J. Korean Math. Soc. **53** (2016), No. 1, pp. 147–159
http://dx.doi.org/10.4134/JKMS.2016.53.1.147

SOME SUBORDINATION PROPERTIES OF THE LINEAR OPERATOR

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Abstract. In this paper, subordination results of analytic function $f \in \mathcal{A}_p$ involving linear operator $\mathcal{K}_{c,p}^{\delta,\lambda}$ are obtained. By applying the differential subordination method, results are derived under some sufficient subordination conditions. On using some hypergeometric identities, corollaries of the main results are derived. Furthermore, convolution preserving properties for a class of multivalent analytic function associated with the operator $\mathcal{K}_{c,p}^{\delta,\lambda}$ are investigated.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.1}$$

that are analytic and p-valent in the *open unit disk* $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For functions $f \in \mathcal{A}_p$ given by (1.1) and $g \in \mathcal{A}_p$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of f and g denoted by $f * g$ is defined as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

Suppose that f and g are analytic in the unit disk $\mathbb{U}$. We say that f is *subordinate* to g (or g is *superordinate* to f), written as

$$f \prec g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

Received September 16, 2014; Revised January 2, 2015.
2010 *Mathematics Subject Classification.* 30C45, 33C05.
Key words and phrases. multivalent functions, differential subordination, Komatu integral operator, Gauss hypergeometric function.

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

It follows from the Schwarz lemma that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function g is *univalent* in $\mathbb{U}$, then the reverse implication holds true (see [7, 8]).

For real parameters $A,\ B\ (-1 \leq B < A \leq 1)$, the function $\frac{1+Az}{1+Bz}$ $(z \in \mathbb{U})$, maps conformally $\mathbb{U}$ onto a disk (whenever $-1 \leq B \leq 1$), symmetrical with respect to the real axis having center at $\frac{1-AB}{1-B^2}$ and radius $\frac{A-B}{1-B^2}$ where $B \neq \pm 1$. Furthermore, the boundary circle of the disk intersects the real axis at the point $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ provided $B \neq \pm 1$.

Motivated essentially by Khairnar and More [5], Salim [10] introduced the integral operator $\mathcal{K}_{c,p}^{\delta} : \mathcal{A}_p \to \mathcal{A}_p$ as follows:

$$\mathcal{K}_{c,p}^{\delta} f(z) = \begin{cases} \frac{(c+p)^{\delta}}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(log \frac{z}{t}\right)^{\delta-1} f(t)dt & (\delta > 0;\ c > -p) \\ f(z) & (\delta = 0). \end{cases} \tag{1.3}$$

For $f \in \mathcal{A}_p$ given by (1.1), it can be easily deduced from (1.3) that

$$\mathcal{K}_{c,p}^{\delta} f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k}\right)^{\delta} a_{k+p} z^{k+p} \quad (\delta \geq 0,\ c > -p;\ z \in \mathbb{U}). \tag{1.4}$$

Define a function $\phi_{c,p}^{\delta}(z)$ by

$$\phi_{c,p}^{\delta}(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{c+p+k}{c+p}\right)^{\delta} z^{k+p} \quad (\delta \geq 0,\ c > -p). \tag{1.5}$$

Corresponding to the function $\phi_{c,p}^{\delta}(z)$ defined by (1.5), we consider the function $\phi_{c,p}^{\delta,+}(z)$, the generalized multiplicative inverse of $\phi_{c,p}^{\delta}(z)$ given by

$$\phi_{c,p}^{\delta}(z) * \phi_{c,p}^{\delta,+}(z) = \frac{z^p}{(1-z)^{\lambda+p}} \qquad (\lambda > -p;\ z \in \mathbb{U}). \tag{1.6}$$

Note that, if $\lambda = -p + 1$, then $\phi_{c,p}^{\delta,+}(z)$ is the inverse of $\phi_{c,p}^{\delta}(z)$ with respect to the Hadamard product (or convolution) $*$.

Using this function we define the following family of operator $\mathcal{K}_{c,p}^{\delta,\lambda} : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$\mathcal{K}_{c,p}^{\delta,\lambda} f(z) = \phi_{c,p}^{\delta,+}(z) * f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k}\right)^{\delta} \frac{(\lambda+p)_k}{(1)_k} a_{k+p} z^{k+p} \tag{1.7}$$

$$(\lambda > -p,\ \delta \geq 0, f \in \mathcal{A}_p;\ z \in \mathbb{U}),$$

where $(\lambda)_k$ is the Pochhammer symbol (or shifted factorial) given by

$$(1.8)\quad (\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

By specializing the parameters $c,\ p, \delta$ and λ we obtain the following operators studied earlier by various researchers:

- $\mathcal{K}_{c,p}^{\delta,-p+1} \equiv \mathcal{K}_{c,p}^{\delta}$ which is the generalized Komatu integral operator [5].
- $\mathcal{K}_{c,1}^{\delta,0} \equiv \mathcal{P}_{c}^{\delta}$ which is the integral operator studied by Komatu [6] and Raina and Bapna [9].
- $\mathcal{K}_{1,p}^{\delta,-p+1} \equiv \mathcal{I}_{p}^{\delta}$ which is the integral operator studied by Shams et al. [11] and Ebadian et al. [3].
- $\mathcal{K}_{c,1}^{1,0} = \mathcal{L}_c$ which is the Bernardi-Libra-Livingston integral operator [1].
- $\mathcal{K}_{1,1}^{\delta,0} \equiv \mathcal{I}^{\delta}$ which is the integral operator studied by Ebadian and Najafzadeh [2].
- $\mathcal{K}_{c,p}^{\delta,-p+1} \equiv \mathcal{J}_{p,c,1}^{\delta}$ which is the generalized differential operator studied by Swamy [14].

It can be easily verified from (1.7) that

$$(1.9)\qquad z\left(\mathcal{K}_{c,p}^{\delta,\lambda} f(z)\right)' = (c+p)\,\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z) - c\,\mathcal{K}_{c,p}^{\delta,\lambda} f(z) \quad (\delta \geq 1).$$

The object of the present paper is to derive new subordination results and convolution preserving properties of multivalent function involving $\mathcal{K}_{c,p}^{\delta,\lambda}$.

2. Preliminaries

In order to derive our main results, we have to recall the following lemmas.

Lemma 2.1 ([4], also see ([8], p. 71))**.** *Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose that the function $\phi(z)$ given by*

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic in $\mathbb{U}$. If

$$(2.1)\qquad \phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (z \in \mathbb{U},\ \Re(\gamma) \geq 0,\ \gamma \neq 0),$$

then

$$(2.2)\qquad \phi(z) \prec \psi(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)dt \prec h(z) \quad (z \in \mathbb{U}),$$

where $\psi(z)$ is the best dominant of (2.2).

For real or complex numbers $a,\ b,\ c\ (c \notin \mathbb{Z}_0^-)$, the Gauss hypergeometric function is defined by:

$$_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots.$$

We note that the above series converges absolutely for $z \in \mathbb{U}$ and hence represents an analytic function in the unit disk $\mathbb{U}$ (see, for details, [15, Chapter 14]). Each of the identities (asserted by Lemma 2.2 below) is fairly well known (see [15, Chapter 14]).

Lemma 2.2 (see [15]). *For real or complex parameters $a,\ b,\ c\ (c \notin \mathbb{Z}_0^-)$, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_2F_1(a,b;c;z), \tag{2.3}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{2.4}$$

$${}_2F_1(1,1;2;z) = -z^{-1}\ln(1-z) \tag{2.5}$$

and

(2.6)
$$c(c-1)(z-1){}_2F_1(a,b;c-1;z) + c[c-1-(2c-a-b-1)z]{}_2F_1(a,b;c;z) + (c-a)(c-b)z{}_2F_1(a,b;c+1;z) = 0.$$

From the identities (2.5) and (2.6), we can easily prove the following:

Lemma 2.3. *For any real number $s \neq 0$, we have*

$${}_2F_1\left(1,1;2;\frac{sz}{sz+1}\right) = \frac{(1+sz)\ln(1+sz)}{sz}, \tag{2.7}$$

$${}_2F_1\left(1,1;3;\frac{sz}{sz+1}\right) = \frac{2(1+sz)}{sz}\left[1-\frac{\ln(1+sz)}{sz}\right], \tag{2.8}$$

$${}_2F_1\left(1,1;4;\frac{sz}{sz+1}\right) = \frac{3(1+sz)}{2(sz)^3}\left[2\ln(1+sz)-sz(2-sz)\right], \tag{2.9}$$

$${}_2F_1\left(1,1;5;\frac{sz}{sz+1}\right) = \frac{2(1+sz)}{(sz)^3}\left[\frac{2(sz)^2-3sz+6}{3}-\frac{2\ln(1+sz)}{sz}\right]. \tag{2.10}$$

With a view to stating a well-known result (Lemma 2.4 below), we denote $\mathcal{P}(\gamma)$ $(0 \leq \gamma < 1)$ by the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{2.11}$$

which is analytic in $\mathbb{U}$ and satisfies the condition

$$\Re(p(z)) > \gamma\ (0 \leq \gamma < 1; z \in \mathbb{U}). \tag{2.12}$$

The relation

$$p \in \mathcal{P}(\gamma) \Longleftrightarrow p(z) \prec \frac{1+(1-2\gamma)z}{1-z}$$

together with Lindelöf's principle of subordination gives the following well known result.

Lemma 2.4 (see [12]). *Let the function $p(z)$ given by* (2.11) *be in the class* $\mathcal{P}(\gamma)$. *Then*

$$\Re(p(z)) \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leq \gamma < 1; z \in \mathbb{U}). \tag{2.13}$$

Lemma 2.5 (see [13]). *If*

$$\psi_j(z) \in \mathcal{P}(\gamma_j) \quad (0 \leq \gamma_j < 1; j = 1, 2), \tag{2.14}$$

then

$$(\psi_1 * \psi_2)(z) \in \mathcal{P}(\gamma_3) \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{2.15}$$

The bound γ_3 is the best possible.

3. Main results

We state and prove our main results.

Theorem 3.1. *Let $\alpha > 0$, $\delta > 2$ and $c > -p$. Suppose that*

$$\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} \left[1 + \alpha \left(\frac{\mathcal{K}_{c,p}^{\delta-2,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)} - \frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} \right) \right] \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{3.1}$$

Then

$$\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{3.2}$$

where

$$q(z) = (1+Bz)^{-1} \left[{}_2F_1 \left(1, 1; \tfrac{c+p}{\alpha} + 1; \tfrac{Bz}{Bz+1}\right) + \tfrac{A(c+p)z}{c+p+\alpha} \; {}_2F_1 \left(1, 1; \tfrac{c+p}{\alpha} + 2; \tfrac{Bz}{Bz+1}\right) \right]$$

and $q(z)$ is the best dominant of (3.2). *Furthermore,*

$$\Re \left\{ \frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} \right\} > \rho, \tag{3.3}$$

where

$$\rho = (1-B)^{-1} \left[{}_2F_1 \left(1, 1; \tfrac{c+p}{\alpha} + 1; \tfrac{B}{B-1}\right) - \tfrac{A(c+p)}{c+p+\alpha} \; {}_2F_1 \left(1, 1; \tfrac{c+p}{\alpha} + 2; \tfrac{B}{B-1}\right) \right].$$

Proof. Suppose that

$$p(z) = \frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)}. \tag{3.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Taking logarithmic differentiation in (3.4) yields

$$\frac{zp'(z)}{p(z)} = \frac{z \left(\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)\right)'}{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)} - \frac{z \left(\mathcal{K}_{c,p}^{\delta,\lambda} f(z)\right)'}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)}. \tag{3.5}$$

By making use of the identity (1.9) in (3.5), we get

$$\frac{\mathcal{K}_{c,p}^{\delta-2,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)} - \frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} = \frac{zp'(z)}{(c+p)\ p(z)}. \tag{3.6}$$

From (3.4) and (3.6), we obtain

$$\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)} \left[1+\alpha\left(\frac{\mathcal{K}_{c,p}^{\delta-2,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)} - \frac{\mathcal{K}_{c,p}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{c,p}^{\delta,\lambda} f(z)}\right)\right] = p(z)+\frac{\alpha}{c+p} zp'(z). \tag{3.7}$$

Thus from (3.1) and (3.7), we have

$$p(z)+\frac{\alpha}{c+p} zp'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{3.8}$$

Applying Lemma 2.1 gives

$$p(z) \prec q(z) = \frac{c+p}{\alpha} z^{-(\frac{c+p}{\alpha})} \int_0^z t^{\frac{c+p}{\alpha}-1}\left(\frac{1+At}{1+Bt}\right) dt \prec \frac{1+Az}{1+Bz}. \tag{3.9}$$

Now using the identities (2.3) and (2.4) of Lemma 2.2, we can rewrite the function $q(z)$ as

(3.10)

$$\begin{aligned} &q(z) \\ &= \frac{c+p}{\alpha}\int_0^1 s^{\frac{c+p}{\alpha}-1}\left(\frac{1+Asz}{1+Bsz}\right) ds \\ &= \frac{c+p}{\alpha}\int_0^1 s^{\frac{c+p}{\alpha}-1}(1+Bsz)^{-1} ds + \frac{c+p}{\alpha} Az \int_0^1 s^{\frac{c+p}{\alpha}}(1+Bsz)^{-1} ds \\ &= (1+Bz)^{-1}\left[{}_2F_1\left(1,1;\tfrac{c+p}{\alpha}+1,\tfrac{Bz}{Bz+1}\right) + \tfrac{(c+p)Az}{c+p+\alpha}\, {}_2F_1\left(1,1;\tfrac{c+p}{\alpha}+2;\tfrac{Bz}{Bz+1}\right)\right]. \end{aligned}$$

This completes the proof of the assertion (3.2) of Theorem 3.1.

To prove (3.3), it is sufficient to show

$$\inf_{|z|<1} q(z) = q(-1). \tag{3.11}$$

Since for $-1 \le B < A \le 1$, $\frac{1+Az}{1+Bz}$ is convex (univalent) in $\mathbb{U}$, we have for $|z| \le r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}. \tag{3.12}$$

Upon setting

$$g(s,z) = \frac{1+Asz}{1+Bsz} \quad (0 \le s \le 1; z \in \mathbb{U}),$$

and

$$dv(s) = s^{\frac{c+p}{\alpha}-1}\left(\frac{c+p}{\alpha}\right) ds,$$

which is a positive measure on $[0,1]$, we get

$$q(z) = \int_0^1 g(s,z)dv(s)$$

so that

$$\Re\{q(z)\} \geq \int_0^1 \left(\frac{1-Asr}{1-Bsr}\right) dv(s) = q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.11). The proof of Theorem 3.1 is thus completed. $\square$

Taking $\alpha = p = 1$ and $c = 0$ in Theorem 3.1 and using the identities (2.7) and (2.8) of Lemma 2.3, we obtain the following results.

Corollary 3.2. *Let $\delta > 2$ and suppose that*

$$\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda} f(z)} \left[1 + \frac{\mathcal{K}_{0,1}^{\delta-2,\lambda} f(z)}{\mathcal{K}_{0,1}^{\delta-1,\lambda} f(z)} - \frac{\mathcal{K}_{0,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda} f(z)}\right] \prec \frac{1+Az}{1+Bz}.$$

Then

$$\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda} f(z)} \prec q_1(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where

$$q_1(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right) \frac{ln(1+Bz)}{Bz} & (B \neq 0) \\ 1 + \frac{A}{2} z & (B = 0) \end{cases}$$

and $q_1(z)$ is the best dominant.

Furthermore,

$$\Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda} f(z)}\right\} > \rho_1$$

where

$$\rho_1 = \begin{cases} \frac{A}{B} - \left(1 - \frac{A}{B}\right) \frac{ln(1-B)}{B} & (B \neq 0) \\ 1 - \frac{A}{2} & (B = 0). \end{cases}$$

Letting $p = c = \alpha = 1$ in Theorem 3.1 and using the identities (2.8) and (2.9) of Lemma 2.3, we get the following corollary.

Corollary 3.3. *Let $\delta > 2$ and suppose that*

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda} f(z)} \left[1 + \frac{\mathcal{K}_{1,1}^{\delta-2,\lambda} f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda} f(z)} - \frac{\mathcal{K}_{1,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda} f(z)}\right] \prec \frac{1+Az}{1+Bz}.$$

Then

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda} f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda} f(z)} \prec q_2(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where

$$q_2(z) = \begin{cases} \frac{A}{B} - \frac{2}{B^2}\left(1 - \frac{A}{B}\right)\left[\frac{\ln(1+Bz)-Bz}{z^2}\right] & (B \neq 0) \\ 1 + \frac{2A}{3}z & (B = 0) \end{cases}$$

and $q_2(z)$ is the best dominant.

Furthermore,

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\} > \rho_2,$$

where

$$\rho_2 = \begin{cases} \frac{A}{B} - \frac{2}{B^2}\left(1 - \frac{A}{B}\right)[\ln(1-B) + B] & (B \neq 0) \\ 1 - \frac{2A}{3} & (B = 0). \end{cases}$$

Putting $\alpha = \frac{2}{3}$ and $c = p = 1$ in Theorem 3.1 and using the identities (2.9) and (2.10) of Lemma 2.3, we have:

Corollary 3.4. *Let $\delta > 2$ and suppose that*

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1 + \frac{2}{3}\left(\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)} - \frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right)\right] \prec \frac{1+Az}{1+Bz}.$$

Then

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)} \prec q_3(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where

$$q_3(z) = \begin{cases} \frac{A}{B} + \frac{3}{(Bz)^3}\left(1 - \frac{A}{B}\right)\left[\ln(1+Bz) - Bz + \frac{(Bz)^2}{2}\right] & (B \neq 0) \\ 1 + \frac{3A}{4}z & (B = 0) \end{cases}$$

and $q_3(z)$ is the best dominant.

Furthermore,

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\} > \rho_3,$$

where

$$\rho_3 = \begin{cases} \frac{A}{B} - \frac{3}{B^3}\left(1 - \frac{A}{B}\right)\left[\ln(1-B) + B + \frac{B^2}{2}\right] & (B \neq 0) \\ 1 - \frac{3A}{4} & (B = 0). \end{cases}$$

Taking $B \neq 0$ in Corollaries 3.2, 3.3 and 3.4 respectively, we obtain the following results.

Corollary 3.5. *Let $\delta > 2$. Then we have the following*:

- *If*

$$\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\left[1 + \frac{\mathcal{K}_{0,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)} - \frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right] \prec \frac{1 + \frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz}$$

which implies

$$\Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

- *If*

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right]\prec\frac{1+\frac{2B[B+\ln(1-B)]}{2[B+\ln(1-B)]+B^2}z}{1+Bz}$$

which implies

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

- *If*

$$\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{2}{3}\left(\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right)\right]$$
$$\prec\frac{1+\frac{3B[\ln(1-B)+B+\frac{B^2}{2}]}{B^3+3[\ln(1-B)+B+\frac{B^2}{2}]}z}{1+Bz}$$

which implies

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

Letting $B=-1$ in Corollary 3.5, we have:

Corollary 3.6. *Let $\delta>2$, then we have the following*:

(i) *If*

$$\Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\left[1+\frac{\mathcal{K}_{0,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right]\right\}>\frac{2\ln 2-1}{2\ln 2-2}\approx-0.61$$
$$\Longrightarrow\Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

(ii) *If*

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right]\right\}>\frac{4\ln 2-3}{4\ln 2-2}\approx-0.29$$
$$\Longrightarrow\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

(iii) *If*

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{2}{3}\left(\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right)\right]\right\}>\frac{6\ln 2-4}{6\ln 2-5}\approx -0.19$$

$$\Longrightarrow \Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>0 \quad (z\in\mathbb{U}).$$

Letting $A = 1-2\eta$ $(0\le\eta<1)$ and $B=-1$ in Corollaries 3.2, 3.3 and 3.4 respectively, we have

Corollary 3.7. *Let* $\delta > 2$. *Then we have the following*:

(i) *If*

$$\Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\left[1+\frac{\mathcal{K}_{0,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right]\right\}>\eta$$

$$\Longrightarrow \Re\left\{\frac{\mathcal{K}_{0,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{0,1}^{\delta,\lambda}f(z)}\right\}>(2\eta-1)+2(1-\eta)\ln 2.$$

(ii) *If*

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right]\right\}>\eta$$

$$\Longrightarrow \Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>(2\eta-1)-4(1-\eta)(\ln 2-1).$$

(iii) *If*

$$\Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\left[1+\frac{2}{3}\left(\frac{\mathcal{K}_{1,1}^{\delta-2,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}-\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right)\right]\right\}>\eta$$

$$\Longrightarrow \Re\left\{\frac{\mathcal{K}_{1,1}^{\delta-1,\lambda}f(z)}{\mathcal{K}_{1,1}^{\delta,\lambda}f(z)}\right\}>(2\eta-1)+3(1-\eta)(2\ln 2-1).$$

4. Convolution properties of $\mathcal{K}_{c,p}^{\delta,\lambda}$

In this section we investigate some new basic properties of the operator $\mathcal{K}_{c,p}^{\delta,\lambda}$ using the principle of differential subordination.

Theorem 4.1. *Let* $\mu > 0$ *and* $-1\le B_j < A_j \le 1$ $(j=1,2)$. *If each of the functions* $f_j\in\mathcal{A}_p$ $(j=1,2)$ *satisfies*

$$(1-\mu)\frac{\mathcal{K}_{c,p}^{\delta,\lambda}f_j(z)}{z^p}+\mu\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda}f_j(z)}{z^p}\prec\frac{1+A_jz}{1+B_jz}\quad (j=1,2;z\in\mathbb{U}), \tag{4.1}$$

then

$$(1-\mu)\frac{\mathcal{K}_{c,p}^{\delta,\lambda}F(z)}{z^p}+\mu\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda}F(z)}{z^p}\prec\frac{1+(1-2\eta)z}{1-z}\quad (z\in\mathbb{U}), \tag{4.2}$$

where

$$F(z)=\mathcal{K}_{c,p}^{\delta,\lambda}(f_1*f_2)(z) \quad (z\in\mathbb{U}), \tag{4.3}$$

and

$$\eta=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\ {}_2F_1\left(1,1,;\frac{c+p}{\mu}+1;\frac{1}{2}\right)\right]. \tag{4.4}$$

The result is sharp when $B_1=B_2=-1$.

Proof. Suppose that each of the functions $f_j\in\mathcal{A}_p$ $(j=1,2)$ satisfies the condition (4.1). Set

$$\phi_j(z)=(1-\mu)\frac{\mathcal{K}_{c,p}^{\delta,\lambda}f_j(z)}{z^p}+\mu\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda}f_j(z)}{z^p} \quad (j=1,2;z\in\mathbb{U}), \tag{4.5}$$

we observe that $\phi_j\in\mathcal{P}(\gamma_j)$ where $\gamma_j=\frac{1-A_j}{1-B_j}$ $(j=1,2)$. Thus, by making use of identity (1.9) in (4.5), we obtain

$$\mathcal{K}_{c,p}^{\delta,\lambda}f_j(z)=\frac{c+p}{\mu}z^{p-\frac{c+p}{\mu}}\int_0^z t^{\frac{c+p}{\mu}-1}\phi_j(t)dt \quad (j=1,2). \tag{4.6}$$

Using (4.3) and (4.6), a simple calculation shows that

$$\mathcal{K}_{c,p}^{\delta,\lambda}F(z)=\left(\frac{c+p}{\mu}z^{p-\frac{c+p}{\mu}}\int_0^z t^{\frac{c+p}{\mu}-1}\phi(t)dt\right) \tag{4.7}$$

where, for convenience,

$$\begin{aligned}\phi(z)&=(1-\mu)\frac{\mathcal{K}_{c,p}^{\delta,\lambda}F(z)}{z^p}+\mu\frac{\mathcal{K}_{c,p}^{\delta-1,\lambda}F(z)}{z^p}\\ &=\frac{c+p}{\mu}z^{-\frac{c+p}{\mu}}\int_0^z t^{\frac{c+p}{\mu}-1}(\phi_1*\phi_2)(t)dt.\end{aligned} \tag{4.8}$$

Since

$$\phi_j\in\mathcal{P}(\gamma_j) \qquad (j=1,2),$$

it follows from Lemma 2.5 that

$$(\phi_1*\phi_2)(z)\in\mathcal{P}(\gamma_3) \quad (\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)) \tag{4.9}$$

and the bound γ_3 is the best possible. Hence applying Lemma 2.4 to (4.8) gives

$$\begin{aligned}\Re(\phi(z))&=\frac{c+p}{\mu}\int_0^1 s^{\frac{c+p}{\mu}-1}\Re(\phi_1*\phi_2(sz))ds\\ &\geq\frac{c+p}{\mu}\int_0^1 s^{\frac{c+p}{\mu}-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+s|z|}\right)ds\\ &>1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\frac{c+p}{\mu}\int_0^1\frac{s^{\frac{c+p}{\mu}-1}}{1+s}ds\right)\\ &=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\ {}_2F_1\left(1,1;\frac{c+p}{\mu}+1;\frac{1}{2}\right)\right]\\ &=\eta \quad (z\in\mathbb{U}).\end{aligned} \tag{4.10}$$

When $B_1 = B_2 = -1$, we consider the functions $f_j(z) \in \mathcal{A}_p$ which satisfy the hypothesis (4.5) and are given by

$$\mathcal{K}_{c,p}^{\delta,\lambda} f_j(z) = \frac{c+p}{\mu} z^{p-\frac{c+p}{\mu}} \int_0^z t^{\frac{c+p}{\mu}-1} \left(\frac{1+A_j t}{1-t} \right) dt \quad (j = 1, 2; z \in \mathbb{U}).$$

Then, it follows from (4.8) and Lemma 2.4 that

$$\begin{aligned}
\phi(z) &= \frac{c+p}{\mu} \int_0^1 s^{\frac{c+p}{\mu}-1} \left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-sz} \right) ds \\
&= 1 - (1+A_1)(1+A_2) \\
&\quad + (1+A_1)(1+A_2)(1-z)^{-1} \ {}_2F_1 \left(1, 1; \frac{c+p}{\mu} + 1; \frac{z}{z-1} \right) \\
&\to 1 - (1+A_1)(1+A_2) \left[1 - \frac{1}{2} \ {}_2F_1 \left(1, 1; \frac{c+p}{\mu} + 1; \frac{1}{2} \right) \right],
\end{aligned}$$

as $z \to -1$.

This completes the proof of Theorem 4.1. □

Upon setting $A_j = 1 - 2\eta_j \ (0 \le \eta_j < 1), \ B_j = -1 \ (j = 1, 2), \delta = c = 0$ and $\lambda = 0$ in Theorem 4.1, we obtain the following results.

Corollary 4.2. *If the functions* $f_j(z) \in \mathcal{A}_p \ (j = 1, 2)$ *satisfy the following inequality*:

$$\Re \left[(1-\mu) \frac{f(z)}{z^p} + \mu \frac{f'(z)}{z^{p-1}} \right] > \eta_j \quad (0 \le \eta_j < 1; \ j = 1, 2),$$

then

$$\Re \left[(1-\mu) \frac{(f_1 * f_2)(z)}{z^p} + \mu \frac{(f_1 * f_2)'(z)}{z^{p-1}} \right] > \rho_1,$$

where

$$\rho_1 = 1 - 4(1-\eta_1)(1-\eta_2) \left[1 - \frac{1}{2} \ {}_2F_1 \left(1, 1; \frac{1}{\mu} + 1; \frac{1}{2} \right) \right].$$

The result is the best possible.

Acknowledgement. The author thanks the reviewer for many useful suggestions for revision which improved the content of the manuscript.

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