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VARIATIONAL ANALYSIS OF AN ELECTRO-VISCOELASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

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Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a piezoelectric body and an electrically conductive obstacle, the so-called foundation. A nonlinear electro-viscoelastic constitutive law is used to model the piezoelectric material. Contact is described with Signorini's conditions and a version of Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account. The evolution of the bonding field is described by a first order differential equation. We derive a variational formulation for the model, in the form of a system for the displacements, the electric potential and the adhesion. Under a smallness assumption which involves only the electrical data of the problem, we prove the existence of a unique weak solution of the model. The proof is based on arguments of time-dependent quasi-variational inequalities, differential equations and Banach's fixed point theorem.

1. Introduction

The piezoelectric effect is characterized by such a coupling between the mechanical and electrical properties of the materials. This coupling, leads to the appearance of electric field in the presence of a mechanical stress, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators.

On a nano-scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal's unit cell. When such a crystal is deformed mechanically, the positive and negative charges are displaced by a different amount causing the appearance of electric polarization. So, while the overall crystal remains electrically neutral, an electric polarization is formed within the crystal. This electric polarization due to mechanical stress is called *piezoelectricity*. A deformable material which exhibits such a behavior is called

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a *piezoelectric material.* Piezoelectric materials for which the mechanical properties are elastic are also called *electro-elastic materials* and piezoelectric materials for which the mechanical properties are viscoelastic are also called *electro-viscoelastic materials.*

Only some materials exhibit sufficient piezoelectricity to be useful in applications. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene flouride (a polymer film), and are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and in measuring equipment. General models for electro-elastic materials can be found in [23, 24] and, more recently, in [3, 19, 25]. A static and a slip-dependent frictional contact problems for electro-elastic materials were studied in [4, 22] and in [31], respectively. A contact problem with normal compliance for electro-viscoelastic materials was investigated in [32]. In the last two references the foundation was assumed to be insulated. The variational formulations of the corresponding problems were derived and existence and uniqueness of weak solutions were obtained.

Here we continue this line of research and study a quasistatic frictionless contact problem for an electro-viscoelastic material, in the framework of the MTCM, when the foundation is conductive; our interest is to describe a physical process in which both contact, friction and piezoelectric effect are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the conductivity of the foundation leads to new and nonstandard boundary conditions on the contact surface, which involve a coupling between the mechanical and the electrical unknowns, and represents the main novelty in this work.

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Basic modelling can be found in [13], [15] and in [9]. Analysis of models for adhesive contact can be found in [2]-[7], [16] and in the recent monographs [29] and [30]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [27] and in [28]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Contact problems for elastic and elastic-viscoelastic bodies with adhesion and friction appear in many applications of solids mechanics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction is proposed by Raous, Cangémi and Cocu in [26]. Adhesive problems have been the subject of some recent publications (see for instance, [1], [6] and [9]). The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [13], [14], the bonding field satisfies the restrictions $0 \le \beta \le 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [15] and in [27].

The aim of this paper is to continue the study of problems begun in [11], [20] and in [21]. The novelty of the present paper is to extend the result when the contact and friction are modelled by Signorini's conditions and a non local Coulomb's friction law. Moreover, the adhesion is taken into account at the interface and the material behavior is assumed to be electro-viscoelastic.

The paper is structured as follows. In Section 2 we present the electro-viscoelastic contact model with friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness results, Theorem 4.1, which states the unique weak solvability of the Signorini's adhesive contact electro-viscoelastic problem with non local Coulomb's friction law conditions. The paper concludes in Section 5.

2. Problem statement

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these conditions, we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$; these conditions allow the use of coercivity arguments which guarantee the uniqueness of the solution for the model. The body is clamped on Γ_1 and, therefore, the displacement field $\boldsymbol{u} = (u_1, \ldots, u_d)$ vanishes there. Surface tractions of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface free electrical charge of density q_2 is prescribed on Γ_b. In the reference configuration the body may come in contact over Γ_3 with a conductive obstacle, which is also called the foundation. The contact is frictional and is modelled with the Signorini's conditions and a version of Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account. Also, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when contact is lost.

We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0, T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic; i.e., the inertial terms in the momentum balance equations are neglected. We denote by $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on $\boldsymbol{x}$ and t. In this paper $i, j, k, l = 1, \ldots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\boldsymbol{x}$. A dot over a variable represents the time derivative.

We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and " $\cdot$ " and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, $\|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, and $\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

With these assumptions, the classical formulation of the electro-viscoelastic contact problem coupling friction and adhesion is the following.

Problem $\boldsymbol{\mathcal{P}}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$, *an electric displacement field* $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) \qquad in\ \Omega \times (0,T), \tag{1}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \qquad in\ \Omega \times (0,T), \tag{2}$$

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \qquad in\ \Omega \times (0,T), \tag{3}$$

$$\operatorname{div} \mathbf{D} = q_0 \qquad in\ \Omega \times (0,T), \tag{4}$$

$$\boldsymbol{u} = \mathbf{0} \qquad on\ \Gamma_1 \times (0,T), \tag{5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad on\ \Gamma_2 \times (0,T), \tag{6}$$

and on $\Gamma_3 \times (0,T)$,

$$\dot{u}_\nu \le 0,\ \boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2 R_\nu(\dot{u}_\nu) \le 0,\ \dot{u}_\nu(\boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2 R_\nu(\dot{u}_\nu)) = 0, \tag{7}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau(\dot{\boldsymbol{u}}_\tau)| \le \mu p(|\mathcal{R}(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\dot{u}_\nu)|), \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau(\dot{\boldsymbol{u}}_\tau)| < \mu p(|\mathcal{R}(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\dot{u}_\nu)|) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau(\dot{\boldsymbol{u}}_\tau)| = \mu p(|\mathcal{R}(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\dot{u}_\nu)|) \\ \Rightarrow \exists\lambda \ge 0 \ such\ that\ \boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau(\dot{\boldsymbol{u}}_\tau) = -\lambda\dot{\boldsymbol{u}}_\tau, \end{cases} \tag{8}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(\dot{u}_\nu)^2 + \gamma_\tau\|\boldsymbol{R}_\tau(\dot{\boldsymbol{u}}_\tau)\|^2) - \epsilon_a), \tag{9}$$

and

$$\varphi = 0 \qquad on\ \Gamma_a \times (0,T),, \tag{10}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = q_2 \qquad on\ \Gamma_b \times (0,T), \tag{11}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \psi(\dot{u}_\nu)\phi(\varphi - \varphi_0) \qquad on\ \Gamma_3 \times (0,T), \tag{12}$$

$$\beta(0) = \beta_0 \qquad on\ \Gamma_3, \tag{13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad in\ \Omega. \tag{14}$$

We now provide some comments on equations and conditions (1)–(14).

Equations (1) and (2) represent the nonlinear electro viscoelastic constitutive law in which $\varepsilon(\boldsymbol{u})$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, where φ is the electric potential, $\mathcal{F}$ is a given nonlinear function, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $\mathcal{B}$ denotes the electric permittivity operator, and $\mathbf{D} = (D_1, \dots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (1) and (2) can be find, for instance, in [3] and in [4]. Next, equations (3) and (4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Equations (5) and (6) represent the displacement and traction boundary conditions.

Conditions (7) represents the Signorini's contact condition with adhesion where $\dot{u}_\nu$ is the normal velocity, $\boldsymbol{\sigma}_\nu$represents the normal stress, γ_ν denote a given adhesion coefficient and R_ν is the truncation operator define by

$$R_\nu(s) = \begin{cases} L_\nu & \text{if } s < -L_\nu, \\ -s & \text{if } -L_\nu \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \tag{15}$$

where $L_\nu > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below, is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(\boldsymbol{u}_\nu) = -\boldsymbol{u}_\nu$ and, therefore, from (7) we recover the contact conditions

$$\dot{u}_\nu \le 0,\ \boldsymbol{\sigma}_\nu + \gamma_\nu\beta^2\dot{u}_\nu \le 0,\ \dot{u}_\nu(\boldsymbol{\sigma}_\nu + \gamma_\nu\beta^2\dot{u}_\nu) = 0 \text{ on } \Gamma_3 \times (0,T).$$

Moreover, Conditions (7) shows when $\dot{u}_\nu < 0$ then the reaction of foundation is uniquely determined by $\boldsymbol{\sigma}_\nu = \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu)$ and, when $\dot{u}_\nu = 0$, the normal stress is not uniquely determined but is submitted to the restriction $\boldsymbol{\sigma}_\nu \le 0$.

Conditions (8) are a non local Coulomb's friction law conditions coupled with adhesion, where $\boldsymbol{u}_\tau$ and $\boldsymbol{\sigma}_\tau$ denote tangential components of vector $\boldsymbol{u}$ and tensor $\boldsymbol{\sigma}$ respectively. R_τ is the truncation operator given by

$$R_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \le L_\tau, \\ L_\tau \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if } \|\boldsymbol{v}\| > L_\tau. \end{cases}$$

This condition shows that the magnitude of the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

The introduction of the nonlocal smoothing operator $\mathcal{R}$ is for technical reasons, since the trace of stress tensor on the boundary is too rough to be defined in the ordinary sense.

We shall need it to regularize the normal trace of the stress witch is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used in some studies with $p(r) = r_+$ where $r_+ = \max\{0, r\}$. Recently, from thermodynamic considerations, a new version of *Coulomb's* law is proposed; its consists to take

$$p(r) = r(1 - \alpha r)_+, \tag{16}$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface.

The evolution of the bonding field is governed by the differential equation (9) with given positive parameters γ_ν, γ_τ and ϵ_a. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$.

Next, (12) is the electrical contact condition on Γ_3 which is the main novelty of this work. It represents a regularized condition which may be obtained as follows.

First, unlike previous papers on piezoelectric contact, we assume that the foundation is electrically conductive and its potential is maintained at φ_0. When $\dot{u}_\nu < 0$ then there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$\dot{u}_\nu < 0 \;\Rightarrow\; \mathbf{D} \cdot \boldsymbol{\nu} = 0. \tag{17}$$

When $\dot{u}_\nu = 0$, the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor. Thus,

$$\dot{u}_\nu = 0 \;\Rightarrow\; \mathbf{D} \cdot \boldsymbol{\nu} = k(\varphi - \varphi_0). \tag{18}$$

We combine (17), (18) to obtain

$$\mathbf{D} \cdot \boldsymbol{\nu} = k\chi_{[0,\infty)}(\dot{u}_\nu)(\varphi - \varphi_0), \tag{19}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval given $[0, \infty)$ that is

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Condition (19) describes perfect electrical contact and is somewhat similar to the Signorini contact condition. Both conditions may be over-idealizations in many applications.

To make it more realistic, we regularize condition (19) and write it as (12) in which $k\chi_{[0,\infty)}$ is replaced with ψ which is a regular function which will be described below, and ϕ is the truncation function

$$\phi(s) = \begin{cases} -L_\phi & \text{if } s < -L_\phi, \\ s & \text{if } -L_\phi \le s \le L_\phi, \\ L_\phi & \text{if } s > L_\phi, \end{cases} \tag{20}$$

where L_ϕ is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since M may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi(\varphi - \varphi_0) = \varphi - \varphi_0$.

The reasons for the regularization (12) of (19) are mathematical. First, we need to avoid the discontinuity in the free electric charge when contact is established and, therefore, we regularize the function $k\chi_{[0,\infty)}$ in (19) with a Lipschitz continuous function ψ_{δ_0}. A possible choice is

$$\psi_{\delta_0}(r) = \begin{cases} 0 & \text{if } r < -1/\delta_0, \\ kr\delta_0 + k & \text{if } -1/\delta_0 \le r \le 0, \\ k & r > 0, \end{cases} \tag{21}$$

where $\delta_0 > 0$ is a large parameter. This choice means that during the contact process, the electrical conductivity increases continuously from $-1/\delta_0$ very close to zero and reaches the value k, very quickly, when $\dot{u}_\nu$ is equal to zero. Secondly, we need the term $\phi(\varphi - \varphi_0)$ to control the boundedness of $\varphi - \varphi_0$.

Note that when $\psi \equiv 0$ in (12) then

$$\mathbf{D} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{22}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (22) models the case when the obstacle is a perfect insulator and was used in [4, 12, 22, 31, 32]. Condition (12), instead of (22), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model.

Because of the friction condition (11), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems. Conditions (11) and (12) represent the electric boundary conditions. Finally, the equations (13) and (14) are the initial conditions.

3. Variational formulations and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation H, H_1 and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces:

$$\begin{aligned}
H &= L^2(\Omega)^d = \{\boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in L^2(\Omega)\},\\
H_1 &= H^1(\Omega)^d = \{\boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in H^1(\Omega)\},\\
\mathcal{H} &= \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},\\
\mathcal{H}_1 &= \{\tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\}.
\end{aligned}$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx, \quad (\boldsymbol{u},\boldsymbol{v})_{H_1} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx + \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx,$$

$$(\boldsymbol{\sigma},\tau)_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma}\cdot\tau\,dx, \quad (\boldsymbol{\sigma},\tau)_{\mathcal{H}_1} = \int_\Omega \boldsymbol{\sigma}\cdot\tau\,dx + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\operatorname{Div}\tau\,dx,$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla\boldsymbol{v} = (\boldsymbol{v}_{i,j}), \quad \boldsymbol{\varepsilon}(\boldsymbol{v}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{v})), \quad \boldsymbol{\varepsilon}_{ij}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}_{i,j} + \boldsymbol{v}_{j,i}) \quad \forall\,\boldsymbol{v} \in H_1,$$

$$\operatorname{Div}\tau = (\tau_{ij,j}) \quad \forall\,\tau \in \mathcal{H}_1.$$

For every element $\boldsymbol{v} \in H_1$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $\boldsymbol{v}_\nu = \boldsymbol{v}\cdot\nu$, $\boldsymbol{v}_\tau = \boldsymbol{v} - \boldsymbol{v}_\nu\nu$.

Let now consider the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K\,\|\boldsymbol{v}\|_{H_1} \quad \forall\,\boldsymbol{v} \in V, \tag{23}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{24}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (23) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (23) and (24), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{v}\|_V \qquad \forall\boldsymbol{v} \in V. \tag{25}$$

Finally, for a real Banach space $(X,\|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leq p \leq \infty$, $k = 1,2,\ldots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X,$$
$$\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.$$

Recall that the dot represents the time derivative, respectively, where the dot represents the time derivative.

We also introduce the following spaces.

$$W = \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\},$$
$$\mathcal{W} = \{D = (D_i) \mid D_i \in L^2(\Omega)\} = L^2(\Omega)^d,$$
$$\mathcal{W}_1 = \{D = (D_i) \mid D_i \in L^2(\Omega),\ D_{i,i} \in L^2(\Omega)\}.$$

Since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_H \geq c_F\,\|\psi\|_{H^1(\Omega)} \quad \forall\,\psi \in W, \tag{26}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\psi = (\psi_{,i})$. Over the space W, we consider the inner product given by

$$(\varphi,\psi)_W = \int_\Omega \nabla\varphi\cdot\nabla\psi\,dx$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (26) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W,\|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\tilde{c}_0$, depending only on Ω, Γ_a and Γ_3, such that

$$\|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0\|\psi\|_W \quad \forall\,\psi \in W. \tag{27}$$

The space $\mathcal{W}_1$ is real Hilbert space with the inner product

$$(D,\boldsymbol{E})_{\mathcal{W}_1} = \int_\Omega D\cdot\boldsymbol{E}\,dx + \int_\Omega \operatorname{div} D\cdot\operatorname{div}\boldsymbol{E}\,dx,$$

where $\operatorname{div} = (D_{i,i})$, and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leq p \leq \infty$, $k \geq 1$ and we also introduce the set

$$\mathcal{Q} = \{\,\theta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \leq \theta(t) \leq 1 \ \ \forall\,t \in [0,T],\ \text{a.e. on } \Gamma_3\}.$$

Finally, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_{X_1}$ and $(\cdot,\cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$ and the associated norm $\|\cdot\|_{X_1\times X_2}$.

We now list the assumptions on the problem's data. The *viscosity operator* $\mathcal{F}$ and the *elasticity operator* $\mathcal{G}$ are assumed to satisfy the conditions:

$$
(28)\quad \begin{cases}
\text{(a)} & \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b)} & \text{There exists } L_{\mathcal{F}} > 0 \text{ such that} \\
& \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
& \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c)} & \text{There exists } m_{\mathcal{F}} > 0 \text{ such that} \\
& (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\
& \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is Lebesgue} \\
& \text{measurable on } \Omega, \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(e)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{cases}
$$

$$
(29)\quad \begin{cases}
\text{(a)} & \mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b)} & \text{There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
& \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{G}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
& \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is measurable on } \Omega, \\
& \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(d)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{cases}
$$

The *piezoelectric tensor* $\mathcal{E}$ and the *electric permittivity tensor* $\mathcal{B}$ satisfy

$$
(30)\quad \begin{cases}
\text{(a)} & \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\
\text{(b)} & \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega).
\end{cases}
$$

$$
(31)\quad \begin{cases}
\text{(a)} & \boldsymbol{\beta} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\
\text{(b)} & \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{E}) = (\beta_{ij}(\boldsymbol{x})E_j) \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c)} & \beta_{ij} = \beta_{ji} \in L^\infty(\Omega). \\
\text{(d)} & \text{There exists } m_{\mathcal{B}} > 0 \text{ such that } \beta_{ij}(\boldsymbol{x})E_iE_j \ge m_{\boldsymbol{\beta}} \|\boldsymbol{E}\|^2 \\
& \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{cases}
$$

In linearized electro viscoelasticity, the constitutive laws (2) and (3) read

$$
\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\boldsymbol{u}}) + g_{ijkl}\varepsilon_{kl}(\boldsymbol{u}) - e_{kij}\varphi_{,k} \,,
$$
$$
D_i = e_{ijk}\varepsilon_{jk}(\boldsymbol{u}) + \beta_{ij}\varphi_{,j} \,,
$$

where a_{ijkl}, g_{ijkl}, β_{ij} and e_{kij} are the components of the tensors $\mathcal{F}$, $\mathcal{G}$, $\boldsymbol{\beta}$ and $\mathcal{E}$ respectively, and $\varphi_{,j} = \partial\varphi/\partial x_j$. Clearly, assumption (28) is satisfied if all the components a_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual properties of symmetry and ellipticity:

$$
a_{ijkl} = a_{jikl} = a_{klij},
$$

and

$$
a_{ijkl}\psi_{ij}\psi_{kl} \ge m_0 \|\psi\|^2
$$

for $m_0 > 0$ and all symmetric tensors ψ. Assumption (29) is satisfied if g_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the same symmetry properties.

A second example is provided by the nonlinear electro viscoelastic constitutive law,

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \varpi(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{P}_K(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \mathcal{E}^*\boldsymbol{E}(\varphi),$$
$$D_i = e_{ijk}\varepsilon_{jk}(\dot{\boldsymbol{u}}) + \beta_{ij}\varphi_{,j}\ .$$

Here $\mathcal{F}$ is a nonlinear fourth-order viscosity tensor that satisfies (28), ϖ is a positive coefficient, K is a closed convex subset of $\mathbb{S}^d$ such that $0 \in K$, and $\mathcal{P}_K : \mathbb{S}^d \to K$ denotes the projection operator. Since the projection operator is nonexpansive, the elasticity operator $\mathcal{G}(x, \boldsymbol{\varepsilon}) = \alpha(\boldsymbol{\varepsilon} - \mathcal{P}_K\boldsymbol{\varepsilon})$ satisfies condition (29).

The friction function p satisfies:

(32)
$$\begin{cases} \quad p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ verifies} \\ \text{(a) there exists } L_p > 0 \text{ such that :} \\ \quad |p\,(x, r_1) - p\,(x, r_2)| \le L_p\,|r_1 - r_2| \\ \quad \text{for every } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(b) } x \mapsto p\,(x, r) \text{ is measurable on } \Gamma_3, \text{ for every } r \in \mathbb{R}; \\ \text{(c) } p\,(x, 0) = 0, \text{ a.e. } x \in \Gamma_3. \end{cases}$$

Using the continuity of the regularization operator $\mathcal{R} :\ H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ and the continuity of the normal trace mapping $\boldsymbol{\sigma} \to \sigma_\nu\ :\ \mathcal{H}_1 \to H^{-\frac{1}{2}}(\Gamma)$, we deduce the existence of a constant $C_{\mathcal{R}}$ depending only on Ω, Γ_1, Γ_3 and $\mathcal{R}$ such that

(33)
$$\|\mathcal{R}(\sigma_\nu)\|_{L^2(\Gamma)} \le C_{\mathcal{R}}\, \|\boldsymbol{\sigma}\|_{\mathcal{H}_1}\ .$$

We note that (32) is satisfied in the case of function p given by (16).

The *surface electrical conductivity* function ψ satisfies:

(34)
$$\begin{cases} \text{(a) } \psi : \Gamma_3 \times \mathbb{R}^- \to \mathbb{R}_+. \\ \text{(b) } \exists\, L_\psi > 0 \text{ such that } |\psi(\boldsymbol{x}, u_1) - \psi(\boldsymbol{x}, u_2)| \le L_\psi |u_1 - u_2| \\ \quad \forall\, u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } \exists\, M_\psi > 0 \text{ such that } |\psi(\boldsymbol{x}, u)| \le M_\psi\ \forall\, u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(e) } \boldsymbol{x} \mapsto \psi(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}. \end{cases}$$

An example of a conductivity function which satisfies condition (34) is given by (21) in which case $M_\psi = k$. Another example is provided by $\psi \equiv 0$, which models the contact with an insulated foundation, as noted in Section 2. We conclude that our results below are valid for the corresponding piezoelectric contact models.

We also suppose that the body forces, surface tractions and surface free charge densities have the regularity

(35)
$$\boldsymbol{f}_0 \in W^{1,\infty}(0, T; H), \quad \boldsymbol{f}_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d),$$

and the densities of electric charges satisfy

(36)
$$q_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_b)).$$

We assume that the electric conductivity coefficient and the potential of the foundation satisfy

$$k \in L^\infty(\Gamma_3), \quad k \geq 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \tag{37}$$

$$\varphi_0 \in L^2(\Gamma_3). \tag{38}$$

We define the function $\boldsymbol{f} : [0,T] \to V$, $q : [0,T] \to W$ and $h : V \times W \to W$ by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega f_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} f_2(t) \cdot \boldsymbol{v}\, da, \tag{39}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da, \tag{40}$$

$$(h(\boldsymbol{u}, \varphi), \psi)_W = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\psi\, da, \tag{41}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0,T]$, and note that conditions (35) and (36) imply that

$$\boldsymbol{f} \in W^{1,\infty}(0,T;V), \qquad q \in W^{1,\infty}(0,T;W). \tag{42}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu,\ \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu,\ \gamma_\tau,\ \epsilon_a \geq 0, \quad \text{a.e. on } \Gamma_3 \tag{43}$$

while the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0, \quad \text{a.e. on } \Gamma_3. \tag{44}$$

We denote by U_{ad} the convex subset of admissible displacements fields given by

$$U_{ad} = \{\boldsymbol{v} \in H_1 \mid \ \boldsymbol{v} = 0 \text{ on } \Gamma_1\,,\ \boldsymbol{v}_\nu \leq 0 \text{ on } \Gamma_3\}\,, \tag{45}$$

and the initial condition β_0 and $\boldsymbol{u}_0$ satisfy

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \quad \text{a.e. on } \Gamma_3, \quad \boldsymbol{u}_0 \in U_{ad}. \tag{46}$$

We define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)\boldsymbol{v}_\nu + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \big)\, da, \tag{47}$$

the friction functional $j_{fr} : L^2(\Gamma_3) \times \mathcal{H}_1 \times V \times V \to \mathbb{R}$ by

$$j_{fr}(\beta, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|) \cdot |\boldsymbol{v}_\tau|\, da. \tag{48}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (1)–(14).

Problem $\mathcal{P}^V$. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\beta : [0,T] \to L^2(\Gamma_3)$ such that $\dot{\boldsymbol{u}}(t) \in U_{ad}$ and*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}\,(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} \tag{49}$$

$$+ (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j_{ad}(\beta(t), \dot{\boldsymbol{u}}(t)(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))$$
$$+ j_{fr}(\boldsymbol{\sigma}(t), \beta(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{\sigma}(t), \beta(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t))$$
$$\geq (f(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V, \quad \forall\, \boldsymbol{v} \in U_{ad},\ \forall t \in [0,T],$$

(50) $$(\mathcal{B}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t), \nabla\psi)_H + (h(\dot{\boldsymbol{u}}(t), \varphi), \psi)_W$$
$$= (q(t), \psi)_W, \quad \forall\psi \in W,\ \forall t \in [0,T],$$

(51) $$\dot\beta(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\dot u_\nu(t))^2 + \gamma_\tau\|R_\tau(\dot{\boldsymbol{u}}_\tau(t))\|^2) - \epsilon_a)_+\ , a.e\ t \in (0,T),$$

(52) $$\boldsymbol{u}(0) = \boldsymbol{u}_0,$$

(53) $$\beta(0) = \beta_0.$$

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, j_{fr} and h which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta,\ \beta_1,\ \beta_2 \leq 1$ a.e. on Γ_3, $\boldsymbol{u}_1$, $\boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$, $\boldsymbol{u}$ and $\boldsymbol{v}$ represent elements of V; φ_1, φ_2 denote elements of W, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ represent elements of $\mathcal{H}_1$ and C is a generic positive constants which may depend on Ω, Γ_1, Γ_3, p, γ_ν, γ_τ, $\mathcal{R}$, L and L_p, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} is linear with respect to the last argument and therefore

(54) $$j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) = -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}).$$

Next, using (47) and the inequalities $|R_\nu(\boldsymbol{u}_{1\nu})| \leq L_\nu,\ \|R_\tau(\boldsymbol{u}_\tau)\| \leq L_\tau,\ |\beta_1| \leq 1,\ |\beta_2| \leq 1$, for the previous inequality, we deduce that

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq C\int_{\Gamma_3} |\beta_1 - \beta_2|\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|\, da,$$

then, we combine this inequality with (25), to obtain

(55) $$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq C\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (55) to find

(56) $$j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0.$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, R_τ show that

(57) $$|j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{v})| \leq C(\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V)\|\boldsymbol{v}\|_V.$$

Also, we take $\boldsymbol{u}_1 = \boldsymbol{v}$ and $\boldsymbol{u}_2 = 0$ in (56), then we use the equalities $R_\nu(0) = 0$, $R_\tau(0) = 0$ and (55) to obtain

(58) $$j_{ad}(\beta, \boldsymbol{v}, \boldsymbol{v}) \geq 0.$$

Next, we use (48), (32)(a), keeping in mind (25), propriety of a normal regularization operator and the inequalities $|R_\nu(\boldsymbol{u}_\nu)| \leq L_\nu,\ |\beta_1| \leq 1,\ |\beta_2| \leq 1$ and the continuity of the operator $\mathcal{R}$ we obtain

(59) $$j_{fr}(\beta_1, \boldsymbol{\sigma}_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - j_{fr}(\beta_1, \boldsymbol{\sigma}_1, \boldsymbol{u}_1, \boldsymbol{v}_1)$$

$$+ j_{fr}(\beta_2, \boldsymbol{\sigma}_2, \boldsymbol{u}_2, \boldsymbol{v}_1 - j_{fr}(\beta_2, \boldsymbol{\sigma}_2, \boldsymbol{u}_2, \boldsymbol{v}_2))$$
$$\leq \|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (C_{\mathcal{R}} ||\boldsymbol{u}_2 - \boldsymbol{u}_1\|_V + L_\nu c_0 \|\beta_2 - \beta_1\|_{L^2(\Gamma_3)}$$
$$+ L_\nu c_0 \|\boldsymbol{\sigma}_{2-}\boldsymbol{\sigma}_1||_{\mathcal{H}_1})||\boldsymbol{v}_2 - \boldsymbol{v}_1\|_V.$$

Now, by using (32)(a) and (44), it follows that the integral in (48) is well defined. Moreover, we have

$$j_{fr}(\beta, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) \leq \|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (L_\nu c_0 ||\boldsymbol{u}||_V\| + (C_{\mathcal{R}} \|\boldsymbol{\sigma}||_{\mathcal{H}_1} + L_\nu c_0 \|\beta\|_{L^2(\Gamma_3)}))\|\boldsymbol{v}\|_V. \tag{60}$$

Next, we use the bounds $|\psi(\boldsymbol{v})| \leq M_\psi$, $|\phi(\varphi_1 - \varphi_0)| \leq L_\phi$, the Lipschitz continuity of the functions ψ and ϕ, and inequality (27) to obtain

$$h(\boldsymbol{u}_1, \varphi_1, \varphi_1)_W - h(\boldsymbol{u}_1, \varphi_1, \varphi_2)_W + h(\boldsymbol{u}_2, \varphi_2, \varphi_1)_W - h(\boldsymbol{u}_2, \varphi_2, \varphi_2)_W \leq M_\psi\, c_0^2 \|\varphi_2 - \varphi_1\|_W^2 + L_\psi\, L_\phi c_0 \widetilde{c}_0 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\varphi_2 - \varphi_1\|_W. \tag{61}$$

The inequalities (55)–(60) combined with equalities (61) will be used in various places in the rest of the paper.

Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 3.1. *Assume that* (28)–(31)*,* (32)–(36) *and* (43)–(46) *hold. Then Problem* $\mathcal{P}^V$ *has a unique solution* $(\boldsymbol{u}, \varphi, \beta)$*. Moreover, the solutions belong to the following spaces*:

$$\boldsymbol{u} \in W^{1,\infty}(0, T; V), \tag{62}$$
$$\varphi \in C([0, T]; W), \tag{63}$$
$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{64}$$

A “quintuple” of functions ($\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, D, β) which satisfy (1), (2), (49)–(53) is called a *weak solution* of the contact problem $\mathcal{P}^V$. We conclude by Theorem 3.1 that, under the stated assumptions, Problem $\mathcal{P}^V$ has a unique weak solution. To precise the regularity of the weak solution we note that the constitutive relations (1) and (2), the assumptions (28), (31) and the regularities (62)–(64) show that $\boldsymbol{\sigma} \in W^{1,\infty}(0, T; \mathcal{H})$, $\mathbf{D} \in W^{1,\infty}(0, T; \mathcal{W})$. We choose as a test function $\mathbf{v} = \dot{\mathbf{u}}(t) \pm \mathbf{z}$ where $\mathbf{z} \in C_0^\infty(\Omega)^d$ in (49) and $\zeta \in C_0^\infty(\Omega)$ in (50) and use the definitions of $\boldsymbol{f}$, q, functionals j_{ad} and j_{fr} to obtain

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0}, \quad \operatorname{div} \mathbf{D}(t) + q_0(t) = 0$$

for all $t \in [0, T]$. It follows now from (42) that $\operatorname{Div} \boldsymbol{\sigma}(t) \in W^{1,\infty}(0, T; H)$ and $\operatorname{div} \mathbf{D} \in W^{1,\infty}(0, T; L^2(\Omega))$ and thus

$$\boldsymbol{\sigma} \in C([0, T]; \mathcal{H}_1), \tag{65}$$
$$\boldsymbol{D} \in C([0, T]; \mathcal{W}_1). \tag{66}$$

We conclude that the weak solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \beta)$ of the piezoelectric contact problem $\mathcal{P}^V$ has the regularity (62)–(64), (65) and (66).

4. Existence and uniqueness results

The proof of Theorem 3.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. Let $Y \subset X$ a non-empty closed convex subset of X. Consider the problem of finding $u(t) \in Y$ such that

$$(Au(t), v - u(t))_X + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_X \quad \forall v \in Y,\ t \in [0,T]. \tag{67}$$

To study problem (67) we need the following assumptions: The operator $A : Y \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\left\{\begin{array}{l} \text{(a) There exists } m_A > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in Y. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in Y. \end{array}\right. \tag{68}$$

The functional $j : Y \times Y \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) } j(u,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m > 0 \text{ such that} \\ \qquad j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \\ \quad \leq m\, \|u_1 - u_2\|_X\, \|v_1 - v_2\|_X \quad \forall\, u_1, u_2, v_1, v_2 \in X. \end{array}\right. \tag{69}$$

Finally, we assume that

$$f \in C([0,T]; X), \tag{70}$$

The following existence, uniqueness and regularity result may be found in [8].

Theorem 4.1. *Let* (68)–(69) *and* (70) *hold. Then, if* $m_A > m$*, there exists a unique solution* u *to the inequality* (67). *Moreover, the solution satisfies* $u \in C([0,T]; X)$.

We turn now to proof of Theorem 3.1. To this end, we assume in the following that (28)–(38) hold. Let $\mathcal{L}$ denotes the closed set of the space $C(0,T; L^2(\Gamma_3))$ defined by

$$\mathcal{L} = \left\{\beta \in C([0,T]; L^2(\Gamma_3)) \cap \mathcal{Q} \mid \beta(0) = \beta_0\right\}. \tag{71}$$

Let $\boldsymbol{\eta} \in C([0,T]; \mathcal{H})$, $\beta \in \mathcal{L}$ and $\boldsymbol{g} \in C([0,T]; \mathcal{H}_1)$ are know, and in the first step consider the following intermediate mechanical problem in which $\boldsymbol{\eta} = \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}\,(t))$ and $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}\,(t)$.

Problem $\mathcal{P}_{\eta\beta g}$. *Find a displacement field* $v_{\eta\beta g}$: $[0,T] \to V$*, an electric potential field* $\varphi_{\eta\beta g} : [0,T] \to W$ *such that*

(72)

$$\boldsymbol{v}_{\eta\beta g}(t) \in U_{ad}, (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}(t)))_{\mathcal{H}} + (\boldsymbol{\eta}, \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}_{\eta\beta g}(t)))_{\mathcal{H}}$$

$$+\,(\mathcal{E}^*\nabla\varphi_{\eta\beta g}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}-\boldsymbol{v}_{\eta\beta g}(t)))_{\mathcal{H}}+j_{ad}(\beta(t),\boldsymbol{v}_{\eta\beta g}(t),\boldsymbol{w}-\boldsymbol{v}_{\eta\beta g}(t))$$
$$+\,j_{fr}(\boldsymbol{g}(t),\beta(t),\boldsymbol{v}_{\eta\beta g}(t),\boldsymbol{w})-j_{fr}(\boldsymbol{g}(t),\beta(t),\boldsymbol{v}_{\eta\beta g}(t),\boldsymbol{v}_{\eta\beta g}(t))$$
$$\geq (f(t),\boldsymbol{w}-\boldsymbol{v}_{\eta\beta g}(t))_V,\quad \forall \boldsymbol{w}\in U_{ad},$$

$$(\mathcal{B}\nabla\varphi_{\eta\beta g}(t),\nabla\psi)_H-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}(t)),\nabla\psi)_H+(h(\boldsymbol{v}_{\eta\beta g}(t),\varphi),\psi)_W = (q(t),\psi)_W \quad \forall\,\psi\in W. \tag{73}$$

In order to solve Problem $\mathcal{P}_{\eta\beta g}$ we consider the product space $X=V\times W$ endowed with the inner product

$$(x,y)_X=(\boldsymbol{v},\boldsymbol{w})_V+(\varphi,\psi)_W \qquad \forall x=(\boldsymbol{v},\varphi),\ y=(\boldsymbol{w},\psi)\in X \tag{74}$$

and the associated norm $||\cdot||_X$. We also introduce the set $K\subset X$ defined by

$$K=U_{ad}\times W. \tag{75}$$

We defined the operator $A_{\eta\beta g}(t):\,X\times X\to\mathbb{R}$, the function $j_{g\beta}(x,y):\,X\times X\to\mathbb{R}$, the elements $\boldsymbol{f}_\eta(t)\in V$ and $\mathbf{f}_\eta(t)\in X$ by qualities

$$\begin{aligned}(A_{\eta\beta g}(t)x_{\eta\beta g},y)_X=&(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}(t))),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}+(\mathcal{B}\nabla\varphi_{\eta\beta g}(t),\nabla\psi)_H\\&+(\mathcal{E}^*\nabla\varphi_{\eta\beta g}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}(t)),\nabla\psi)_H\\&+j_{ad}(\beta(t),\boldsymbol{v}_{\eta\beta g}(t),\boldsymbol{w}),\\ &\forall x_{\eta\beta g}=(\boldsymbol{v}_{\eta\beta g}(t),\varphi_{\eta\beta g}(t)),\ y=(\boldsymbol{w},\psi)\in K,\quad t\in[0,T],\end{aligned} \tag{76}$$

$$\begin{aligned}j_{\eta g\beta}(x_{\eta\beta g}(t),y)=&j_{fr}(\boldsymbol{g}(t),\beta(t),\boldsymbol{v}_{\eta\beta g}(t),\mathbf{w})+(h(\boldsymbol{v}_{\eta\beta g}(t),\varphi_{\eta\beta g}(t)),\psi)_W,\\&\forall x_{\eta\beta g}=(\boldsymbol{v}_{\eta\beta g}(t),\varphi_{\eta\beta g}(t)),\ y=(\boldsymbol{w},\psi)\in K,\end{aligned} \tag{77}$$

$$(\boldsymbol{f}_\eta(t),\boldsymbol{w})_V=(\boldsymbol{f}(t),\boldsymbol{w})_V-(\boldsymbol{\eta}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}\ \forall\ \boldsymbol{w}\in U_{ad},\ \forall t\in[0,T], \tag{78}$$

$$\mathbf{f}_\eta(t)=(\boldsymbol{f}_\eta(t),q(t)). \tag{79}$$

We start with the following equivalence result.

Lemma 4.2. *The couple $x_{\eta\beta g}=(\boldsymbol{v}_{\eta\beta g}(t),\varphi_{\eta\beta g}):[0,T]\to V\times W$ is a solution to Problem $\mathcal{P}_{\eta\beta g}$ if and only if $x_{\eta\beta g}\in C([0,T];K)$ and satisfies*

$$\begin{aligned}&x_{\eta\beta g}(t)\in K,\quad (A_{\eta\beta g}(t)x_{\eta\beta g}(t),y-x_{\eta\beta g}(t))_X+j_{\eta\beta g}(x_{\eta\beta g}(t),y)\\&-j_{\eta\beta g}(x_{\eta\beta g}(t),x_{\eta\beta g}(t))\geq(\mathbf{f}_\eta(t),y-x_{\eta\beta g}(t))_X\quad \forall y\in K,\ t\in[0,T],\end{aligned} \tag{80}$$

Proof. Let $x_{\eta\beta g}=(\boldsymbol{v}_{\eta\beta g},\varphi_{\eta\beta g}):\ [0,T]\to K$ be a solution to Problem $\mathcal{P}_{\eta\beta g}$. Let $y=(\boldsymbol{w},\psi)\in K$ and let $t\in[0,T]$. We use the test function $\psi-\varphi_{\eta\beta g}(t)$ in (73), add the corresponding inequality to (72), and use (74)–(78) to obtain (80). Conversely, assume that $x_{\eta\beta g}=(\boldsymbol{v}_{\eta\beta g},\varphi_{\eta\beta g}):\ [0,T]\to K$ satisfies (80) and let $t\in[0,T]$. For any $\boldsymbol{w}\in U_{ad}$, we take $y=(\boldsymbol{w},\varphi_{\eta\beta g}(t))$ in (80) to obtain (72). Then, for any $\psi\in W$, we take successively $y=(\boldsymbol{v}_{\eta\beta g},\varphi_{\eta\beta g}(t)+\psi)$ and $y=(\boldsymbol{v}_{\eta\beta g},\varphi_{\eta\beta g}(t)-\psi)$ in (80) to obtain (73). □

We use now Lemma 4.2 to obtain the following existence and uniqueness result.

Lemma 4.3. *There exists $\mu_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$ and p such that, if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, Problem $\mathcal{P}_{\eta\beta g}$ has a unique solution $(\boldsymbol{v}_{\eta\beta g}, \varphi_{\eta\beta g}) \in C([0,T]; U_{ad} \times W)$.*

Proof. We apply Theorem 4.1 where $X = V \times W$ and $Y = K = U_{ad} \times W$. Let $t \in [0,T]$, we use (28)–(31), (54), and (58) to see that $A_{\eta\beta g}(t)$ is a strongly monotone Lipschitz continuous operator on X and it satisfies

$$(A_\eta(t)x_{\eta\beta_1 g_1}(t) - A_{\eta\beta g}(t)x_{\eta\beta_2 g_2}(t), x_{\eta\beta_1 g_1}(t) - x_{\eta\beta_2 g_2}(t))_X$$
$$\geq \min(m_{\mathcal{F}}, m_{\mathcal{B}})||x_{\eta\beta_1 g_1}(t) - x_{\eta\beta_2 g_2}(t)||_X^2. \tag{81}$$

Using (60), (61) and (77), we can easily check that the functional $j_{\eta\beta g}$ satisfies

$$\begin{aligned} & j_{\eta\beta_1 g_1}(x_{\eta\beta_1 g_1}, x_{\eta\beta_1 g_1}) - j_{\eta\beta_1 g_1}(x_{\eta\beta_1 g_1}, x_{\eta\beta_2 g_2}) \\ & \quad + j_{\eta\beta_2 g_2}(x_{\eta\beta_2 g_2}, x_{\eta_1\beta_1 g_1}) - j_{\eta\beta_2 g_2}(x_{\eta\beta_2 g_2}, x_{\eta_2\beta_2 g_2}) \\ & \leq \|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (C_{\mathcal{R}} ||x_{\eta\beta_2 g_2} - x_{\eta_1\beta_1 g_1}\|_X \\ & \quad + L_\nu c_0 \|\beta_2 - \beta_1\|_{L^2(\Gamma_3)}) ||x_{\eta\beta_2 g_2} - x_{\eta\beta_1 g_1}\|_X \\ & \quad + (\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0^2 L_\nu + M_\psi\, \tilde{c}_0^2 + L_\psi\, L_\phi c_0 \widetilde{c}_0)\|x_{\eta\beta_2 g_2} - x_{\eta\beta_1 g_1}||_X \end{aligned} \tag{82}$$

is a continuous seminorm on X and moreover, it satisfies condition (69) on X when $\beta_1 = \beta_2$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2$.

Let

$$\mu_0 = \frac{\min(m_{\mathcal{F}}, m_{\mathcal{B}})}{c_0^2 L_p L_\nu + M_\psi\, \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0},$$

where $m_{\mathcal{F}}$, $m_{\mathcal{B}}$, c_0, L_p, L_ν, M_ψ, $\tilde{c}_0$, L_ψ and L_ϕ are given in (28), (29), (25), (32), (15), (34), (27) and (20), respectively. We note that μ_0 depends on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$, R_ν, ψ, ϕ and p. Assume that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then

$$\|\mu\|_{L^\infty(\Gamma_3)} c_0^2 L_p L_\nu + M_\psi\, \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0 < \min(m_{\mathcal{F}}, m_{\mathcal{B}}), \tag{83}$$

and note that this smallness assumption involves of the geometry, the electrical, and on the mechanical data of problem.

Using (81), (82) and (83), the existence and uniqueness part in Lemma 4.3 is now a consequence of Lemma 4.2 and Theorem 4.1.

For $t_1, t_2 \in [0,T]$, an argument based on (28), (81), the inequalities involving the functionals j_{ad}, h and j_{fr} presented at the end of Section 3, (82) and (83) shows that

$$\begin{aligned} & ||x_{\eta\beta g}(t_2) - x_{\eta\beta g}(t_1)||_X \\ & \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (C_{\mathcal{R}} ||\boldsymbol{g}(t_2) - \boldsymbol{g}(t_1)\|_V + L_\nu c_0 \|\beta(t_2) - \beta(t_1)\|_{L^2(\Gamma_3)})}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} \\ & \quad + \frac{1}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} ||\mathbf{f}_\eta(t_2) - \mathbf{f}_\eta(t_1)||_X. \end{aligned} \tag{84}$$

The last inequality implies that

$$||\boldsymbol{v}_{\eta\beta g}(t_2) - \boldsymbol{v}_{\eta\beta g}(t_1)||_V$$

$$\leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (C_{\mathcal{R}} \|\boldsymbol{g}(t_2) - \boldsymbol{g}(t_1)\|_V + L_\nu c_0 \|\beta(t_2) - \beta(t_1)\|_{L^2(\Gamma_3)})}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0}$$

$$+ \frac{1}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} ||\mathbf{f}_\eta(t_2) - \mathbf{f}_\eta(t_1)||_X \tag{85}$$

and

$$||\varphi_{\eta\beta g}(t_2) - \varphi_{\eta\beta g}(t_1)||_V$$

$$\leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 (C_{\mathcal{R}} \|\boldsymbol{g}(t_2) - \boldsymbol{g}(t_1)\|_V + L_\nu c_0 \|\beta(t_2) - \beta(t_1)\|_{L^2(\Gamma_3)})}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0}$$

$$+ \frac{1}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} ||\mathbf{f}_\eta(t_2) - \mathbf{f}_\eta(t_1)||_X. \tag{86}$$

Keeping in mind that $\boldsymbol{f}_\eta \in W^{1,\infty}(0,T;V)$, $q \in W^{1,\infty}(0,T;W)$ and recall that $\boldsymbol{\eta} \in C([0,T];\mathcal{H})$, we deduce from (79) that $\mathbf{f}_\eta \in C([0,T];X)$. Knowing that $\beta \in \mathcal{L}$ and $\boldsymbol{g} \in C([0,T];\mathcal{H}_1)$, it follows now from (84) that the mapping $t \to x_{\eta\beta g} = (\boldsymbol{v}_{\eta\beta g}, \varphi_{\eta\beta g}) : [0,T] \to X$ is continuous, this implies that $(\boldsymbol{u}_{\eta\beta g}, \varphi_{\eta\beta g}) \in W^{1,\infty}(0,T;V) \times \in C([0,T];W)$.

We assume in what follows that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ and therefore (83) is valid. In the next step, we use the displacement field $\boldsymbol{v}_{\eta\beta g}$ obtained in Lemma 4.3, denote by $\boldsymbol{v}_{\eta\beta g\nu}$, $\boldsymbol{v}_{\eta\beta g\tau}$ its normal and tangential components, and we consider the following initial value problem. □

Problem $\mathcal{P}^{\boldsymbol{\theta}}_{\boldsymbol{\eta\beta g}}$. *Find a bonding field $\theta_{\eta\beta g}$: $[0,T] \to L^2(\Gamma_3)$ such that*

$$\dot{\theta}_{\eta\beta g}(t) = -\Big(\theta_{\eta\beta g}(t)\big(\gamma_\nu R_\nu(\boldsymbol{v}_{\eta\beta g\nu}(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{v}_{\eta\beta g\tau}(t))\|^2\big) - \epsilon_a\Big)_+ \quad a.e.\ t \in (0,T), \tag{87}$$

$$\theta_{\eta\beta g}(0) = \beta_0. \tag{88}$$

We obtain the following result.

Lemma 4.4. *There exists a unique solution to Problem $\mathcal{P}^\theta_{\eta\beta g}$ and it satisfies $\theta_{\eta\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$*

Proof. Consider the mapping $F_{\eta\beta g} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_{\eta\beta g}(t,\theta) = -(\theta(t)(\gamma_\nu R_\nu((\boldsymbol{v}_{\eta\beta g})_\gamma(t))^2 + \gamma_\tau \|R_\tau((\boldsymbol{v}_{\eta\beta g})_\tau(t))\|^2) - \epsilon_a)_+ \tag{89}$$

for all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and R_τ that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_{\eta\beta g}(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem, we obtain the existence of a unique function $\theta_{\eta\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ which solves (87), (88). We note that the restriction $0 \leq \beta \leq 1$ is implicitly included in the variational problem $\mathcal{P}_{\boldsymbol{v}}$. Indeed, (50) and (52) guarantee that $\beta(t) \leq \beta_0$ and, therefore, assumption (46) shows that $\beta(t) \leq 1$ for $t \geq 0$, a.e. on Γ_3. On the other hand, if $\beta(t_0) = 0$ at $t = t_0$, then it follows from (50) and (52) that $\dot{\beta}(t) = 0$ for all $t \geq t_0$ and therefore, $\beta(t) = 0$ for all $t \geq 0$, a.e. on Γ_3. We conclude that $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Q}$, we find that $\theta_{\eta\beta g} \in \mathcal{Q}$, which concludes the proof of lemma. □

It follows from Lemma 4.4 that for all $\boldsymbol{\eta} \in C([0, T]; \mathcal{H})$, $\beta \in \mathcal{L}$ and $\boldsymbol{g} \in C([0, T]; \mathcal{H}_1)$, the solution $\theta_{\eta\beta g}$ of Problem $\mathcal{P}^{\theta}_{\eta\beta g}$ belongs to $\mathcal{L}$, see (71).

We denote now by $\boldsymbol{\sigma}_{\eta\beta g}$ the tensor given by

$$\sigma_{\eta\beta g} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta\beta g}) + \mathcal{E}^* \nabla(\varphi_{\eta\beta g}), \tag{90}$$

where $(\boldsymbol{v}_{\eta\beta g}, \varphi_{\eta\beta g})$ is the solution of Problem $\mathcal{P}_{\eta\beta g}$. From (28), (30) and Lemma 4.3, its follows that $\boldsymbol{\sigma}_{\eta\beta g} \in C([0, T]; \mathcal{H}_1)$. Therefore, we may consider the operator $\Lambda_\eta : \mathcal{L} \times C([0, T]; \mathcal{H}_1) \to \mathcal{L} \times C([0, T]; \mathcal{H}_1)$ given by

$$\Lambda_\eta(\beta, \boldsymbol{g}) = (\theta_{\eta\beta g}, \boldsymbol{\sigma}_{\eta\beta g}). \tag{91}$$

The third step consists in the following result.

Lemma 4.5. *There exists a unique element* $(\beta^*, \boldsymbol{g}^*) \in \mathcal{L} \times C([0, T]; \mathcal{H}_1)$ *such that*

$$\Lambda_\eta(\beta^*, \boldsymbol{g}^*) = (\beta^*, \boldsymbol{g}^*).$$

Proof. Suppose that $(\beta_i, \boldsymbol{g}_i)$ are two couples of functions in $\mathcal{L} \times C([0, T]; \mathcal{H}_1)$ and denote by $(\boldsymbol{u}_i, \varphi_i)$, θ_i the functions obtained in Lemmas 4.3 and 4.4, respectively, for $(\beta, \boldsymbol{g})$=$(\beta_i, \boldsymbol{g}_i)$, $i = 1, 2$. Let $t \in [0, T]$. We use arguments similar to those used in the proof of (84) to deduce that

$$\begin{aligned} ||x_{\eta\beta_1 g_1}(t) - x_{\eta\beta_2 g_2}(t)||_X \leq & \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 C_{\mathcal{R}} ||\boldsymbol{g}_2(t) - \boldsymbol{g}_1(t)\|_{\mathcal{H}_1}}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} \\ & + \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 L_\nu c_0 \|||\beta_2(t) - \beta_1(t)\|_{L^2(\Gamma_3)}}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} \end{aligned} \tag{92}$$

which implies

$$\begin{aligned} ||\boldsymbol{v}_{\eta\beta_1 g_1}(t) - \boldsymbol{v}_{\eta_2\beta_2 g_2}(t)||_V \leq & \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 C_{\mathcal{R}} ||\boldsymbol{g}_2(t) - \boldsymbol{g}_1(t)\|_{\mathcal{H}_1}}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0} \\ & + \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 L_\nu c_0 \|||\beta_2(t) - \beta_1(t)\|_{L^2(\Gamma_3)}}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0}. \end{aligned} \tag{93}$$

On the other hand, it follows from (87) and (88) that

$$\theta_{\eta\beta igi}(t) = \beta_0 - \int_0^t (\theta_{\eta\beta igi}(s)(\gamma_\nu R_\nu(\boldsymbol{v}_{\eta\beta igi\nu}(s))^2 + \gamma_\tau \|R_\tau(\boldsymbol{v}_{\eta\beta igi\tau}(s))\|^2) - \epsilon_a)_+ ds \tag{94}$$

and then

(95)

$$||\theta_{\eta\beta_2 g_2}(t) - \theta_{\eta\beta_1 g_1}(t)||_{L^2(\Gamma_3)}$$

$$\leq c\Big(\int_0^t ||\theta_{\eta\beta_2 g_2}(s)R_\nu(\boldsymbol{v}_{\eta\beta_2 g_2\nu}(s))^2 - \theta_{\eta\beta_1 g_1}(s)R_\nu(\boldsymbol{v}_{\eta\beta_1 g_1\nu}(s))^2||_{L^2(\Gamma_3)}ds$$
$$+\int_0^t ||\theta_{\eta\beta_2 g_2}(s)||R_\tau(\boldsymbol{v}_{\eta\beta_2 g_2\tau}(s))^{||2} - \theta_{\eta\beta_1 g_1}(s)||R_\tau(\boldsymbol{v}_{\eta\beta_1 g_1\tau}(s))||^2||_{L^2(\Gamma_3)}ds\Big).$$

Using the definition of R_ν and R_τ and writing , $\theta_{\eta\beta_1 g_1} = \theta_{\eta\beta_1 g_1} - \theta_{\eta\beta_2 g_2} + \theta_{\eta\beta_2 g_2}$ we get

$$||\theta_{\eta\beta_2 g_2}(t) - \theta_{\eta\beta_1 g_1}(t)||_{L^2(\Gamma_3)} \leq c\Big(\int_0^t ||\theta_{\eta\beta_2 g_2}(s) - \theta_{\eta\beta_1 g_1}(s)||_{L^2(\Gamma_3)}ds$$
$$+\int_0^t ||\boldsymbol{v}_{\eta\beta_1 g_1}(s) - \boldsymbol{v}_{\eta\beta_2 g_2}(s)||_{L^2(\Gamma_3)}ds\Big). \tag{96}$$

By Gronwall's inequality, it follows that

$$||\theta_{\eta\beta_2 g_2}(t) - \theta_{\eta\beta_1 g_1}(t)||_{L^2(\Gamma_3)} \leq C_{Gr}\int_0^t ||\boldsymbol{v}_{\eta\beta_1 g_1}(s) - \boldsymbol{v}_{\eta\beta_2 g_2}(s)||_{L^2(\Gamma_3)}ds, \tag{97}$$

where C_{Gr} is the Gronwall's constant.

Using (25), we obtain

$$||\theta_{\eta\beta_2 g_2}(t) - \theta_{\eta\beta_1 g_1}(t)||_{L^2(\Gamma_3)} \leq C_{Gr}c_0\int_0^t ||\boldsymbol{v}_{\eta\beta_1 g_1}(s) - \boldsymbol{v}_{\eta\beta_2 g_2}(s)||_V ds. \tag{98}$$

Let

$$N = \frac{C_{Gr}c_0^2\|\mu\|_{L^\infty(\Gamma_3)}L_p\max(C_{\mathcal{R}}, L_\nu c_0)\max(L_{\mathcal{F}}, c_{\mathcal{E}}\widetilde{c}_0)}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2L_pL_\nu + M_\psi\tilde{c}_0^2 + L_\psi L_\phi c_0\widetilde{c}_0}.$$

We now combine (93) and (98) to see that

$$||\theta_{\eta\beta_2 g_2}(t) - \theta_{\eta\beta_1 g_1}(t)||_{L^2(\Gamma_3)}$$
$$\leq N\int_0^t (||\boldsymbol{g}_2(s) - \boldsymbol{g}_1(s)\|_{\mathcal{H}_1} + ||\beta_2(s) - \beta_1(s)\|_{L^2(\Gamma_3)})ds. \tag{99}$$

Using now (28), (30) and (90) (92) it is easy to see that

$$||\sigma_{\eta\beta_1 g_1}(t) - \sigma_{\eta\beta_2 g_2}(t)||_{\mathcal{H}} \leq N(||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1}), \tag{100}$$

where $c_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$.

From (91), (99) and the last inequality, it results that

$$||\Lambda_\eta(\beta_1, g_1)(t) - \Lambda_\eta(\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}$$
$$\leq N||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}$$
$$+c\int_0^t ||(\beta_1, g_1)(s) - (\beta_2, g_2)(s)||_{L^2(\Gamma_3)\times\mathcal{H}_1}ds. \tag{101}$$

Using the following notations

(102)

$$I_0(t) = ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1},$$

$$I_1(t) = \int_0^t ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} ds,$$
$$I_k(t) = \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} dr ds_1 \cdots ds_{k-1}, \quad \forall k \geq 2,$$

and denoting now by Λ_η^p the powers of operator Λ_η, (101) and (102) imply by recurrence that

$$\left\|\Lambda_\eta^p(\beta_1, g_1)(t) - \Lambda_\eta^p(\beta_2, g_2)(t)\right\|_{L^2(\Gamma_3)\times\mathcal{H}_1}$$

$$\leq (\sum_{k=0}^{p} C_p^k \frac{N^{p-k} M^p T^p}{k!}) \left\|(\beta_1, g_1)(t) - (\beta_2, g_2)(t)\right\|_{L^2(\Gamma_3)\times\mathcal{H}_1} \tag{103}$$

$$\leq \frac{(Np + MT)^p}{p!} \left\|(\beta_1, g_1)(t) - (\beta_2, g_2)(t)\right\|_{L^2(\Gamma_3)\times\mathcal{H}_1}. \tag{104}$$

Using the Stirling's formula, we obtain under the condition $N \leq \frac{1}{e}$ that

$$\lim_{p\to\infty} \frac{(Np + MT)^p}{p!} = 0,$$

which shows that for p sufficiently large $\Lambda_\eta^p : \mathcal{L}\times C([0,T];\mathcal{H}_1) \to \mathcal{L}\times C([0,T];\mathcal{H}_1)$ is a contraction. Then, we conclude by using the Banach fixed point theorem that Λ_η has a unique fixed point $(\beta^*, g^*) \in \mathcal{L} \times C([0,T];\mathcal{H}_1)$ such that $\Lambda_\eta(\beta^*, g^*) = (\beta^*, g^*)$. Hence, from (91) it results for all $t \in [0,T]$,

$$(\beta^*, g^*)(t) = (\theta_{\eta\beta^*g^*}(t), \sigma_{\eta\beta^*g^*}(t)). \tag{105}$$

□

From (29) and Lemma 4.3, its follows that $\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta\beta^*g^*}(t)) \in C([0,T];\mathcal{H}_1)$. We now consider the operator $\Lambda : C([0,T];\mathcal{H}) \to C([0,T];\mathcal{H})$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta\beta^*g^*}(t)) \qquad \forall\, \boldsymbol{\eta} \in C([0,T];\mathcal{H}),\ t \in [0,T], \tag{106}$$

where (β^*, g^*) is a fixed point of Λ_η and $\boldsymbol{u}_{\eta\beta^*g^*} = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{\eta\beta^*g^*}(s)ds$.

We show that Λ has a unique fixed point.

Lemma 4.6. *There exists a unique $\boldsymbol{\eta}^* \in C([0,T];\mathcal{H})$ such that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.*

Proof. Let $\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2 \in C([0,T];\mathcal{H})$, the fixed point (β^*, g^*) of the operator $\Lambda_\eta(\beta^*, g^*)$ and denote by $\boldsymbol{v}_i$ the function $\boldsymbol{v}_{\eta_i\beta^*g^*}$ obtained in Lemma 4.3 and by $\boldsymbol{u}_i$ the function $\boldsymbol{u}_{\eta_i\beta^*g^*}$ for $i = 1, 2$. Let $t \in [0,T]$. Using (106) and (29) we obtain

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}} \leq L_{\mathcal{G}} \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V. \tag{107}$$

We use arguments similar to those used in the proof of (84) to deduce that for $\beta_1 = \beta_2 = \beta^*$ and $g_1 = g_2 = g^*$

$$\|\boldsymbol{v}_1(t) - \mathbf{v}_2(t)\|_V \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 ||\mathbf{f}_{\eta_1}(t) - \mathbf{f}_{\eta_2}(t)||_V}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0}, \tag{108}$$

and using this inequality in (79), its easy to obtain

$$||\mathbf{f}_{\eta_1}(t) - \mathbf{f}_{\eta_2}(t)||_V \le c\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}}. \tag{109}$$

Combining (107)–(109) leads to

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}} \le \frac{L_{\mathcal{G}}\|\mu\|_{L^\infty(\Gamma_3)} L_p c_0 \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}} ds}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 L_p L_\nu + M_\psi \tilde{c}_0^2 + L_\psi L_\phi c_0 \widetilde{c}_0}.$$

Reiterating this inequality n times results, it yields

$$\|\Lambda^n\boldsymbol{\eta}_1 - \Lambda^n\boldsymbol{\eta}_2\|_{C([0,T];\mathcal{H})} \le \frac{c^n}{n!}\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{C([0,T];\mathcal{H})}.$$

This inequality shows that for a sufficiently large n the operator Λ^n is a contraction on the Banach space $C([0,T];\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\eta}^* \in C([0,T];\mathcal{H})$ such that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$. □

Now, we have all the ingredients to provide the proof of Theorem 3.1.

Proof of Theorem 3.2. Existence. Let $\boldsymbol{\eta}^* \in C([0,T];\mathcal{H})$ and $(\beta^*, \boldsymbol{g}^*) \in \mathcal{L} \times C([0,T];\mathcal{H}_1)$ are the fixed points of Λ and $\Lambda_{\boldsymbol{\eta}}$, respectively. Let $(\boldsymbol{v}^*, \varphi^*)$ be the solution of Problem $\mathcal{P}_{\eta\beta g}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and $(\beta, \boldsymbol{g}) = (\beta^*, \boldsymbol{g}^*)$, that is, $\boldsymbol{v}^* = \boldsymbol{u}_{\widetilde{\boldsymbol{\eta}}^*\beta^* g^*}$ and $\varphi^* = \varphi_{\widetilde{\boldsymbol{\eta}}^*\beta^* g^*}$. Since $\theta_{\widetilde{\boldsymbol{\eta}}^*\beta^* g^*} = \beta^*$, we conclude by (72), (73), (87), (88) and the fact that $\dot{\boldsymbol{u}}^* = \boldsymbol{v}^*$ that $(\boldsymbol{u}^*, \varphi^*, \beta^*)$ is a solution of Problem $\mathcal{P}^V$, and, moreover, β^* satisfies the regularity (64). Also, since $\boldsymbol{\eta} \in C([0,T];\mathcal{H})$, $\mathbf{f}_{\boldsymbol{\eta}^*} \in C([0,T];X)$, by (42) and (78) it is easy to see that the function $\mathbf{f}_\eta$ defined by (79) satisfies . Inequality (85), (86) imply that the function $x^* = (\boldsymbol{v}^*, \varphi^*) : [0,T] \to X$ is Lipschitz continuous; therefore, x^* belongs to $C([0,T];X)$, which shows that the functions $\boldsymbol{u}^*$ and φ^* have the regularity expressed in (62), (63).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ and $\Lambda_{\boldsymbol{\eta}}$ defined by (91) and (106), respectively. Indeed, let $(\boldsymbol{u}, \varphi, \beta)$ be a another solution of Problem $\mathcal{P}^V$ which satisfies (62)–(64).

We denote by $\boldsymbol{\eta} \in C([0,T];\mathcal{H}_1)$ and $(\beta, \boldsymbol{g}) \in \mathcal{L} \times C([0,T];\mathcal{H}_1)$ the functions defined by

$$\dot{\beta}(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\dot{\boldsymbol{u}}_\nu(t))^2 + \gamma_\tau\|R_\tau(\dot{\boldsymbol{u}}_\tau(t))\|^2) - \epsilon_a)_+ \text{ a.e. } t \in (0,T), \tag{110}$$

$$\beta(0) = \beta_0, \tag{111}$$

$$\boldsymbol{g}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^*\nabla(\varphi(t)), \tag{112}$$

$$\boldsymbol{\eta}(t) = \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}\,(t)). \tag{113}$$

It follows from (72), (73) that $(\dot{\boldsymbol{u}}, \varphi)$ is a solution to Problem $\mathcal{P}_{\eta\beta g}$ and, since by Lemma 4.3 this problem has a unique solution denoted by $(\dot{\boldsymbol{u}}_{\eta\beta g}, \varphi_{\eta\beta g})$, we obtain

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_{\eta\beta g}, \tag{114}$$

$$\varphi = \varphi_{\eta\beta g}. \tag{115}$$

Then, we replace $\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_{\eta\beta g}$ in (51) and use the initial condition (53) to see that β is a solution to Problem $\mathcal{P}^{\theta}_{\eta\beta g}$. Since by Lemma 4.4 this last problem has a unique solution denoted by $\theta_{\eta\beta g}$, we find

$$\beta = \theta_{\eta\beta g}. \tag{116}$$

We use now (91), (106), (116) (114), (112), (115), and Lemma 4.5 that

$$(\beta, \boldsymbol{g}) = (\beta^*, \boldsymbol{g}^*). \tag{117}$$

We use now (106), (114) and Lemma 4.6 that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*. \tag{118}$$

The uniqueness part of the theorem is now a consequence of (114), (115), (117) and the last inequality. □

5. Conclusions

We presented a model for the quasistatic process of frictional contact between a deformable body made of a piezoelectric material, more precisely, an electro-viscoelastic material, and an electrically conductive rigid foundation. The contact and friction are modelled by Signorini's conditions and a non local Coulomb's friction law in which the adhesion of contact surfaces is taken into account.

The problem was set as a variational inequality for the displacements, a variational equality for the electric potential and a first order differential equation for the adhesion. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities involving nonlinear strongly monotone Lipschitz continuous operators, and a fixed-point theorem. It was obtained under a smallness assumption which involves the mechanical and electric data of the problem. This work opens the way to study further problems with other conditions of contact.

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