

ASYMPTOTIC BEHAVIORS OF SOLUTIONS FOR AN AEROTAXIS MODEL COUPLED TO FLUID EQUATIONS

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ABSTRACT. We consider a coupled system of Keller-Segel type equations and the incompressible Navier-Stokes equations in spatial dimension two. We show temporal decay estimates of solutions with small initial data and obtain their asymptotic profiles as time tends to infinity.

1. Introduction

In this paper, we consider a mathematical model describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in \mathbb{R}^2 .

$$(1.1) \quad \begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (\chi(c)n\nabla c), \\ \partial_t c + u \cdot \nabla c - \Delta c = -k(c)n, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -n\nabla\phi, \quad \nabla \cdot u = 0 \end{cases} \quad \text{in } Q_T := (0, T) \times \mathbb{R}^2,$$

where $c(t, x) : Q_T \rightarrow \mathbb{R}^+$, $n(t, x) : Q_T \rightarrow \mathbb{R}^+$, $u(t, x) : Q_T \rightarrow \mathbb{R}^d$ and $p(t, x) : Q_T \rightarrow \mathbb{R}$ denote the oxygen concentration, cell concentration, fluid velocity, and scalar pressure, respectively. Here \mathbb{R}^+ indicates the set of non-negative real numbers. Such a model was proposed by Tuval et al. [25], formulating the dynamics of swimming bacteria, *Bacillus subtilis* (see [25] for more details on biological phenomena).

The nonnegative functions $k(c)$ and $\chi(c)$ denote the oxygen consumption rate and the aerobic sensitivity, respectively, i.e., $k, \chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $k(c) = k(c(x, t))$ and $\chi(c) = \chi(c(x, t))$. Initial data are given by $(n_0(x), c_0(x), u_0(x))$ with $n_0(x), c_0(x) \geq 0$ and $\nabla \cdot u_0 = 0$. To describe the fluid motions, Boussinesq approximation is used to denote the effect due to heavy bacteria. The time-independent function $\phi = \phi(x)$ denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force.

We can compare the above system (1.1) to the classical Keller-Segel model,

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suggested by Patlak [22] and Keller-Segel [16, 17], which is given as

$$(1.2) \quad \begin{cases} n_t = \Delta n - \nabla \cdot (n\chi \nabla c), \\ c_t = \Delta c - \alpha c + \beta n, \end{cases}$$

where $n = n(t, x)$ is the cell density and $c = c(t, x)$ is the concentration of chemical attractant substance. Here, χ is the chemotactic sensitivity, and $\alpha \geq 0$ and $\beta \geq 0$ are the decay and production rate of the chemical, respectively. The system (1.2) has been comprehensively studied and we will not try to give list of results here (see e.g. [12, 15, 19, 21, 26] and the survey papers [13, 14]).

In the absence of effect of fluids, i.e., $u = 0$ and $\phi = 0$, the system (1.1) is reduced to a type of Keller-Segel model with the negative term $-k(c)n$ (compare to the classical Keller-Segel equations (1.2) with the positive term βn). It is due to different biological contexts; the oxygen concentration in (1.1) is consumed and the chemical substance, meanwhile, is produced by n in (1.2). It is known for the Keller-Segel equations (1.2) that there exists an initial datum of n such that blow-up occurs in a finite time if its initial mass is larger than a critical value, 8π (see e.g. [12] and [23], and also compare to [15] for the parabolic-elliptic case). It is not, however, clear due to the opposite effect of $-k(c)n$ whether or not solutions of the system (1.1) exist globally in general, or some initial data would develop a blow up, even in the case of the absence of u .

We review some known results regarding the system (1.1). In [18] existence of solutions was shown locally in time for bounded domains in \mathbb{R}^3 and [7] proved that smooth solutions are globally extended in time if initial data are sufficiently close to constant steady states and if $\chi(\cdot)$ and $k(\cdot)$ satisfy the following conditions on $[0, \infty)$:

$$(1.3) \quad \chi'(\cdot) \geq 0, \quad k'(\cdot) > 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)} \right)'' < 0.$$

It was also shown in [7] that weak solutions exist globally in time in \mathbb{R}^2 , provided that the initial chemical concentration is small. [27] proved the global existence of regular solutions without smallness assumptions on initial data for bounded domains in \mathbb{R}^2 with boundary conditions $\partial_\nu n = \partial_\nu c = u = 0$ under the following sign conditions on $\chi(\cdot)$ and $k(\cdot)$ on $[0, \infty)$:

$$(1.4) \quad \left(\frac{k(\cdot)}{\chi(\cdot)} \right)' > 0, \quad (\chi(\cdot)k(\cdot))' \geq 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)} \right)'' \leq 0.$$

In [3] the authors established global existence of smooth solutions in \mathbb{R}^2 with no smallness of the initial data and certain conditions, motivated by experimental results in [5] and [25], on $\chi(\cdot)$ and $k(\cdot)$ (compare to (1.4)), that is,

$$(1.5) \quad \chi(c), k(c), \chi'(c), k'(c) \geq 0, \text{ and } \sup |\chi(c) - \mu k(c)| < \epsilon \text{ for some } \mu > 0.$$

Construction of weak solutions in \mathbb{R}^3 was also established in [3] in case that $|\chi(c) - \mu k(c)| = 0$ in (1.5). The authors also studied the time decay of regular

solution in [4]. More precisely, it was shown that if L^∞ -norm of c_0 is sufficiently small, then regular solution exists globally and, furthermore, n and c satisfy the following time decay:

$$(1.6) \quad \|n(t)\|_{L^\infty(\mathbb{R}^d)} + \|c(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(1+t)^{-\frac{d}{4}}, \quad d = 2, 3.$$

For bounded convex domains with smooth boundary, [28] showed that (n, c, u) converges to $((n)_a, 0, 0)$ in L^∞ -norm under the assumption (1.4), where $(n)_a$ indicates the mean value of n_0 . We consult [6], [8] and [24] with reference therein for the nonlinear diffusion models of a porous medium type.

Our main objective of this paper is to obtain asymptotic profiles of temporal decaying solutions of (1.1). To be more precise, if certain norms of initial data are sufficiently small, we prove existence of global regular solutions, which show certain degree of temporal decay, and in addition, asymptotic profiles of n and u can be obtained.

Before we state our main result, since the vorticity equation is rather convenient than the equation of velocity, we consider from now on

$$(1.7) \quad \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (\chi(c)n\nabla c),$$

$$(1.8) \quad \partial_t c + u \cdot \nabla c - \Delta c = -k(c)n,$$

$$(1.9) \quad \partial_t \omega + u \cdot \nabla \omega - \Delta \omega = -\nabla^\perp(n\nabla \phi),$$

where u is given as a Biot-Savart law, namely

$$(1.10) \quad u = K * \omega, \quad K(x) = \nabla^\perp \log |x| = \left\langle -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right\rangle.$$

We denote by m and γ the total mass of n and total circulation of ω , respectively, i.e.,

$$(1.11) \quad \int_{\mathbb{R}^2} n_0(x) dx = m, \quad \int_{\mathbb{R}^2} \omega_0(x) dx = \gamma.$$

If we consider only the Navier-Stokes equations, i.e., $n = 0$ and $c = 0$, in (1.7)-(1.10), it is known that vorticity field converges asymptotically to the radial self-similar solution at time infinity. To be more precise, in such case the following asymptotic formula holds:

$$(1.12) \quad \lim_{t \rightarrow \infty} t \|\omega(\cdot, t) - \gamma \Gamma(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} = 0,$$

where $\Gamma(x, t)$ is the two dimensional heat kernel, i.e.,

$$\Gamma(x, t) = (4\pi t)^{-1} \exp(-|x|^2/4t).$$

The formula (1.12) was first proved in [11] when $\|\omega_0\|_{L^1}$ is small. Such smallness assumption of $\|\omega_0\|_{L^1}$ was replaced by the smallness of γ in [2] and [9] removed the smallness assumption of γ (see [10] for further references related to asymptotics of the vorticity equations). Our approach is similar to that shown in [10] and it is, however, quite difficult to analyze the asymptotics of solutions for the system (1.7)-(1.10), because of the coupled structure of the

system. Nevertheless, we obtain asymptotic behaviors of solutions under the smallness assumptions of initial data. To be more precise, our main result reads as follows:

Theorem 1. *Let the initial data (n_0, c_0, u_0) be given in $H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ for $m \geq 3$ with $n_0 \geq 0$, $c_0 \geq 0$ and $\operatorname{div} u_0 = 0$. Assume that χ, k, χ', k' are all non-negative and $\chi, k \in C^m(\mathbb{R}^+)$ and $k(0) = 0$, $\|\nabla^l \phi\|_{L^1 \cap L^\infty} < \infty$ for $1 \leq |l| \leq m$. There exists a constant $\epsilon_1 > 0$ such that if*

$$(1.13) \quad \|n_0\|_{L^1(\mathbb{R}^2)} + \|c_0\|_{(L^\infty \cap \dot{H}^1)(\mathbb{R}^2)} + \|\omega_0\|_{L^1(\mathbb{R}^2)} < \epsilon_1,$$

then unique classical solutions (n, c, ω) of (1.7)-(1.10) exist globally and (n, c, ω) satisfy the following asymptotics: for any $R < \infty$ and for all $1 < r < \infty$

$$\lim_{t \rightarrow \infty} t \|n(\cdot, t) - m\Gamma(\cdot, t)\|_{L^\infty(B_{t,R})} = 0,$$

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}} \|\nabla c(\cdot, t)\|_{L^\infty(B_{t,R})} = 0,$$

$$\lim_{t \rightarrow \infty} t^{1-\frac{1}{r}} \|\omega(\cdot, t) - \gamma\Gamma(\cdot, t)\|_{L^r(B_{t,R})} = 0,$$

where $B_{t,R} := \{x \in \mathbb{R}^2 : |x| < Rt^{\frac{1}{2}}\}$ and $\Gamma(x, t)$ is the two dimensional heat kernel.

Remark 1. We note that Theorem 1 implies the following temporal decay of (n, c, ω) for large t :

$$\|n(t)\|_{L^\infty(\mathbb{R}^2)} \sim \frac{m}{t} + \frac{o(1)}{t}, \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \sim \frac{o(1)}{t^{\frac{1}{2}}},$$

$$\|\omega(t)\|_{L^r(\mathbb{R}^2)} \sim \frac{\gamma}{t^{1-\frac{1}{r}}} + \frac{o(1)}{t^{1-\frac{1}{r}}}, \quad 1 < r < \infty.$$

Remark 2. The existence of unique classical solution was proved previously in [4] assuming either $\|n_0\|_{L^1(\mathbb{R}^2)} < \epsilon_1$ or $\|c_0\|_{L^\infty} < \epsilon_1$. Further smallness assumption of ∇c_0 and ω_0 , i.e., $\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + \|\omega_0\|_{L^1(\mathbb{R}^2)} < \epsilon$ enables us to obtain the time decay and asymptotic behaviors. We do not know whether or not the assumption (1.13) can be removed, and thus we leave it as an open question.

This paper is organized as follows. Section 2 is devoted to obtaining decay rate of solutions in case that certain norm of initial data are sufficiently small. In Section 3, we present the proof of Theorem 1.

2. Estimates of temporal decay

We first introduce the notation and present preparatory results that are useful to our analysis. We start with the notation. For $1 \leq q \leq \infty$, we mean by $W^{k,q}(\Omega)$ the usual Sobolev spaces, namely $W^{k,q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$. We write the homogeneous Sobolev spaces as $\dot{W}^{k,q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), |\alpha| = k\}$. In case $q = 2$, we denote

$W^{k,2}(\Omega)$ and $\dot{W}^{k,2}(\Omega)$ by $H^k(\Omega)$ and $\dot{H}^k(\Omega)$, respectively. The letter C is used to represent a generic constant, which may change from line to line, and $C(*, \dots, *)$ is considered a positive constant depending on $*, \dots, *$. Sometimes, we use $A \lesssim B$, which means the inequality $A \leq CB$, where C is a generic constant. For convenience we mention the elementary inequalities which are repeatedly used;

$$(2.1) \quad \int_0^t \frac{1}{(t-s)^{1-a}} \frac{1}{s^{1-b}} ds \leq \frac{C}{t^{1-(a+b)}} \quad (a > 0, b > 0),$$

$$(2.2) \quad \int_0^{\frac{t}{2}} \frac{1}{(t-s)^b} \frac{1}{s^{1-a}} ds \leq \frac{C}{t^{b-a}}, \quad \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{1-a}} \frac{1}{s^b} ds \leq \frac{C}{t^{b-a}} \quad (a > 0, b \geq 0).$$

We remind a lemma in [10, Section 2.2.5] and the following is its slight modified version.

Lemma 2. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be C^1 and radial in \mathbb{R}^2 . Then,*

$$((K * g)\nabla) f = 0 \quad \text{in } \mathbb{R}^2,$$

where $K(x) = \langle -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \rangle$.

Proof. The proof can be similarly proved by the same arguments as the lemma in [10, Section 2.2.5], and therefore, we skip its details. \square

In this section, we are concerned with optimal temporal decays of solutions (n, c, ω) of (1.7)-(1.10), and our main goal is to prove the next proposition. Let us recall the smallness assumption in Theorem 1:

$$(2.3) \quad \|n_0\|_{L^1(\mathbb{R}^2)} + \|c_0\|_{(L^\infty \cap \dot{H}^1)(\mathbb{R}^2)} + \|\omega_0\|_{L^1(\mathbb{R}^2)} < \epsilon_1,$$

where $\omega_0 = \nabla \times u_0$.

Proposition 1. *Assume the condition of Theorem 1 holds. The classical solutions (n, c, ω) of (1.7)-(1.10) exist globally and (n, c, ω) satisfy the following time decay:*

$$(2.4) \quad \|n(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t}, \quad \|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{3}{2}}},$$

$$(2.5) \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{1}{2}}}, \quad \|\nabla^2 c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t},$$

$$(2.6) \quad \|\omega(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{1-\frac{1}{r}}}, \quad 1 < r < \infty, \quad \|\nabla \omega(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}, \quad 1 \leq r < 2.$$

The proof of Proposition 1 will be presented in the series of lemmas. Lemma 4 considers the decays of $\|n\|_{L^\infty(t)}$, $\|\nabla c\|_{L^\infty(t)}$, $\|\omega\|_{L^r(t)}$, and Lemma 5 shows the decays of quantities with derivatives. Notice that the decay rates in (2.4) and (2.6) are the same as in the $L^q - L^1$ estimate for the two dimensional heat equation. In this regard our approach is to see the system (1.1) as the

perturbed heat equations with the smallness assumption (1.13), and to apply the linear heat kernel estimates

$$(2.7) \quad \|\nabla^\alpha e^{-\Delta t} u\|_{L^q(\mathbb{R}^2)} \leq C t^{-(1/r-1/q)-|\alpha|/2} \|u\|_{L^r(\mathbb{R}^2)}, \quad 1 \leq r \leq q \leq \infty.$$

In doing so, we need an intermediate step (Lemma 3 shown below), which establishes $(n, \nabla c, \omega)$ to be small in a weighted norms in time variable (Lemma 4 and Lemma 5 shown below). This types of estimates for weighted norms can be found in [20]. Due to Lemma 3 we work out Lemma 4 and Lemma 5 so that the nonlinear terms in the Duhamel's formula are estimated by either quadratic terms or terms multiplied with small parameter ϵ_1 (see e.g. (2.35)). Let us introduce some spaces of functions defined as follows:

$$(2.8) \quad \|n\|_{\mathcal{K}_p(\mathbb{R}^2)} := \sup_{t \geq 0} t^{1-\frac{1}{p}} \|n(t)\|_{L^p(\mathbb{R}^2)},$$

$$(2.9) \quad \|c\|_{\mathcal{N}_q(\mathbb{R}^2)} := \sup_{t \geq 0} t^{\frac{1}{2}-\frac{1}{q}} \|\nabla c(t)\|_{L^q(\mathbb{R}^2)},$$

$$(2.10) \quad \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)} := \sup_{t \geq 0} t^{1-\frac{1}{r}} \|\omega(t)\|_{L^r(\mathbb{R}^2)}.$$

For convenience, we denote

$$\|(n, c, u)\|_{\mathcal{K}_{p,q,r}} := \|n\|_{\mathcal{K}_p} + \|c\|_{\mathcal{N}_q} + \|\omega\|_{\mathcal{K}_r}.$$

Lemma 3. *Let n, c and ω be solutions of (1.7)-(1.10). Suppose that the assumptions in Theorem 1 are satisfied, and p, q, r are in the range of*

$$(2.11) \quad \frac{4}{3} < p < 2, \quad 2 < q < 4, \quad 1 < r < 2.$$

Then, we have

$$(2.12) \quad \|(n, c, \omega)\|_{\mathcal{K}_{p,q,r}} \leq C(\|n_0\|_{L^1} + \|c_0\|_{L^\infty \cap \dot{H}^1} + \|\omega_0\|_{L^1}) \leq C\epsilon_1.$$

Proof. First, we write the equations as integral representation.

$$(2.13) \quad n(t) = e^{t\Delta} n_0 - \int_0^t \nabla e^{(t-s)\Delta} (\chi(c)n(s)\nabla c(s)) ds \\ - \int_0^t \nabla e^{(t-s)\Delta} (u(s)n(s)) ds,$$

$$(2.14) \quad c(t) = e^{t\Delta} c_0 - \int_0^t e^{(t-s)\Delta} (k(c)n(s)) ds - \int_0^t e^{(t-s)\Delta} (u(s)\nabla c(s)) ds,$$

$$(2.15) \quad \omega(t) = e^{t\Delta} \omega_0 - \int_0^t \nabla^\perp e^{(t-s)\Delta} (n(s)\nabla \phi) ds - \int_0^t \nabla e^{(t-s)\Delta} (u(s)\omega(s)) ds,$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Using the estimate of the heat kernel, we obtain

$$\|n(t)\|_{L^p} \lesssim t^{-1+\frac{1}{p}} \|n_0\|_{L^1} + \int_0^t \left\| \nabla e^{(t-s)\Delta} n(s)\nabla c(s) \right\|_{L^p} ds$$

$$\begin{aligned}
& + \int_0^t \left\| \nabla e^{(t-s)\Delta} u(s) n(s) \right\|_{L^p} ds \\
& \lesssim t^{-1+\frac{1}{p}} \|n_0\|_{L^1} + \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha}}} \|n(s)\|_{L^p} \|\nabla c(s)\|_{L^q} ds \\
& + \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha'}}} \|u(s)\|_{L^{\frac{2r}{2-r}}} \|n(s)\|_{L^p} ds \\
(2.16) \quad & := t^{-1+\frac{1}{p}} \|n_0\|_{L^1} + I_1 + I_2,
\end{aligned}$$

where $1 + \frac{1}{p} = \frac{1}{\alpha} + \frac{1}{p} + \frac{1}{q}$ and $\frac{3}{2} - \frac{1}{r} = \frac{1}{\alpha'}$. We estimate I_1 and I_2 as follows:

$$\begin{aligned}
I_1 & \lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha}}} \cdot \frac{1}{s^{\frac{3}{2}-\frac{1}{p}-\frac{1}{q}}} ds \|n\|_{\mathcal{K}_p} \|c\|_{\mathcal{N}_q} \lesssim \frac{1}{t^{1-\frac{1}{p}}} \|n\|_{\mathcal{K}_p} \|c\|_{\mathcal{N}_q}, \\
I_2 & \lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha'}}} \cdot \frac{1}{s^{2-\frac{1}{r}-\frac{1}{p}}} ds \|\omega\|_{\mathcal{K}_r} \|n\|_{\mathcal{K}_p} \lesssim \frac{1}{t^{1-\frac{1}{p}}} \|\omega\|_{\mathcal{K}_r} \|n\|_{\mathcal{K}_p},
\end{aligned}$$

where we used (2.1). Therefore, we obtain

$$(2.17) \quad \|n\|_{\mathcal{K}_p} \leq C \|n_0\|_{L^1} + C \|n\|_{\mathcal{K}_p} \|c\|_{\mathcal{N}_q} + C \|\omega\|_{\mathcal{K}_r} \|n\|_{\mathcal{K}_p}.$$

We first recall that

$$(2.18) \quad \|\nabla e^{t\Delta} c_0\|_{L^q} \leq C t^{-(\frac{1}{2}-\frac{1}{q})} \|\nabla c_0\|_{L^2}.$$

Using the above estimate (2.18), we obtain

$$\begin{aligned}
& \|c\|_{\mathcal{N}_q} \leq C \|\nabla c_0\|_{L^2} + C \sup |k(c)| \|n\|_{\mathcal{K}_p} + C \|c\|_{\mathcal{N}_q} \|\omega\|_{\mathcal{K}_r} \\
(2.19) \quad & \leq C \|\nabla c_0\|_{L^2} + C \|k(c)\|_{L^\infty} \|n\|_{\mathcal{K}_p} + C \|c\|_{\mathcal{N}_q} \|\omega\|_{\mathcal{K}_r}.
\end{aligned}$$

Next, we estimate the vorticity. Using estimates of the heat kernel as in (2.16), we obtain

$$\begin{aligned}
\|\omega(t)\|_{L^r} & \lesssim t^{-1+\frac{1}{r}} \|\omega_0\|_{L^1} + \int_0^t \left\| \nabla e^{(t-s)\Delta} n(s) \nabla \phi \right\|_{L^r} \\
& + \int_0^t \left\| \nabla e^{(t-s)\Delta} u(s) \omega(s) \right\|_{L^r} \\
& \lesssim t^{-1+\frac{1}{r}} \|\omega_0\|_{L^1} + \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha}}} \|n(s)\|_{L^p} \|\nabla \phi\|_{L^2} \\
& + \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha'}}} \|u\|_{L^{\frac{2r}{2-r}}} \|\omega\|_{L^r} \\
& = t^{-1+\frac{1}{r}} \|\omega_0\|_{L^1} + J_1 + J_2,
\end{aligned}$$

where $\frac{1}{r} = \frac{1}{\alpha} + \frac{1}{p} - \frac{1}{2}$ and $\frac{1}{\alpha'} = \frac{3}{2} - \frac{1}{r}$. Similar estimates as above yield

$$J_1 \lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha}}} \frac{1}{s^{1-\frac{1}{p}}} ds \|\nabla \phi\|_{L^2} \|n\|_{\mathcal{K}_p} \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\nabla \phi\|_{L^2} \|n\|_{\mathcal{K}_p}.$$

On the other hand, via $\|u(t)\|_{L^s} \lesssim \|\omega(t)\|_{L^r}$ with $1/r = 1/s + 1/2$, we obtain

$$J_2 \lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha'}}} \frac{1}{s^{2(1-\frac{1}{r})}} ds \|\omega\|_{\mathcal{K}_r}^2 \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\omega\|_{\mathcal{K}_r}^2.$$

Thus, we have

$$(2.20) \quad \|\omega\|_{\mathcal{K}_r} \leq C \|\omega_0\|_{L^1} + C \|\nabla\phi\|_{L^2} \|n\|_{\mathcal{K}_p} + C \|\omega\|_{\mathcal{K}_r}^2.$$

Here we set $M_1 := C \|k(c)\|_{L^\infty}$ and $M_2 := C \|\nabla\phi\|_{L^2}$, where C are the constants in (2.19) and (2.20). Multiplying (2.17) with $2(M_1 + M_2)$ and summing up the above estimates,

$$(2.21) \quad \begin{aligned} & (M_1 + M_2) \|n\|_{\mathcal{K}_p} + \|c\|_{\mathcal{N}_q} + \|\omega\|_{\mathcal{K}_r} \\ & \leq C(2(M_1 + M_2) \|n_0\|_{L^1} + \|c_0\|_{L^\infty \cap \dot{H}^1} + \|\omega_0\|_{L^1}) \\ & \quad + 2C(M_1 + M_2) \|n\|_{\mathcal{K}_p} \|c\|_{\mathcal{N}_q} + 2C(M_1 + M_2) \|\omega\|_{\mathcal{K}_r} \|n\|_{\mathcal{K}_p} \\ & \quad + C \|c\|_{\mathcal{N}_q} \|\omega\|_{\mathcal{K}_r} + C \|\omega\|_{\mathcal{K}_r}^2. \end{aligned}$$

Therefore, we obtain

$$(2.22) \quad \|(n, c, \omega)\|_{\mathcal{K}_{p,q,r}} \leq C(\|n_0\|_{L^1} + \|c_0\|_{L^\infty \cap \dot{H}^1} + \|\omega_0\|_{L^1}) + C \|(n, c, \omega)\|_{\mathcal{K}_{p,q,r}}^2.$$

We deduce the lemma by the standard theory of local well-posedness argument. \square

Next we show the decay of $(n, \nabla c, \omega)$ in $L^\infty \times L^\infty \times L^r$ for $2 \leq r < \infty$.

Lemma 4. *Let n, c and ω be solutions of (1.7)-(1.10). If the assumptions in Theorem 1 are satisfied, then*

$$(2.23) \quad \|n(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t}, \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{1}{2}}},$$

$$(2.24) \quad \|\omega(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{1-\frac{1}{r}}}, \quad 2 \leq r < \infty.$$

Proof. For convenience, we denote

$$\begin{aligned} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} &:= \sup_{t \geq 0} t \|n(t)\|_{L^\infty(\mathbb{R}^2)}, \quad \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)} := \sup_{t \geq 0} t^{\frac{1}{2}} \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)}, \\ \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)} &:= \sup_{t \geq 0} t^{1-\frac{1}{r}} \|\omega(t)\|_{L^r(\mathbb{R}^2)}, \quad 1 < r < \infty. \end{aligned}$$

Using the estimate of heat kernel, we obtain

$$\begin{aligned} \|n\|_{L^\infty}(t) &\lesssim \frac{1}{t} \|n_0\|_{L^1} + \int_0^t \left\| \nabla e^{(t-s)\Delta} n \nabla c \right\|_{L^\infty}(s) ds \\ &\quad + \int_0^t \left\| \nabla e^{(t-s)\Delta} \omega n \right\|_{L^\infty}(s) ds \\ &= \frac{1}{t} \|n_0\|_{L^1} + I_1 + I_2. \end{aligned}$$

We first estimate I_1 .

$$\begin{aligned}
I_1 &\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|n \nabla c\|_{L^1}(s) ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2}}} \|n \nabla c\|_{L^\infty}(s) ds \\
&\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|n\|_{L^1} \|\nabla c\|_{L^\infty} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2}}} \|n\|_{L^\infty} \|\nabla c\|_{L^\infty} ds \\
(2.25) \quad &\lesssim \frac{\epsilon_1}{t} \|\nabla c\|_{\mathcal{N}_\infty(\mathbb{R}^2)} + \frac{1}{t} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)},
\end{aligned}$$

where we used (2.2). For convenience, we introduce Hölder conjugate numbers 2^+ and 2^- so that

$$1/2^+ = 1/2 - 1/\alpha, \quad 1/2^- = 1/2 + 1/\alpha, \quad 2 < \alpha < \infty.$$

We then estimate I_2 as follows:

$$\begin{aligned}
(2.26) \quad I_2 &\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|un\|_{L^1}(s) ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|un\|_{L^{2^+}}(s) ds \\
&\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|u\|_{L^{2^+}} \|n\|_{L^{2^-}} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|u\|_{L^{2^+}} \|n\|_{L^\infty} ds \\
&\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|u\|_{L^{2^+}} \|n\|_{L^{2^-}} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|u\|_{L^{2^+}} \|n\|_{L^\infty} ds \\
&\lesssim \frac{1}{t^{\frac{3}{2}}} \int_0^{t/2} \|\omega\|_{L^{\frac{\alpha}{\alpha-1}}} \|n\|_{L^{2^-}} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|\omega\|_{L^{\frac{\alpha}{\alpha-1}}} \|n\|_{L^\infty} ds \\
&\lesssim \frac{1}{t} \|n\|_{\mathcal{K}_{2^-}(\mathbb{R}^2)} \|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}(\mathbb{R}^2)} + \frac{1}{t} \|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}(\mathbb{R}^2)} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} \\
&\lesssim \frac{\epsilon_1^2}{t} + \frac{\epsilon_1}{t} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)},
\end{aligned}$$

where we used the result in Lemma 3. Adding the estimates, we obtain

$$(2.27) \quad \|n\|_{L^\infty}(t) \lesssim \frac{\epsilon_1}{t} + \frac{\epsilon_1}{t} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} + \frac{\epsilon_1}{t} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)} + \frac{1}{t} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)}.$$

On the other hand, ∇c is computed as follows:

$$\begin{aligned}
\|\nabla c\|(t) &\lesssim \frac{1}{t^{\frac{1}{2}}} \|c_0\|_{L^\infty} + \int_0^t \left\| \nabla e^{(t-s)\Delta} kn \right\|_{L^\infty}(s) ds \\
&\quad + \int_0^t \left\| \nabla e^{(t-s)\Delta} (u \nabla c) \right\|_{L^\infty}(s) ds \\
&= \frac{1}{t^{\frac{1}{2}}} \|c_0\|_{L^\infty} + J_1 + J_2.
\end{aligned}$$

Firstly, we estimate J_1 .

$$\begin{aligned}
J_1 &\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}}} \|kn(s)\|_{L^1} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2}}} \|kn(s)\|_{L^\infty} ds \\
&\lesssim \frac{1}{t^{\frac{1}{2}}} \|k(c)\|_{L^\infty} \|n\|_{L^1} + \frac{1}{t^{\frac{1}{2}}} \|k(c)\|_{L^\infty} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} \\
(2.28) \quad &\lesssim \frac{\epsilon_1}{t^{\frac{1}{2}}} + \frac{\epsilon_1}{t^{\frac{1}{2}}} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)}.
\end{aligned}$$

Before we estimate J_2 , we set $1/4^+ = 1/4 - 1/\beta$ and $1/4^- = 1/4 + 1/\beta$ with $\beta > 4$. We then estimate J_2 .

$$\begin{aligned}
(2.29) \quad J_2 &\lesssim \int_0^{t/2} \frac{1}{t-s} \|u\nabla c\|_{L^2} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|u\nabla c\|_{L^{2^+}}(s) ds \\
&\lesssim \frac{1}{t} \int_0^{t/2} \|u\|_{L^{4^+}} \|\nabla c\|_{L^{4^-}} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|u\|_{L^{2^+}} \|\nabla c\|_{L^\infty}(s) ds \\
&\lesssim \frac{1}{t} \int_0^{t/2} \|\omega\|_{L^{\frac{4\beta}{3\beta-4}}} \|\nabla c\|_{L^{4^-}} ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}} \|\omega\|_{L^{\frac{\alpha}{\alpha-1}}} \|\nabla c\|_{L^\infty}(s) ds \\
&\lesssim \frac{1}{t^{\frac{1}{2}}} \|\omega\|_{\mathcal{K}^{\frac{4\beta}{3\beta-4}}(\mathbb{R}^2)} \|c\|_{\mathcal{N}_{4^-}(\mathbb{R}^2)} + \frac{1}{t^{\frac{1}{2}}} \|\omega\|_{\mathcal{K}^{\frac{\alpha}{\alpha-1}}(\mathbb{R}^2)} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)} \\
&\lesssim \frac{\epsilon_1^2}{t^{\frac{1}{2}}} + \frac{\epsilon_1}{t^{\frac{1}{2}}} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)},
\end{aligned}$$

where the result in Lemma 3 is used. Combining (2.28) and (2.29), we have

$$(2.30) \quad \|\nabla c\|_{L^\infty}(t) \lesssim \frac{\epsilon_1}{t^{\frac{1}{2}}} + \frac{\epsilon_1}{t^{\frac{1}{2}}} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} + \frac{\epsilon_1}{t^{\frac{1}{2}}} \|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)}.$$

Next, we estimate the vorticity. For any $1 \leq r < \infty$

$$\begin{aligned}
\|\omega(t)\|_{L^r} &\lesssim t^{-1+\frac{1}{r}} \|\omega_0\|_{L^1} + \int_0^t \left\| \nabla^\perp e^{(t-s)\tilde{\Delta}} (n(s)\nabla\phi) \right\|_{L^r} ds \\
&\quad + \int_0^t \left\| \nabla e^{(t-s)\tilde{\Delta}} (u\omega) \right\|_{L^r} ds \\
&= t^{-1+\frac{1}{r}} \|\omega_0\|_{L^1} + L_1 + L_2.
\end{aligned}$$

If we restrict $2 \leq r$, we have

$$\begin{aligned}
L_1 &\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{r}}} \|n(s)\|_{L^2} \|\nabla\phi\|_{L^2} ds \\
&\quad + \int_{t/2}^t \frac{1}{(t-s)^{1-\frac{1}{r}}} \|n(s)\|_{L^\infty} \|\nabla\phi\|_{L^2} ds \\
&\lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{r}}} \|n(s)\|_{L^1}^{\frac{1}{2}} \|n(s)\|_{L^\infty}^{\frac{1}{2}} \|\nabla\phi\|_{L^2} ds
\end{aligned}$$

$$(2.31) \quad \begin{aligned} & + \int_{t/2}^t \frac{1}{(t-s)^{1-\frac{1}{r}}} \|n(s)\|_{L^\infty} \|\nabla\phi\|_{L^2} \\ & \lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}} + \frac{1}{t^{1-\frac{1}{r}}} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)}, \end{aligned}$$

where we used the Hölder's inequality and Young's inequality. The exponents r^*, \tilde{r} are defined by $1/r^* = 1/2 - 1/r$ and $1/\tilde{r} = 1/\tilde{r} - 1/2$. Now we estimate L_2 .

$$(2.32) \quad \begin{aligned} L_2 & \lesssim \int_0^{t/2} \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2r}}} \|u\|_{L^{2+}} \|\omega\|_{L^r} + \int_{t/2}^t \frac{1}{(t-s)^{1-\frac{1}{r}}} \|u\|_{L^{r^*}} \|\omega\|_{L^r} \\ & \lesssim \frac{1}{t^{\frac{3}{2}-\frac{1}{2r}}} \int_0^{t/2} \|\omega\|_{L^{\frac{\alpha}{\alpha-1}}} \|\omega\|_{L^r} + \int_{t/2}^t \frac{1}{(t-s)^{1-\frac{1}{r}}} \|\omega\|_{L^{\tilde{r}}} \|\omega\|_{L^r} \\ & \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}(\mathbb{R}^2)} \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)} + \frac{1}{t^{1-\frac{1}{r}}} \|\omega\|_{\mathcal{K}_{\tilde{r}}(\mathbb{R}^2)} \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)} \\ & \lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}} \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)}, \end{aligned}$$

where the result in Lemma 3 is used. Therefore, we have

$$(2.33) \quad \|\omega(t)\|_{L^r} \lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}} + \frac{1}{t^{1-\frac{1}{r}}} \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} + \frac{\epsilon_1}{t^{1-\frac{1}{r}}} \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)}.$$

Using the estimate (2.27), we obtain

$$(2.34) \quad \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)} \lesssim \epsilon_1 + \epsilon_1 \|n\|_{\mathcal{K}_\infty(\mathbb{R}^2)} + \epsilon_1 \|c\|_{\mathcal{N}_\infty} + \|n\|_{\mathcal{K}_\infty} \|c\|_{\mathcal{N}_\infty} + \epsilon_1 \|\omega\|_{\mathcal{K}_r(\mathbb{R}^2)}.$$

Combining estimates (2.27), (2.30) and (2.34), we obtain

$$(2.35) \quad \|n\|_{\mathcal{K}_\infty} + \|c\|_{\mathcal{N}_\infty} + \|\omega\|_{\mathcal{K}_r} \lesssim \epsilon_1 + \epsilon_1 (\|n\|_{\mathcal{K}_\infty} + \|c\|_{\mathcal{N}_\infty} + \|\omega\|_{\mathcal{K}_r}) + \|n\|_{\mathcal{K}_\infty} \|c\|_{\mathcal{N}_\infty}.$$

This completes the proof. \square

We remark that the case $r = \infty$ in (2.24) is missing due to Sobolev embedding inequalities.

Next we show estimates of higher derivatives. For convenience, we denote

$$\begin{aligned} \|\nabla n\|_{\mathcal{K}_\infty^1(\mathbb{R}^2)} & := \sup_{t \geq 0} t^{\frac{3}{2}} \|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)}, \quad \|\nabla^2 c\|_{\mathcal{K}_\infty(\mathbb{R}^2)} := \sup_{t \geq 0} t \|\nabla^2 c(t)\|_{L^\infty(\mathbb{R}^2)}, \\ \|\nabla\omega\|_{\mathcal{K}_r^1(\mathbb{R}^2)} & := \sup_{t \geq 0} t^{\frac{3}{2}-\frac{1}{r}} \|\nabla\omega(t)\|_{L^r(\mathbb{R}^2)}, \quad 1 \leq r < 2. \end{aligned}$$

Lemma 5. *Let n, c and ω be solutions of (1.7)-(1.10). If the assumptions in Theorem 1 are satisfied, then*

$$(2.36) \quad \|\nabla^2 c\|_{L^\infty}(t) \leq \frac{C\epsilon_1}{t}, \quad \|\nabla n\|_{L^\infty}(t) \leq \frac{C\epsilon_1}{t^{\frac{3}{2}}},$$

$$(2.37) \quad \|\nabla\omega\|_{L^r}(t) \leq \frac{C\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}, \quad 1 \leq r < 2.$$

Proof. We first estimate $\nabla^2 c$.

$$\begin{aligned} \|\nabla^2 c\|(t) &\lesssim \frac{1}{t} \|c_0\|_{L^\infty} + \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} kn \right\|_{L^\infty}(s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla(kn) \right\|_{L^\infty}(s) ds + \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} u \nabla c \right\|_{L^\infty}(s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla(u \nabla c) \right\|_{L^\infty}(s) ds. \end{aligned}$$

Consider the second term in the rightside.

$$\int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} kn \right\|_{L^\infty}(s) ds \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^2} \|n\|_{L^1} \|k(c)\|_{L^\infty} ds \lesssim \frac{\epsilon_1}{t}.$$

The third term is estimated as follows:

$$\begin{aligned} &\int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla(kn) \right\|_{L^\infty}(s) ds \\ &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}}} [\|k'(c)\|_{L^\infty} \|\nabla cn\|_{L^\infty} + \|k(c)\|_{L^\infty} \|\nabla n\|_{L^\infty}](s) ds \\ &\lesssim \|k'(c)\|_{L^\infty} \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{2}}} ds + \|k(c)\|_{L^\infty} \|\nabla n\|_{\mathcal{K}_\infty^1} \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{2}}} ds \\ &\lesssim \frac{\epsilon_1}{t} + \frac{\epsilon_1}{t} \|\nabla n\|_{\mathcal{K}_\infty^1}. \end{aligned}$$

We estimate the fourth and fifth terms.

$$\begin{aligned} \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} u \nabla c \right\|_{L^\infty}(s) ds &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|u \nabla c\|_{L^{\frac{3}{2}}}(s) ds \\ &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|u\|_{L^3} \|\nabla c\|_{L^3}(s) ds \\ &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|\omega\|_{L^{\frac{6}{5}}} \|\nabla c\|_{L^3}(s) ds \leq \frac{\epsilon_1}{t}. \end{aligned}$$

For $p > 2$ and $1 < q < 2$ with $1/p + 1/q = 1$

$$\begin{aligned} &\int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla(u \nabla c) \right\|_{L^\infty}(s) ds \\ &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2} - \frac{1}{q}}} (\|\nabla u \nabla c\|_{L^p} + \|u \nabla^2 c\|_{L^p})(s) ds \\ &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2} - \frac{1}{q}}} (\|\omega\|_{L^p} \|\nabla c\|_{L^\infty} + \|u\|_{L^p} \|\nabla^2 c\|_{L^\infty})(s) ds \\ &\lesssim \frac{\epsilon_1}{t} + \frac{\epsilon_1}{t} \|\nabla^2 c\|_{\mathcal{K}_\infty}. \end{aligned}$$

Summing up all estimates, we obtain

$$(2.38) \quad \|\nabla^2 c\|_{\mathcal{K}_\infty} \lesssim \epsilon_1 + \epsilon_1 \|\nabla^2 c\|_{\mathcal{K}_\infty} + \epsilon_1 \|\nabla n\|_{\mathcal{K}_\infty^1}.$$

Next we consider ∇n .

$$\begin{aligned} \|\nabla n\|_{L^\infty}(t) &\lesssim \frac{1}{t^{\frac{3}{2}}} \|n_0\|_{L^1} + \int_1^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [\chi n \nabla c] \right\|_{L^\infty}(s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla [\chi n \nabla c] \right\|_{L^\infty}(s) ds \\ &\quad + \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [un] \right\|_{L^\infty}(s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [u \nabla n] \right\|_{L^\infty}(s) ds. \end{aligned}$$

First, we compute

$$\begin{aligned} \int_1^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [\chi n \nabla c] \right\|_{L^\infty}(s) ds &\leq \int_1^{\frac{t}{2}} \frac{1}{(t-s)^2} \|n \nabla c\|_{L^1} ds \\ &\lesssim \frac{\epsilon_1}{t^2} \int_1^{\frac{t}{2}} \frac{1}{s^{\frac{1}{2}}} ds \lesssim \frac{\epsilon_1}{t^{\frac{3}{2}}}. \end{aligned}$$

Secondly,

$$\begin{aligned} &\int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla [\chi n \nabla c] \right\|_{L^\infty}(s) ds \\ &\leq \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}}} (\| \nabla n \nabla c \|_{L^\infty} + \| n \nabla^2 c \|_{L^\infty} + \| n |\nabla c|^2 \|_{L^\infty})(s) ds \\ &\lesssim \|\nabla n\|_{\mathcal{K}_\infty^1} \int_{\frac{t}{2}}^t \frac{\epsilon_1}{(t-s)^{\frac{1}{2}} s^2} ds + \|\nabla^2 c\|_{\mathcal{K}_\infty} \int_{\frac{t}{2}}^t \frac{\epsilon_1}{(t-s)^{\frac{1}{2}} s^2} ds + \int_{\frac{t}{2}}^t \frac{\epsilon_1}{(t-s)^{\frac{1}{2}} s^2} ds \\ &\lesssim \frac{\epsilon_1}{t^{\frac{3}{2}}} \|\nabla n\|_{\mathcal{K}_\infty^1} + \frac{\epsilon_1}{t^{\frac{3}{2}}} \|\nabla^2 c\|_{\mathcal{K}_\infty} + \frac{\epsilon_1}{t^{\frac{3}{2}}}. \end{aligned}$$

Thirdly,

$$\begin{aligned} \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [un] \right\|_{L^\infty}(s) ds &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|un\|_{L^{\frac{3}{2}}}(s) ds \\ &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|u\|_{L^3} \|n\|_{L^3}(s) ds \\ &\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{3}}} \|\omega\|_{L^{\frac{6}{5}}} \|n\|_{L^3}(s) ds \leq \frac{\epsilon_1}{t^{\frac{3}{2}}}. \end{aligned}$$

Lastly, for $p > 2$ and $1 < q < 2$ with $1/p + 1/q = 1$

$$\begin{aligned} \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [u \nabla n] \right\|_{L^\infty} (s) ds &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} \|u \nabla n\|_{L^p} (s) ds \\ &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} \|u\|_{L^p} (s) \|\nabla n\|_{L^\infty} (s) ds \\ &\lesssim \frac{\epsilon_1}{t^{\frac{3}{2}}} \|\nabla n\|_{\mathcal{K}_\infty^1}. \end{aligned}$$

Summing up, we obtain

$$(2.39) \quad \|\nabla n\|_{\mathcal{K}_\infty^1} \lesssim \epsilon_1 \|\nabla n\|_{\mathcal{K}_\infty^1} + \epsilon_1 \|\nabla^2 c\|_{\mathcal{K}_\infty} + \epsilon_1.$$

Combining (2.38) and (2.39), we obtain the first assertion of the lemma:

$$(2.40) \quad \|\nabla n\|_{\mathcal{K}_\infty^1} + \|\nabla^2 c\|_{\mathcal{K}_\infty} \leq C\epsilon_1.$$

With the above estimate in hands, it is easy to show $\|\nabla n\|_{L^2}$ satisfy the following decay:

$$(2.41) \quad \|\nabla n\|_{L^2}(t) \leq \frac{\epsilon_1}{t}.$$

We consider the vorticity equation. Using the integral representation, we compute

$$\begin{aligned} \|\nabla \omega\|_{L^r}(t) &\lesssim \frac{1}{t^{\frac{3}{2}-\frac{1}{r}}} \|\omega_0\|_{L^1} + \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [u\omega] \right\|_{L^r} (s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [u \nabla \omega] \right\|_{L^r} (s) ds + \int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [n \nabla \phi] \right\|_{L^r} (s) ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla [n \nabla \phi] \right\|_{L^r} (s) ds. \end{aligned}$$

First, for $p > 2$ and $1 < q < 2$ with $1/p + 1/q = 1$

$$\int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [u\omega] \right\|_{L^r} (s) ds \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{2-\frac{1}{r}}} \|u\|_{L^p} \|\omega\|_{L^q} (s) ds \lesssim \frac{\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}.$$

Secondly,

$$\begin{aligned} \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [u \nabla \omega] \right\|_{L^r} (s) ds &\lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} \|u\|_{L^p} \|\nabla \omega\|_{L^r} (s) ds \\ &\lesssim \frac{\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}} \|\nabla \omega\|_{\mathcal{K}_r^1}. \end{aligned}$$

Thirdly,

$$\int_0^{\frac{t}{2}} \left\| \nabla^2 e^{(t-s)\Delta} [n \nabla \phi] \right\|_{L^r} (s) ds \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{2-\frac{1}{r}}} \|n\|_{L^2} \|\nabla \phi\|_{L^2} (s) ds \leq \frac{\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}.$$

Lastly,

$$\begin{aligned}
& \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} \nabla [n \nabla \phi] \right\|_{L^r} (s) ds \\
& \lesssim \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [\nabla n \nabla \phi + n \nabla^2 \phi] \right\|_{L^r} (s) ds \\
& \lesssim \int_{\frac{t}{2}}^t \left\| \nabla e^{(t-s)\Delta} [\nabla n \nabla \phi + n \nabla^2 \phi] \right\|_{L^r} (s) ds \\
& \lesssim \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{r}}} \|\nabla n\|_{L^2} \|\nabla \phi\|_{L^2} ds + \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{r}}} \|n\|_{L^\infty} \|\nabla^2 \phi\|_{L^1} ds \\
& \lesssim \frac{\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}
\end{aligned}$$

by (2.41) and Lemma 4. Summing up, we obtain

$$(2.42) \quad \|\nabla \omega\|_{\mathcal{K}_r^1} \lesssim \epsilon_1 + \epsilon_1 \|\nabla \omega\|_{\mathcal{K}_r^1}.$$

This completes the proof of the second assertion of Lemma 5. \square

Remark 3. The restriction that $r < 2$ in (2.37) is due to absence of temporal decay of ϕ , since ϕ is independent of time. We leave it open question whether or not the estimate (2.37) is available for $r \geq 2$.

Proof of Propostion 1. The decay estimate of solutions is the consequence of consecutive Lemma 3-Lemma 5. \square

3. Proof of Theorem 1

In this section, we present the proof of Theorem 1.

Proof of Theorem 1. We define the family of rescaled solutions in \mathbb{R}^2 ¹

$$\begin{aligned}
n_k(x, t) &= k^2 n(kx, k^2 t), & c_k(x, t) &= c(kx, k^2 t), \\
u_k(x, t) &= ku(kx, k^2 t), & \phi_k(x) &= \phi(kx)
\end{aligned}$$

with (sufficiently regular) initial data

$$n_{k,0}(x) = k^2 n_0(kx), \quad c_{k,0}(x) = c_0(kx), \quad u_{k,0}(x) = ku_0(kx).$$

For the vorticity field, we have following rescaled solutions and initial data

$$\omega_k(x, t) = k^2 \omega(kx, k^2 t), \quad \omega_{k,0}(x) = k^2 \omega_0(kx).$$

We recall some invariant quantities (independent of k), which are

$$\begin{aligned}
\|n_k(t)\|_{L^1} &= \|n(t)\|_{L^1} = \|n_0\|_{L^1}, \quad \|c_{k,0}\|_{L^\infty} = \|c_0\|_{L^\infty}, \quad \|\nabla c_{k,0}\|_{L^2} = \|\nabla c_0\|_{L^2}, \\
\|\omega_{k,0}\|_{L^1} &= \|\omega_0\|_{L^1}, \quad \int_{\mathbb{R}^2} \omega(t) dx = \int_{\mathbb{R}^2} \omega_k(t) dx = \int_{\mathbb{R}^2} \omega_0 dx.
\end{aligned}$$

¹ (n_k, c_k, u_k) solve system (1.1) with the potential ϕ_k , instead of ϕ .

Therefore, the smallness assumption (1.13) is likewise valid for $(n_{k,0}, c_{k,0}, \omega_{k,0})$, namely

$$\|n_{k,0}\|_{L^1(\mathbb{R}^2)} + \|c_{k,0}\|_{(L^\infty \cap \dot{H}^1)(\mathbb{R}^2)} + \|\omega_{k,0}\|_{L^1(\mathbb{R}^2)} < \epsilon_1.$$

We also note that the potential ϕ_k also remains invariant by norm of

$$(3.1) \quad \|\nabla \phi_k\|_{L^2} = \|\nabla \phi\|_{L^2}.$$

From now on, we consider the vorticity equation, instead equation of velocity fields. We then have global existence and time decay of solutions (n_k, c_k, ω_k) and sequence of functions also solves the system in a weak sense as follows: (possibly subsequence) for $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$ it holds

$$(3.2) \quad \begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) n_k + n_k u_k \cdot \nabla \varphi + \chi(c_k) n_k \nabla c_k \nabla \varphi \, dx dt = \int_{\mathbb{R}^2} n_{k,0} \varphi(x, 0) dx, \\ & \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) c_k + c_k u_k \cdot \nabla \varphi - k(c_k) n_k \varphi \, dx dt = \int_{\mathbb{R}^2} c_{k,0} \varphi(x, 0) dx, \\ & \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) \omega_k + \omega_k u_k \cdot \nabla \varphi + n_k \nabla \phi_k \nabla^\perp \varphi \, dx dt = \int_{\mathbb{R}^2} \omega_k(x, 0) \varphi(x, 0) dx. \end{aligned}$$

In particular the time decay rates in Proposition 1 are scaling invariant, so rescaled solutions also satisfy uniform estimates

$$(3.3) \quad \|n_k(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t}, \quad \|\nabla n_k(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{3}{2}}},$$

$$(3.4) \quad \|c_k(t)\|_{(L^\infty \cap \dot{H}^1)(\mathbb{R}^2)} \leq \epsilon_1, \quad \|\nabla c_k(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{1}{2}}}, \quad \|\nabla^2 c_k(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t},$$

$$(3.5) \quad \|\omega_k(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{1-\frac{1}{r}}}, \quad 1 < r < \infty, \quad \|\nabla \omega_k(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C\epsilon_1}{t^{\frac{3}{2}-\frac{1}{r}}}, \quad 1 \leq r < 2.$$

Therefore, we have strong convergence of (n_k, c_k, ω_k) in $L^p \times W^{1,p} \times L^r$ with $1 \leq p < \infty$ and $1 \leq r < \infty$ in any compact set in $\mathbb{R}^2 \times (0, \infty)$. Let us denote limit functions by $(\tilde{n}, \tilde{c}, \tilde{\omega})$ as $k \rightarrow \infty$ (possibly subsequence of k). To be more precise, there is a subsequence such that as k_j tends to infinity, for any $1 \leq p < \infty$, $1 \leq r < \infty$ and for all $R, \eta_\epsilon > 0$

$$\begin{aligned} n_{k_j} & \longrightarrow \tilde{n} \quad \text{strongly in } L^p(B_R \times (\eta_\epsilon, \eta_\epsilon^{-1})), \\ \nabla c_{k_j} & \longrightarrow \nabla \tilde{c} \quad \text{strongly in } L^p(B_R \times (\eta_\epsilon, \eta_\epsilon^{-1})), \\ \omega_{k_j} & \longrightarrow \tilde{\omega} \quad \text{strongly in } L^r(B_R \times (\eta_\epsilon, \eta_\epsilon^{-1})). \end{aligned}$$

We observe that $(\tilde{n}, \tilde{c}, \tilde{\omega})$ satisfy the estimates (3.3)-(3.5). Similarly we denote by $\tilde{\phi}$ the weak limit of ϕ_k , then $\nabla \tilde{\phi} \in L^2(\mathbb{R})$ due to (3.1). Combining the strong convergence in any compact domain of $\mathbb{R}^2 \times (0, \infty)$ with these time decays, we

can take the limit $k \rightarrow \infty$ to (3.2), and show that $(\tilde{n}, \tilde{c}, \tilde{\omega})$ solve the following equations in a weak sense:

$$(3.6) \quad \begin{cases} \partial_t \tilde{n} + \tilde{u} \cdot \nabla \tilde{n} - \Delta \tilde{n} = -\nabla \cdot (\chi(\tilde{c}) \tilde{n} \nabla \tilde{c}), \\ \partial_t \tilde{c} + \tilde{u} \cdot \nabla \tilde{c} - \Delta \tilde{c} = -k(\tilde{c}) \tilde{n}, \\ \partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} - \Delta \tilde{\omega} = -\nabla \times (\tilde{n} \nabla \tilde{\phi}) \end{cases}$$

with initial data

$$(3.7) \quad \tilde{n}_0 = m\delta_0, \quad \tilde{c}_0 = 0, \quad \tilde{\omega}_0 = \gamma\delta_0,$$

where m is the total mass of n and γ is total circulation of ω . While the proof for passing to limit goes on closely following [10, Section 2.5.1], for the sake of concreteness we take some terms, say, $\int_0^\infty \int_{\mathbb{R}^2} \chi(c_k) n_k \nabla c_k \nabla \varphi \, dx dt$ and $\int_0^\infty \int_{\mathbb{R}^2} \omega_k u_k \nabla \varphi \, dx dt$ to show

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^2} \chi(c_k) n_k \nabla c_k \nabla \varphi \, dx dt &= \int_0^\infty \int_{\mathbb{R}^2} \chi(\tilde{c}) \tilde{n} \nabla \tilde{c} \nabla \varphi \, dx dt, \\ \lim_{k \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^2} \omega_k u_k \nabla \varphi \, dx dt &= \int_0^\infty \int_{\mathbb{R}^2} \tilde{\omega} \tilde{u} \nabla \varphi \, dx dt. \end{aligned}$$

Let $\text{supp } \varphi \in B_R \times [0, T]$. We define

$$F_k(t) = \int_{B_R} \chi(c_k) n_k \nabla c_k \nabla \varphi \, dx, \quad F(t) = \int_{B_R} \chi(\tilde{c}) \tilde{n} \nabla \tilde{c} \nabla \varphi.$$

Due to strong convergence we have $F_k(t) \rightarrow F(t)$ for $t > 0$. Using the decay estimate (3.4), it holds that $F_k(t) \leq C(R)t^{-\frac{1}{2}}$, and we then have $\lim_{k \rightarrow 0} F_k(t) = F(t)$ via the dominated convergence theorem. For the second example we also have

$$\int_{B(R)} \omega_k u_k \nabla \varphi \, dx \leq \|\omega_k\|_{L^{\frac{4}{3}}} \|u_k\|_{L^4} \|\nabla \varphi\|_{L^\infty} \leq C(R)t^{-\frac{1}{2}},$$

where we used the embedding $\|u_k\|_{L^4} \leq C\|\omega_k\|_{L^{\frac{4}{3}}}$ and the estimate (3.5). In fact, it holds that

$$(3.8) \quad \tilde{c} = 0, \quad \nabla \tilde{\phi} = 0.$$

Indeed, from the c_k equation we have

$$\|c_k(t)\|_{L^p} \leq \|c_{k,0}\|_{L^p} = k^{-\frac{2}{p}} \|c_0\|_{L^p}, \quad 1 \leq p \leq \infty.$$

It implies $\tilde{c} = 0$. Next we show that $\tilde{\phi}$ is a function of homogeneity zero. If $l > 0$ is fixed and $\psi \in C_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \phi_k(lx) \psi(x) \, dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \phi(klx) \psi(x) \, dx = \int_{\mathbb{R}^2} \tilde{\phi}(x) \psi(x) \, dx.$$

On the other hand, denoting $\psi_l(y) := \psi(l^{-1}y)$, we see that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \phi_k(lx) \psi(x) \, dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} l^{-2} \phi_k(y) \psi(l^{-1}y) \, dy$$

$$= l^{-2} \int_{\mathbb{R}^2} \tilde{\phi}(y) \psi_l(y) dy = \int_{\mathbb{R}^2} \tilde{\phi}(lx) \psi(x) dx.$$

Therefore, $\nabla \tilde{\phi}$ is a function of homogeneity 1, namely $\nabla \phi(x) = l \nabla \phi(lx)$, which implies $\nabla \tilde{\phi} = 0$, since $\nabla \tilde{\phi} \in L^2(\mathbb{R}^2)$. On account of (3.8), the system (3.6)-(3.7) is reduced to

$$\begin{cases} \partial_t \tilde{n} + \tilde{u} \cdot \nabla \tilde{n} - \Delta \tilde{n} = 0, \\ \partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} - \Delta \tilde{\omega} = 0 \end{cases}$$

with initial data

$$\tilde{n}_0 = m\delta_0, \quad \tilde{\omega}_0 = \gamma\delta_0.$$

It is well established that the vorticity equation of Navier-Stokes equation with the dirac-delta initial data has the unique solution

$$\tilde{w}(x, t) = \gamma \Gamma(x, t).$$

We refer to [9] and [10], and references cited therein. In particular

$$\tilde{u} = K * \tilde{\omega}, \quad K(x) = \nabla^\perp \log |x| = \left\langle -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right\rangle,$$

which implies $\tilde{u} \cdot \nabla \tilde{n} = 0$ by Lemma 2. Then \tilde{n} equation is reduced to

$$\partial_t \tilde{n} - \Delta \tilde{n} = 0$$

with initial data $\tilde{n}_0 = m\delta_0$. As a direct application of Theorem 4.4.2 in [10], the above equation has the unique solution

$$\tilde{n}(x, t) = m\Gamma(x, t).$$

The asymptotics are obtained as follows. When $t = 1$, tending to zero as $k_j \rightarrow \infty$, we have

$$(3.9) \quad \lim_{k_j \rightarrow \infty} \|n_{k_j}(\cdot, 1) - \tilde{n}(\cdot, 1)\|_{L^\infty(B_R)} = 0.$$

Using $\tilde{n} = m\Gamma$ is self-similar, we observe that

$$n_{k_j}(x, 1) - \tilde{n}(x, 1) = k_j^2 n(k_j x, k_j^2) - k_j^2 \tilde{n}(k_j x, k_j^2).$$

Setting $t = k_j^2$, (3.9) can be rewritten as

$$(3.10) \quad t \|(n(\cdot, t) - \tilde{n}(\cdot, t))\|_{L^\infty(B_{t,R})} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $B_{t,R} = \{x : |x| < \sqrt{t}R\}$. Similarly, for any $r < \infty$ we obtain

$$(3.11) \quad t^{1-\frac{1}{r}} \|(\omega(\cdot, t) - \tilde{\omega}(\cdot, t))\|_{L^r(B_{t,R})} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Since we also have a convergence of ∇c to $\nabla \tilde{c} = 0$, we can see that

$$(3.12) \quad t^{\frac{1}{2}} \|\nabla c(\cdot, t)\|_{L^\infty(B_{t,R})} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Here the point is that the decay estimates are independent of k . Since $\tilde{n}(x, t) = m\Gamma(x, t)$ and $\tilde{\omega}(x, t) = \gamma\Gamma(x, t)$, we complete the proof. \square

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