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SECOND-ORDER SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING OVER CONES

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Abstract. In this paper, some omissions in Mishra and Lai [13], have been pointed out and their corrective measures have been discussed briefly.

1. Introduction

A pair of primal and dual problems in mathematical programming is called symmetric if the dual of the dual is the primal problem. Dorn [6] introduced the concept of symmetric duality in quadratic programming. His results were extended to nonlinear convex programming problems by Dantzig et al. [4] and later by Bazaraa and Goode [3] over arbitrary cones.

Mangasarian [11] introduced the concept of second-order duality for nonlinear problems. Since then, many authors [1, 2, 7, 9, 15, 16] have worked on second-order symmetric duality. Mishra and Lai [13] studied Mond-Weir type second-order multiobjective symmetric duality for the following pair of problems:

$$
\begin{aligned}
(\mathbf{P}) \quad & K\text{−minimize} && f(x,y) - \frac{1}{2} p^T \nabla_{yy} f(x,y) p \\
& \text{subject to} && -\nabla_y(\lambda^T f)(x,y) - \nabla_{yy}(\lambda^T f)(x,y) \in C_2^*, \\
& && y^T[\nabla_y(\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)] \geqslant 0, \\
& && \lambda \in K^*,\ x \in C_1. \\
(\mathbf{D}) \quad & K\text{−maximize} && f(u,v) - \frac{1}{2} q^T \nabla_{xx} f(u,v) q \\
& \text{subject to} && \nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(\lambda^T f)(u,v) \in C_1^*, \\
& && u^T[\nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(\lambda^T f)(u,v)] \leqslant 0, \\
& && \lambda \in K^*,\ v \in C_2,
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a twice differentiable function of x and y, C_1 and C_2 are closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively,

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K is a closed convex pointed cone in $\mathbb{R}^k$ such that int $K \neq \phi$ and K^* is its positive polar cone.

The first term in the objectives of (P) and (D) is a k-vector, while the second term is not a k-vector. Therefore the models and so the results in [13] seem to be erroneous. Some of the other observations are as follows:

(i) For a vector function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, it is not clear that what do the authors mean by $\nabla_{xx} f(x, y)$ and $\nabla_{yy} f(x, y)$.

(ii) It is well known that the weak duality theorem gives a relation between the objective functions of the primal and dual problems. It is not so in [13] as the second-order terms in the two objective functions are missing from the conclusion of the weak duality theorem.

(iii) In the strong duality theorem, the assumption that $\nabla_{yyy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is negative definite is meaningless since $\nabla_{yyy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is not a matrix.

(iv) The authors simply state that the proof of their strong duality theorem follows on the lines of [5], while the proof in [5] is full of errors (see [7]).

(v) The definitions of K-strongly K-second-order pseudoinvex functions seem to be inappropriate due to the absence of a second-order derivative term (see [1, 7, 15]).

2. Notations and preliminaries

Let C_1 and C_2 be closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with nonempty interiors. Let $\nabla_x f_i$ ($\nabla_y f_i$) denote $n \times 1$ ($m \times 1$) gradient vector with respect to first (second) vector variable and let $\nabla_{xy} f_i$ denote the $n \times m$ matrix. All vectors shall be considered as column vectors.

Definition 2.1 ([14])**.** The positive polar cone C^* of a cone C is defined by

$$C^* = \{z : x^T z \geqq 0 \text{ for all } x \in C\}.$$

We consider the following multiobjective programming problem:

$$\textbf{(P1)} \quad K\text{−minimize} \quad f(x)$$
$$\text{subject to} \quad x \in X^\circ = \{x \in S : -g(x) \in Q\},$$

where $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^k$, $g : S \to \mathbb{R}^m$, K and Q are closed convex pointed cones with nonempty interiors in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively.

Definition 2.2 ([10])**.** A point $\bar{x} \in X^o$ is an efficient solution of (P1) if there exists no $x \in X^o$ such that $f(\bar{x}) - f(x) \in K \backslash \{0\}$.

For the definitions of K-η-bonvex, K-η-pseudobonvex and second-order F-pseudoconvex functions, refer to [8].

3. Mond-Weir type second-order symmetric duality

We consider the following pair of Mond-Weir type second-order multiobjective symmetric dual programming problems which also aims to correct the dual pair considered in [13]:

Primal(MP)

$$\begin{aligned}
K\text{-minimize}\quad & \{f_1(x,y)-\tfrac{1}{2}p_1^T\nabla_{yy}f_1(x,y)p_1,\ldots,f_k(x,y)-\tfrac{1}{2}p_k^T\nabla_{yy}f_k(x,y)p_k\}\\
\text{subject to}\quad & -\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)\in C_2^*,\\
& y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)\geqq 0,\\
& \lambda\in \operatorname{int}K^*, x\in C_1,
\end{aligned}$$

Dual(MD)

$$\begin{aligned}
K\text{-maximize}\quad & \{f_1(u,v)-\tfrac{1}{2}q_1^T\nabla_{xx}f_1(u,v)q_1,\ldots,f_k(u,v)-\tfrac{1}{2}q_k^T\nabla_{xx}f_k(u,v)q_k\}\\
\text{subject to}\quad & \sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)\in C_1^*,\\
& u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)\leqq 0,\\
& \lambda\in \operatorname{int}K^*, v\in C_2,
\end{aligned}$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)^T\in\mathbb{R}^k,\ C_1\times C_2\subset S_1\times S_2$ and for $i=1,2,\ldots,k$,

(i) $f_i: S_1\times S_2\to\mathbb{R}$ is a thrice differentiable function of x and y, and
(ii) p_i and q_i are vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

We will use $p=(p_1,p_2,\ldots,p_k)$ and $q=(q_1,q_2,\ldots,q_k)$.

Duality theorems

We do not need to restrict $x\in C_1$ and $v\in C_2$ in the programs (MP) and (MD) respectively for proving Theorems 3.1 and 3.2. However, these restrictions are required in the proof of strong and converse duality theorems.

Theorem 3.1 (Weak duality)**.** *Let* (x,y,λ,p) *be feasible for* (MP) *and* (u,v,λ,q) *be feasible for* (MD)*. Let*

(i) $f(\cdot,v)$ *be* K-η_1-*bonvex in the first variable at* u*,*
(ii) $-f(x,\cdot)$ *be* K-η_2-*bonvex in the second variable at* y*, and*
(iii) $\eta_1(x,u)+u\in C_1$ *and* $\eta_2(v,y)+y\in C_2$.

Then

$$\begin{aligned}
&\{f_1(u,v)-\tfrac{1}{2}q_1^T\nabla_{xx}f_1(u,v)q_1,\ldots,f_k(u,v)-\tfrac{1}{2}q_k^T\nabla_{xx}f_k(u,v)q_k\}\\
&-\{f_1(x,y)-\tfrac{1}{2}p_1^T\nabla_{yy}f_1(x,y)p_1,\ldots,f_k(x,y)-\tfrac{1}{2}p_k^T\nabla_{yy}f_k(x,y)p_k\}\notin K\backslash\{0\}.
\end{aligned}$$

Proof. The proof follows on the lines of Theorem 3.2 [8]. □

The following weak duality theorem can also be proved.

Theorem 3.2 (Weak duality)**.** *Let* (x, y, λ, p) *be feasible for* (MP) *and* (u, v, λ, q) *be feasible for* (MD). *Let*

(a$_1$) $\lambda^T f(\cdot, v)$ *be* η_1*-pseudobonvex in the first variable at* u,
(a$_2$) $-\lambda^T f(x, \cdot)$ *be* η_2*-pseudobonvex in the second variable at* y,
(a$_3$) $\eta_1(x, u) + u \in C_1$ *and* $\eta_2(v, y) + y \in C_2$,
or
(b$_1$) $\lambda^T f(\cdot, v)$ *be second-order* F*-pseudoconvex at* u,
(b$_2$) $-\lambda^T f(x, \cdot)$ *be second-order* F*-pseudoconvex at* y,
(b$_3$) $F_{x,u}(\xi) + u^T\xi \geqq 0$ *for* $\xi \in C_1^*$ *and* $F_{v,y}(\eta) + y^T\eta \geqq 0$ *for* $\eta \in C_2^*$.

Then

$$\{f_1(u, v) - \tfrac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1, \dots, f_k(u, v) - \tfrac{1}{2}q_k^T\nabla_{xx}f_k(u, v)q_k\}$$
$$-\{f_1(x, y) - \tfrac{1}{2}p_1^T\nabla_{yy}f_1(x, y)p_1, \dots, f_k(x, y) - \tfrac{1}{2}p_k^T\nabla_{yy}f_k(x, y)p_k\} \notin K\backslash\{0\}.$$

In the following theorems $(\mathrm{MP})_{\bar\lambda}$ and $(\mathrm{MD})_{\bar\lambda}$ respectively denote the problems (MP) and (MD) when λ is fixed to be $\bar\lambda$.

Theorem 3.3 (Strong duality)**.** *Let* $(\bar x, \bar y, \bar\lambda, \bar p)$ *be an efficient solution for* (MP). *Suppose that*

(i) $\nabla_{yy}f_i(\bar x, \bar y)$ *is positive definite for* $i = 1, 2, \dots, k$ *and* $\sum_{i=1}^{k} \bar\lambda_i \bar p_i^T \nabla_y f_i(\bar x, \bar y) \geqq 0$ *or* $\nabla_{yy}f_i(\bar x, \bar y)$ *is negative definite for* $i = 1, 2, \dots, k$ *and* $\sum_{i=1}^{k} \bar\lambda_i \bar p_i^T \nabla_y f_i(\bar x, \bar y) \leqq 0$,
(ii) *the set* $\{\nabla_y f_i(\bar x, \bar y) + \nabla_{yy}f_i(\bar x, \bar y)\bar p_i, i = 1, 2, \dots, k\}$ *is linearly independent, and*
(iii) $R_+^k \subseteq K$.

Then, $(\bar x, \bar y, \bar q = 0)$ *is feasible for* $(\mathrm{MD})_{\bar\lambda}$, *and the objective function values of* (MP) *and* $(\mathrm{MD})_{\bar\lambda}$ *are equal. Furthermore, if the hypotheses of Theorem 3.1 or Theorem 3.2 are satisfied for all feasible solutions of* $(\mathrm{MP})_{\bar\lambda}$ *and* $(\mathrm{MD})_{\bar\lambda}$, *then* $(\bar x, \bar y, \bar q)$ *is an efficient solution for* $(\mathrm{MD})_{\bar\lambda}$.

Proof. Since $(\bar x, \bar y, \bar\lambda, \bar p)$ is an efficient solution for (MP), by the Fritz-John necessary optimality conditions [14], there exist $\alpha \in K^*$, $\beta \in C_2$, $\gamma \in \mathbb{R}_+$, such that the following conditions are satisfied at $(\bar x, \bar y, \bar\lambda, \bar p)$ (for simplicity, we write $\nabla_x f_i, \nabla_{xy} f_i$ instead of $\nabla_x f_i(\bar x, \bar y), \nabla_{xy} f_i(\bar x, \bar y)$ etc.):

$$(x - \bar x)^T[\sum_{i=1}^{k} \alpha_i(\nabla_x f_i - \frac{1}{2}(\nabla_x(\nabla_{yy}f_i\bar p_i))^T\bar p_i)$$
$$+\sum_{i=1}^{k} \bar\lambda_i(\nabla_{yx}f_i + \nabla_x(\nabla_{yy}f_i\bar p_i))^T(\beta - \gamma\bar y)] \geqq 0 \quad \text{for all} \quad x \in C_1,$$

$$(y-\bar{y})^T[\sum_{i=1}^{k}\alpha_i(\nabla_y f_i - \frac{1}{2}(\nabla_y(\nabla_{yy}f_i\bar{p}_i))^T\bar{p}_i)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i + \nabla_y(\nabla_{yy}f_i\bar{p}_i))(\beta-\gamma\bar{y})$$

$$-\gamma\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i)] \geqq 0 \quad \text{for all} \quad y \in \mathbb{R}^m,$$

$$[(\beta-\gamma\bar{y})^T(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i)](\lambda_i - \bar{\lambda}_i) \geqq 0,$$

$$i = 1,2,\ldots,k \quad \text{for all} \quad \lambda \in \text{int}K^*,$$

$$[(\beta-\gamma\bar{y})\bar{\lambda}_i - \alpha_i\bar{p}_i]^T\nabla_{yy}f_i = 0, \ i = 1,2,\ldots,k,$$

$$\beta^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i) = 0,$$

$$\gamma\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i) = 0,$$

$$(\alpha,\beta,\gamma) \neq 0.$$

□

Following the proof of Theorem 3.4 [8], it can be proved that $(\bar{x},\bar{y},\bar{q})$ is an efficient solution of $(\text{MD})_{\bar{\lambda}}$.

Remark 3.1. In multiobjective programming for weak duality theorems, one requires same λ for the primal and dual feasible solutions and so for strong duality theorems, $\bar{\lambda}$ corresponding to the optimal (weak efficient, efficient or properly efficient) solution of the primal problem is required to be fixed in the dual problem. Therefore the above proof gives that $(\bar{x},\bar{y},\bar{q}=0)$ is an efficient solution for $(\text{MD})_{\bar{\lambda}}$ and not that $(\bar{x},\bar{y},\bar{\lambda},\bar{q}=0)$ is an efficient solution for (MD). In the literature [12, 13, 16] optimality for the dual problem (Wolfe or Mond-Weir type) has been claimed, which is not correct.

Theorem 3.4 (Converse duality)**.** *Let* $(\bar{u},\bar{v},\bar{\lambda},\bar{q})$ *be an efficient solution for* (MD). *Suppose that*

(i) $\nabla_{xx}f_i(\bar{u},\bar{v})$ *is positive definite for all* $i = 1,2,\ldots,k$ *and* $\sum_{i=1}^{k}\bar{\lambda}_i\bar{q}_i^T\nabla_x f_i(\bar{u},\bar{v}) \geqq 0$ *or* $\nabla_{xx}f_i(\bar{u},\bar{v})$ *is negative definite for all* $i = 1,2,\ldots,k$ *and* $\sum_{i=1}^{k}\bar{\lambda}_i\bar{q}_i^T\nabla_x f_i(\bar{u},\bar{v}) \leqq 0$,

(ii) *the set* $\{\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i, i = 1,2,\ldots,k\}$ *is linearly independent, and*

(iii) $\mathbb{R}^k_+ \subseteq K$.

Then $(\bar{u}, \bar{v}, \bar{p} = 0)$ *is feasible for* $(\mathrm{MP})_{\bar{\lambda}}$*, and the objective function values of* $(\mathrm{MP})_{\bar{\lambda}}$ *and* (MD) *are equal. Furthermore, if the hypotheses of Theorem 3.1 or Theorem 3.2 are satisfied for all feasible solutions of* $(\mathrm{MP})_{\bar{\lambda}}$ *and* $(\mathrm{MD})_{\bar{\lambda}}$*, then* $(\bar{u}, \bar{v}, \bar{p})$ *is an efficient solution for* $(\mathrm{MP})_{\bar{\lambda}}$*.*

Proof. Follows on the lines of Theorem 3.3. □

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References

[1] I. Ahmad, *Second order symmetric duality in nondifferentiable multiobjective programming*, Inform. Sci. **173** (2005), no. 1-3, 23–34.

[2] I. Ahmad and Z. Husain, *Nondifferentiable second order symmetric duality in multiobjective programming*, Appl. Math. Lett. **18** (2005), no. 7, 721–728.

[3] M. S. Bazaraa and J. J. Goode, *On symmetric duality in nonlinear programming*, Operations Res. **21** (1973), 1–9.

[4] G. B. Dantzig, E. Eisenberg, and R. W. Cottle, *Symmetric dual nonlinear programs*, Pacific J. Math. **15** (1965), 809–812.

[5] G. Devi, *Symmetric duality for nonlinear programming problem involving η-bonvex functions*, European J. Operational Research **104** (1998), 615–621.

[6] W. S. Dorn, *A symmetric dual theorem for quadratic programs*, J. Operations Research Society of Japan **2** (1960), 93–97.

[7] T. R. Gulati, S. K. Gupta, and I. Ahmad, *Second-order symmetric duality with cone constraints*, J. Comput. Appl. Math. **220** (2008), no. 1-2, 347–354.

[8] T. R. Gulati and G. Mehndiratta, *Nondifferentiable multiobjective Mond-Weir type second-order symmetric duality over cones*, Optim. Lett. **4** (2010), no. 2, 293–309.

[9] S. H. Hou and X. M. Yang, *On second-order symmetric duality in nondifferentiable programming*, J. Math. Anal. Appl. **255** (2001), no. 2, 491–498.

[10] S. Khurana, *Symmetric duality in multiobjective programming involving generalized cone-index functions*, European J. Oper. Res. **165** (2005), no. 3, 592–597.

[11] O. L. Mangasarian, *Second and higher-order duality in nonlinear programming*, J. Math. Anal. Appl. **51** (1975), no. 3, 607–620.

[12] S. K. Mishra, *Multiobjective second order symmetric duality with cone constraints*, European J. Oper. Res. **126** (2000), no. 3, 675–682.

[13] S. K. Mishra and K. K. Lai, *Second order symmetric duality in multiobjective programming involving generalized cone-invex functions*, European J. Oper. Res. **178** (2007), no. 1, 20–26.

[14] S. K. Suneja, S. Aggarwal, and S. Davar, *Multiobjective symmetric duality involving cones*, European J. Oper. Res. **141** (2002), no. 3, 471–479.

[15] S. K. Suneja, C. S. Lalitha, and S. Khurana, *Second order symmetric duality in multiobjective programming*, European J. Oper. Res. **144** (2003), no. 3, 492–500.

[16] X. M. Yang, X. Q. Yang, K. L. Teo, and S. H. Hou, *Multiobjective second-order symmetric duality with F-convexity*, European J. Oper. Res. **165** (2005), no. 3, 585–591.

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