Linear Operators*, Academic Press, London, 1973.
- [9] N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274.
- [10] J. Eschmeier, *Invariant subspaces for subscalar operators*, Arch. Math. (Basel) **52** (1989), no. 6, 562–570.
- [11] J. Eschmeier, K. B. Laursen, and M. M. Neumann, *Multipliers with natural local spectra on commutative Banach algebras*, J. Funct. Anal. **138** (1996), no. 2, 273–294.

- [12] J. Eschmeier and M. Putinar, *Spectral Decompositions and Analytic Sheaves*, London Mathematical Society Monographs, No. 10, Clarendon Press, Oxford, 1996.
- [13] J. K. Han, H. Y. Lee, and W. Y. Lee, *Invertible completions of 2×2 upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (2000), no. 1, 119–123.
- [14] J. S. Han, S. H. Lee, and W. Y. Lee, *On M -hyponormal weighted shifts*, J. Math. Anal. Appl. **286** (2003), no. 1, 116–124.
- [15] R. E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Roy. Irish Acad. **85A** (1985), 151–176.
- [16] ———, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, 1988.
- [17] J. C. Hou and X. L. Zhang, *On the Weyl spectrum: Spectral mapping theorem and Weyl's theorem*, J. Math. Anal. Appl. **220** (1998), no. 2, 760–768.
- [18] K. B. Laursen and M. M. Neumann, *Automatic continuity of intertwining linear operators on Banach spaces*, Rend. Circ. Mat. Palermo **40** (1991), no. 2, 325–341.
- [19] ———, *An Introduction to Local Spectral Theory*, London Math. Soc. Monogr. (N.S) 20, Clarendon Press, Oxford, 2000.
- [20] W. Y. Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 131–138.
- [21] S. Mecheri, *Bishop's property, SVEP and Dunford property (C)*, Electron. J. Linear Algebra **23** (2012), 523–529.
- [22] ———, *Bishop's property and Riesz idempotent for k -quasi-paranormal operators*, Banach J. Math. Anal. **6** (2012), no. 1, 147–154.
- [23] ———, *Isolated points of spectrum of k -quasi- $*$ -class A operators*, Studia Math. **208** (2012), no. 1, 87–96.
- [24] ———, *On k -quasi- M -hyponormal operators*, Math. Inequal. Appl. **16** (2013), no. 3, 895–902.
- [25] K. K. Oberai, *On the Weyl spectrum*, Illinois J. Math. **18** (1974), 208–212.
- [26] M. Oudghiri, *Weyl's and Browder's theorems for operators satisfying the SVEP*, Studia Math. **163** (2004), no. 1, 85–101.
- [27] M. Putinar, *Hyponormal operators are subscalar*, J. Operator Theory **12** (1984), no. 2, 385–395.
- [28] ———, *Hyponormal operators and eigendistribution*, Advances in invariant subspaces and other results of operator theory (Timioara and Herculane, 1984), 249–273, Oper. Theory Adv. Appl., 17, Birkhuser, Basel, 1986.
- [29] ———, *Quasimilarity of tuples with Bishop's property (β)*, Integral Equations Operator Theory **15** (1992), no. 6, 1047–1052.

SALAH MECHERI
 COLLEGE OF SCIENCE DEPARTMENT OF MATHEMATICS
 TAIBAH UNIVERSITY
 P.O.BOX 30002, AL MADINAH AL MUNAWARAH, SAUDI ARABIA
E-mail address: mecherisalah@hotmail.com

FEI ZUO
 COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
 HENAN NORMAL UNIVERSITY
 XINXIANG 453007, HENAN, P. R. CHINA
E-mail address: zuofei2008@126.com