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$\mathcal{W}$-RESOLUTIONS AND GORENSTEIN CATEGORIES WITH RESPECT TO A SEMIDUALIZING BIMODULES

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Abstract. Let $\mathcal{W}$ be an additive full subcategory of the category R-Mod of left R-modules. We provide a method to construct a proper $\mathcal{W}_C^H$-resolution (resp. coproper $\mathcal{W}_C^T$-coresolution) of one term in a short exact sequence in R-Mod from those of the other two terms. By using these constructions, we introduce and study the stability of the Gorenstein categories $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ with respect to a semidualizing bimodule C, and investigate the 2-out-of-3 property of these categories of a short exact sequence by using these constructions. Also we prove how they are related to the Gorenstein categories $\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$ and $\mathcal{G}(\mathrm{Hom}_R(R\ltimes C,\mathcal{W}))_C$ over $R\ltimes C$.

1. Introduction

Auslander and Bridger [1] introduced the modules of finite G-dimension over a commutative Noetherian ring R, in part, to identify a class of finitely generated R-modules with particularly nice duality properties with respect to R. They are exactly the R-modules which admit a finite resolution by modules of G-dimension 0. As a special case, the duality theory for these modules recovers the well-known duality theory for finitely generated modules over a Gorenstein ring. This notion has been extended in several directions. For instance, Enochs and Jenda [3] introduced the Gorenstein projective modules and the Gorenstein injective modules; these are analogues of modules of G-dimension 0 for the non-finitely generated arena. Foxby [5], Golod [6] and Vasconcelos [12] independently initiated the study of semidualizing modules (under different names) over a commutative Noetherian ring. White and other peoples investigated them in commutative (possibly non-Noetherian) rings. Recently, Holm, Jørgensen [7] and Sather-Wagstaff, Sharif, White [11] unified these approaches with the G_C-projective modules, the G_C-injective modules and the G_C-flat modules.

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Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ an additive full subcategory of $\mathcal{A}$. Sather-Wagstaff, Sharif and White introduced the Gorenstein category $\mathcal{G}(\mathcal{C})$, which unifies the following notions: modules of Gorenstein dimension zero [1], Gorenstein projective modules, Gorenstein injective modules [3], V-Gorenstein projective modules, V-Gorenstein injective modules [4], and so on. Huang [9] provided a method to construct a proper $\mathcal{C}$-resolution (resp. coproper $\mathcal{C}$-coresolution) of one term in a short exact sequence in $\mathcal{A}$ from those of the other two terms. By using these constructions, he answered affirmatively an open question on the stability of the Gorenstein category $\mathcal{G}(\mathcal{C})$ posed by Sather-Wagstaff, Sharif and White [10], and also proved that $\mathcal{G}(\mathcal{C})$ is closed under direct summands.

Let R be a ring, and let R-Mod be the category of left R-modules and $\mathcal{W}$ an additive full subcategory of R-Mod. Inspired by the works of Huang [9], in Section 2 we provide a method to construct a proper $\mathcal{W}_C^H$-resolution (resp. coproper $\mathcal{W}_C^T$-coresolution) of one term in a short exact sequence in R-Mod from those of the other two terms. Section 3 is devoted to introducing and studying the stability of the Gorenstein categories $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ with respect to a semidualizing bimodule C, and investigating the 2-out-of-3 property of these categories of a short exact sequence by using these constructions. Also we prove how they are related to the Gorenstein categories $\mathcal{G}((R \ltimes C)\otimes_R \mathcal{W})_C$ and $\mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$ over $R \ltimes C$.

2. Preliminaries

This section is devoted to recalling some definitions and notations. For terminology we shall follow [7], [8] and [9].

Definition 2.1. An (R, R)-bimodule C is semidualizing if it satisfies the following:

(1) ${}_RC$ admits a degreewise finite R-projective resolution.
(2) C_R admits a degreewise finite R-projective resolution.
(3) The homothety map ${}_RR_R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.
(4) The homothety map ${}_RR_R \to \mathrm{Hom}_{R^{\mathrm{op}}}(C, C)$ is an isomorphism.
(5) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.
(6) $\mathrm{Ext}_{R^{\mathrm{op}}}^{\geqslant 1}(C, C) = 0$.

Let C be a semidualizing (R, R)-bimodule, and set

$$\mathcal{W}_C^T = \{C \otimes_R W \,|\, W \in \mathcal{W}\} \text{ and } \mathcal{W}_C^H = \{\mathrm{Hom}_R(C, W) \,|\, W \in \mathcal{W}\}.$$

The Auslander class of C is the subcategory $\mathcal{A}_C(R)$ of left R-modules M such that

(1) $\mathrm{Tor}^R_{\geq 1}(C, M) = 0 = \mathrm{Ext}_R^{\geq 1}(C, C \otimes_R M)$, and
(2) The natural map $M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The Bass class of C is the subcategory $\mathcal{B}_C(R)$ of left R-modules N such that

(1) $\mathrm{Ext}_R^{\geq 1}(C, N) = 0 = \mathrm{Tor}^R_{\geq 1}(C, \mathrm{Hom}_R(C, N))$, and
(2) The natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Definition 2.2. Let C be a semidualizing (R, R)-bimodule. Then the direct sum $R \oplus C$ can be equipped with the product:

$$(r, c) \cdot (r', c') = (rr', rc' + cr').$$

This turns $R \oplus C$ into a ring which is called the trivial extension of R by C and denoted by $R \ltimes C$. There are canonical ring homomorphisms $R \rightleftarrows R \ltimes C$, which enable us to view R-modules as $R \ltimes C$-modules, and vice versa. This will be used frequently.

Definition 2.3. Let $\mathcal{X}$ be a class of left R-modules. An exact sequence of R-modules is called $\mathrm{Hom}_R(\mathcal{X}, -)$-exact if it remains still exact after applying the functor $\mathrm{Hom}_R(\mathcal{X}, -)$. Let M be a left R-module. An exact sequence $\cdots \to X_1 \to X_0 \to M \to 0$ of R-modules with all X_i in $\mathcal{X}$ is called an $\mathcal{X}$-resolution of M. The above exact sequence is called a proper $\mathcal{X}$-resolution of M if it is an $\mathcal{X}$-resolution of M and is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact. Dually, the notions of a $\mathrm{Hom}_R(-, \mathcal{X})$-exact exact sequence, an $\mathcal{X}$-coresolution and a coproper $\mathcal{X}$-coresolution of M are defined.

Notation 2.4. Throughout this paper, R is an associative ring with identity, R-Mod is the category of left R-modules, $\mathcal{W}$ is an additive full subcategory of R-Mod that is closed under isomorphisms and C is a fixed semidualizing (R, R)-bimodule of R.

We work within the derived category $D(R)$ of the category of left R-modules; cf. e.g. [2], and consistently use the hyper-homological notation from [2, Appendix], in particular we use $\mathrm{RHom}_R(-, -)$ for the right derived Hom functor and $- \otimes_R^{L} -$ for the left derived tensor product functor.

3. $\mathcal{W}$-resolutions and coresolutions with respect to a semidualizing bimodule

In this section, we give a method to construct a proper $\mathcal{W}$-resolution (resp. coproper $\mathcal{W}$-coresolution) of the last (resp. first) term in a short exact sequence from those of the other two terms, and we also give a method to construct a proper $\mathcal{W}_C^H$-resolution (resp. coproper $\mathcal{W}_C^T$-coresolution) of the last (resp. first) term in a short exact sequence from those of the other two terms.

The following result provides a method to construct a proper $\mathcal{W}$-resolution of the last term in a short exact sequence from those of the first two terms.

Theorem 3.1 ([9, Theorem 3.6])**.** *Given a short exact sequence of left R-modules*

$$0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0. \tag{3.1}$$

Assume that

$$W_0^n \to \cdots \to W_0^1 \to W_0^0 \to X_0 \to 0, \tag{3.2}$$

$$W_1^{n-1} \to \cdots \to W_1^1 \to W_1^0 \to X_1 \to 0 \tag{3.3}$$

are proper $\mathcal{W}$-resolutions of X_0 and X_1 respectively.

(1) *If the exact sequence* (3.1) *is* $\mathrm{Hom}_R(\mathcal{W},-)$*-exact, then we get the following proper $\mathcal{W}$-resolution of X*

$$W_0^n \oplus W_1^{n-1} \to \cdots \to W_0^1 \oplus W_1^0 \to W_0^0 \to X \to 0. \tag{3.4}$$

(2) *If both the exact sequences* (3.2), (3.3) *and* (3.1) *are* $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$*-exact, then so is the exact sequence* (3.4).

Corollary 3.2 ([9, Corollary 3.7]). *Given an exact sequence of left R-modules*

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0. \tag{3.5}$$

Assume that

$$W_j^{n-j} \to \cdots \to W_j^1 \to W_j^0 \to X_j \to 0 \tag{3.6(j)}$$

is a proper $\mathcal{W}$-resolution of X_j for $0 \leq j \leq n$. If the exact sequence (3.5) *is* $\mathrm{Hom}_R(\mathcal{W},-)$*-exact, then*

$$\bigoplus_{i=0}^{n} W_i^{n-i} \to \cdots \to W_0^1 \oplus W_1^0 \to W_0^0 \to X \to 0 \tag{3.7}$$

is a proper $\mathcal{W}$-resolution of X. Furthermore, if the exact sequence (3.5) *and all* (3.6(j)) *are* $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$*-exact, then so is the exact sequence* (3.7).

The next two results which are dual to Theorem 3.1 and Corollary 3.2 respectively provide a method to construct a coproper $\mathcal{W}$-coresolution of the first term in a short exact sequence from those of the last two terms.

Theorem 3.3 ([9, Theorem 3.8]). *Given a short exact sequence of left R-modules*

$$0 \longrightarrow Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow 0. \tag{3.8}$$

Assume that

$$0 \to Y^0 \to W_0^0 \to W_1^0 \to \cdots \to W_n^0, \tag{3.9}$$

$$0 \to Y^1 \to W_0^1 \to W_1^1 \to \cdots \to W_{n-1}^1 \tag{3.10}$$

are coproper $\mathcal{W}$-coresolutions of Y^0 and Y^1 respectively.

(1) *If the exact sequence* (3.8) *is* $\mathrm{Hom}_R(-,\mathcal{W})$*-exact, then we get the following coproper $\mathcal{W}$-coresolution of Y*

$$0 \to Y \to W_0^0 \to W_0^1 \oplus W_1^0 \to \cdots \to W_{n-1}^1 \oplus W_n^0. \tag{3.11}$$

(2) *If both the exact sequences* (3.9), (3.10) *and* (3.8) *are* $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$*-exact, then so is the exact sequence* (3.11).

Corollary 3.4 ([9, Corollary 3.9]). *Given an exact sequence of left R-modules*

$$0 \longrightarrow Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^n. \tag{3.12}$$

Assume that

$$0 \to Y^j \to W_0^j \to W_1^j \to \cdots \to W_{n-j}^j \tag{3.13(j)}$$

is a coproper $\mathcal{W}$-coresolution of Y^j for $0 \leq j \leq n$. If the exact sequence (3.12) *is* $\mathrm{Hom}_R(-,\mathcal{W})$*-exact, then*

$$0 \to Y \to W_0^0 \to W_1^0 \oplus W_0^1 \to \cdots \to \bigoplus_{i=0}^{n} W_{n-i}^i \tag{3.14}$$

is a coproper $\mathcal{W}$-coresolution of Y. Furthermore, if the exact sequence (3.12) *and all* (3.13(j)) *are* $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$*-exact, then so is the exact sequence* (3.14).

The following result provides a method to construct a proper $\mathcal{W}_C^H$-resolution of the last term in a short exact sequence from those of the first two terms.

Theorem 3.5 ([9, Theorem 3.6]). *Given a short exact sequence of left R-modules*

$$0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0. \tag{3.15}$$

Assume that

$$\mathrm{Hom}_R(C,W_0^n) \to \cdots \to \mathrm{Hom}_R(C,W_0^1) \to \mathrm{Hom}_R(C,W_0^0) \to X_0 \to 0, \tag{3.16}$$

$$\mathrm{Hom}_R(C,W_1^{n-1}) \to \cdots \to \mathrm{Hom}_R(C,W_1^1) \to \mathrm{Hom}_R(C,W_1^0) \to X_1 \to 0 \tag{3.17}$$

are proper $\mathcal{W}_C^H$-resolutions of X_0 and X_1 respectively.

(1) *If the exact sequence* (3.15) *is* $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$*-exact, then we get the following proper $\mathcal{W}_C^H$-resolution of X*

$$\begin{aligned}\mathrm{Hom}_R(C,W_0^n \oplus W_1^{n-1}) \to \cdots \to \mathrm{Hom}_R(C,W_0^1 \oplus W_1^0)\\ \to \mathrm{Hom}_R(C,W_0^0) \to X \to 0.\end{aligned} \tag{3.18}$$

(2) *If both the exact sequences* (3.16), (3.17) *and* (3.15) *are* $\mathrm{Hom}_R(-,\mathcal{W})$*-exact, then so is the exact sequence* (3.18).

Corollary 3.6 ([9, Corollary 3.7]). *Given an exact sequence of left R-modules*

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0. \tag{3.19}$$

Assume that

$$\mathrm{Hom}_R(C,W_j^{n-j}) \to \cdots \to \mathrm{Hom}_R(C,W_j^1) \to \mathrm{Hom}_R(C,W_j^0) \to X_j \to 0 \tag{3.20(j)}$$

is a proper $\mathcal{W}_C^H$-resolution of X_j for $0 \leq j \leq n$. If the exact sequence (3.19) *is* $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$*-exact, then*

(3.21)

$$\mathrm{Hom}_R(C,\bigoplus_{i=0}^{n} W_i^{n-i}) \to \cdots \to \mathrm{Hom}_R(C,W_0^1 \oplus W_1^0) \to \mathrm{Hom}_R(C,W_0^0) \to X \to 0$$

is a proper $\mathcal{W}_C^H$-resolution of X. Furthermore, if the exact sequence (3.19) *and all* (3.20(j)) *are* $\mathrm{Hom}_R(-,\mathcal{W})$*-exact, then so is the exact sequence* (3.21).

The next two results which are dual to Theorem 3.5 and Corollary 3.6 respectively provide a method to construct a coproper $\mathcal{W}_C^T$-coresolution of the first term in a short exact sequence from those of the last two terms.

Theorem 3.7 ([9, Theorem 3.8]). *Given a short exact sequence of left R-modules*

$$0 \longrightarrow Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow 0. \tag{3.22}$$

Assume that

$$0 \to Y^0 \to C \otimes_R W_0^0 \to C \otimes_R W_1^0 \to \cdots \to C \otimes_R W_n^0, \tag{3.23}$$

$$0 \to Y^1 \to C \otimes_R W_0^1 \to C \otimes_R W_1^1 \to \cdots \to C \otimes_R W_{n-1}^1 \tag{3.24}$$

are coproper $\mathcal{W}_C^T$-coresolutions of Y^0 and Y^1 respectively.

(1) *If the exact sequence* (3.22) *is* $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$*-exact, then we get the following coproper $\mathcal{W}_C^T$-coresolution of Y*

$$0 \to Y \to C \otimes_R W_0^0 \to C \otimes_R (W_0^1 \oplus W_1^0) \to \cdots \to C \otimes_R (W_{n-1}^1 \oplus W_n^0). \tag{3.25}$$

(2) *If both the exact sequences* (3.23), (3.24) *and* (3.22) *are* $\mathrm{Hom}_R(\mathcal{W},-)$*-exact, then so is the exact sequence* (3.25).

Corollary 3.8 ([9, Corollary 3.9]). *Given an exact sequence of left R-modules*

$$0 \longrightarrow Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^n. \tag{3.26}$$

Assume that

$$0 \to Y^j \to C \otimes_R W_0^j \to C \otimes_R W_1^j \to \cdots \to C \otimes_R W_{n-j}^j \tag{3.27(j)}$$

is a coproper $\mathcal{W}_C^T$-coresolution of Y^j for $0 \leq j \leq n$. If the exact sequence (3.26) *is* $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$*-exact, then*

$$0 \to Y \to C \otimes_R W_0^0 \to C \otimes_R (W_1^0 \oplus W_0^1) \to \cdots \to C \otimes_R (\bigoplus_{i=0}^{n} W_{n-i}^i) \tag{3.28}$$

is a coproper $\mathcal{W}_C^T$-coresolution of Y. Furthermore, if the exact sequence (3.26) *and all* (3.27(j)) *are* $\mathrm{Hom}_R(\mathcal{W},-)$*-exact, then so is the exact sequence* (3.28).

4. Stability of Gorenstein categories with respect to a semidualizing bimodule

In this section, we give some applications of the results in the above section. We introduce and show the stability of the Gorenstein categories $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ with respect to a semidualizing bimodule C, and study the 2-out-of-3 property of these categories of a short exact sequence in R-Mod. Also we prove how they are related to the Gorenstein categories $\mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$ and $\mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$ over $R \ltimes C$.

We begin this section with the following definition.

Definition 4.1. A complete $\mathcal{W}\mathcal{W}_C^T$-resolution is a both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact exact sequence of left R-modules

$$X: \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow C \otimes_R W^0 \longrightarrow C \otimes_R W^1 \longrightarrow \cdots$$

with all W_i and W^i in $\mathcal{W}$, in which case X is a complete $\mathcal{W}\mathcal{W}_C^T$-resolution of $\mathrm{Im}(W_0 \to C \otimes_R W^0)$. The Gorenstein subcategory $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ of R-Mod is defined as

$$\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T) = \{G \in R\text{-Mod} \mid G \text{ admits a complete } \mathcal{W}\mathcal{W}_C^T\text{-resolution}\}.$$

A complete $\mathcal{W}_C^H\mathcal{W}$-resolution is a both $\mathrm{Hom}_R(\mathcal{W}_C^H, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W})$-exact exact sequence of left R-modules

$$Y: \cdots \longrightarrow \mathrm{Hom}_R(C, W_1) \longrightarrow \mathrm{Hom}_R(C, W_0) \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

with all W_i and W^i in $\mathcal{W}$, in which case Y is a complete $\mathcal{W}_C^H\mathcal{W}$-resolution of $\mathrm{Im}(\mathrm{Hom}_R(C, W_0) \to W^0)$. The Gorenstein subcategory $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ of R-Mod is defined as

$$\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W}) = \{G \in R\text{-Mod} \mid G \text{ admits a complete } \mathcal{W}_C^H\mathcal{W}\text{-resolution}\}.$$

Remark 4.2. If $\mathcal{W}$ is the class of projective (resp. injective) left R-modules, then Gorenstein category $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ (resp. $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$) coincides with the class of G_C projective (resp. G_C injective) left R-modules [11].

Definition 4.3. Let $\mathcal{X}$ be a class of left $(R \ltimes C)$-modules and M a left $(R \ltimes C)$-module. A complete $\mathcal{X}$-resolution of M over $R \ltimes C$ is a both $\mathrm{Hom}_{R\ltimes C}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{R\ltimes C}(-, \mathcal{X})$-exact exact sequence

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

of left $(R \ltimes C)$-modules in $\mathcal{X}$ such that $M = \mathrm{Im}(X_0 \to X^0)$. The Gorenstein subcategory $\mathcal{G}(\mathcal{X})_C$ of $(R \ltimes C)$-Mod is defined as

$$\mathcal{G}(\mathcal{X})_C = \{G \in (R \ltimes C)\text{-Mod} | G \text{ admits a complete } \mathcal{X}\text{-resolution over } R \ltimes C\}.$$

As an application of the results in the above section, we get the following result.

Theorem 4.4. *Let $M \in R$-Mod. If $\mathcal{W}$ is closed under countable direct sums, then*

(1) *$M \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ if and only if there exists a both* $\mathrm{Hom}_R(\mathcal{W}, -)$*-exact and* $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$*-exact exact sequence*

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

of left R-modules in $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ such that $M = \mathrm{Im}(G_0 \to G^0)$.

(2) *$M \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ if and only if there exists a both* $\mathrm{Hom}_R(\mathcal{W}_C^H, -)$*-exact and* $\mathrm{Hom}_R(-, \mathcal{W})$*-exact exact sequence*

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

of left R-modules in $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ such that $M = \mathrm{Im}(G_0 \to G^0)$.

Proof. We just prove the first statement since the second is proved dually.

(1) The "only if" part is clear.

"If" part. By assumption, there is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences $\cdots \to G_1 \to G_0 \to M \to 0$ and $0 \to M \to G^0 \to G^1 \to \cdots$. Then for any $j \geq 0$, there exist both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences:

$$\cdots \longrightarrow W_j^i \longrightarrow \cdots \longrightarrow W_j^1 \longrightarrow W_j^0 \longrightarrow G_j \longrightarrow 0,$$

$$0 \longrightarrow G^j \longrightarrow C \otimes_R V_0^j \longrightarrow C \otimes_R V_1^j \longrightarrow \cdots \longrightarrow C \otimes_R V_i^j \longrightarrow \cdots$$

with all W_j^i and V_i^j in $\mathcal{W}$. By Corollaries 3.2 and 3.8, we get exact sequences:

$$\cdots \longrightarrow \textstyle\bigoplus_{i=0}^n W_i^{n-i} \longrightarrow \cdots \longrightarrow W_0^1 \oplus W_1^0 \longrightarrow W_0^0 \to M \longrightarrow 0,$$

$$0 \longrightarrow M \longrightarrow C \otimes_R V_0^0 \longrightarrow C \otimes_R (V_1^0 \oplus V_0^1) \longrightarrow \cdots \longrightarrow C \otimes_R (\textstyle\bigoplus_{i=0}^n V_{n-i}^i) \longrightarrow \cdots$$

which are both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. So

$$\cdots \longrightarrow \textstyle\bigoplus_{i=0}^n W_i^{n-i} \longrightarrow \cdots \longrightarrow W_0^1 \oplus W_1^0 \longrightarrow W_0^0 \longrightarrow C \otimes_R V_0^0 \longrightarrow C \otimes_R (V_1^0 \oplus V_0^1) \longrightarrow \cdots \longrightarrow C \otimes_R (\textstyle\bigoplus_{i=0}^n V_{n-i}^i) \longrightarrow \cdots$$

is a complete $\mathcal{W}\mathcal{W}_C^T$-resolution of M, and hence $M \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. □

Theorem 4.5. (1) *If $\mathcal{W}$ is closed under countable direct sums, then $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ are closed under direct summands.*

Given a short exact sequence of left R-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0. \tag{4.1}$$

(2) *Assume the sequence* (4.1) *is both* $\mathrm{Hom}_R(\mathcal{W},-)$*-exact and* $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$*-exact. If any two of X, Y and Z are in $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$, then the third term is also in $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.*

(3) *Assume the sequence* (4.1) *is both* $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$*-exact and* $\mathrm{Hom}_R(-,\mathcal{W})$*-exact. If any two of X, Y and Z are in $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$, then the third term is also in $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$.*

Proof. (1) We just prove the first statement since the second is proved dually.

Let $X_1 \oplus X_2 = X \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\cdots \to W_1 \to W_0 \to C \otimes_R W^0 \to C \otimes_R W^1 \to \cdots$ be a complete $\mathcal{W}\mathcal{W}_C^T$-resolution of X. Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_1 & = & K_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & D & \longrightarrow & W_0 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Applying to the above diagram the functors $\mathrm{Hom}_R(W,-), \mathrm{Hom}_R(-, C\otimes_R W)$ for any $W \in \mathcal{W}$, a simple diagram chasing argument shows that the middle row is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. Similarly, we have an exact sequence $0 \to D' \to W_0 \to X_2 \to 0$ which is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. Consider the exact sequence $0 \to X_i \to X \to X_j \to 0$ for $i, j = 1, 2$. By Theorem 3.1, we get both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences $W_0 \oplus W_1 \to W_0 \to X_1 \to 0$ and $W_0 \oplus W_1 \to W_0 \to X_2 \to 0$. Again by Theorem 3.1, we get both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences $W_0 \oplus W_1 \oplus W_2 \to W_0 \oplus W_1 \to W_0 \to X_1 \to 0$ and $W_0 \oplus W_1 \oplus W_2 \to W_0 \oplus W_1 \to W_0 \to X_2 \to 0$. Continuing this process, we finally get the following both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences

$$\cdots \longrightarrow W_0 \oplus W_1 \oplus W_2 \longrightarrow W_0 \oplus W_1 \longrightarrow W_0 \longrightarrow X_1 \longrightarrow 0,$$

$$\cdots \longrightarrow W_0 \oplus W_1 \oplus W_2 \longrightarrow W_0 \oplus W_1 \longrightarrow W_0 \longrightarrow X_2 \longrightarrow 0.$$

Dually, by Theorem 3.7, we get the following both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences

$$0 \longrightarrow X_1 \longrightarrow C\otimes_R W^0 \longrightarrow C\otimes_R (W^0 \oplus W^1) \longrightarrow C\otimes_R (W^0 \oplus W^1 \oplus W^2) \longrightarrow \cdots,$$

$$0 \longrightarrow X_2 \longrightarrow C\otimes_R W^0 \longrightarrow C\otimes_R (W^0 \oplus W^1) \longrightarrow C\otimes_R (W^0 \oplus W^1 \oplus W^2) \longrightarrow \cdots.$$

Consequently, X_1 and X_2 are in $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

We just prove the statement (2) since the statement (3) is proved dually.

(2) First assume that $X, Z \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. There exist complete $\mathcal{W}\mathcal{W}_C^T$-resolutions of X and Z respectively,

$$\cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow C \otimes_R W^0 \longrightarrow C \otimes_R W^1 \longrightarrow \cdots,$$
$$\cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow C \otimes_R V^0 \longrightarrow C \otimes_R V^1 \longrightarrow \cdots.$$

Consider the both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequences $0 \to X_1 \to W_0 \to X \to 0$ and $0 \to Z_1 \to V_0 \to Z \to 0$. By assumption and [9, 3.1(1)], we get a commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_0 & \longrightarrow & W_0 \oplus V_0 & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By assumption, both the first and third columns and the second and third rows in the above diagram are both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact, we have the middle column and the first row in this diagram are also both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. Consider the both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact exact sequences $0 \to X \to C \otimes_R W^0 \to X^1 \to 0$ and $0 \to Z \to C \otimes_R V^0 \to Z^1 \to 0$. By assumption and [9, 3.1(2)], we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C \otimes_R W^0 & \longrightarrow & C \otimes_R (W^0 \oplus V^0) & \longrightarrow & C \otimes_R V^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^1 & \longrightarrow & Y^1 & \longrightarrow & Z^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By assumption, both the first and third columns and the first and second rows in the above diagram are both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact, so the middle column and the third row in this diagram are both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact. Continuing this process, we get that

$$\cdots \longrightarrow W_1 \oplus V_1 \longrightarrow W_0 \oplus V_0 \longrightarrow C \otimes_R (W^0 \oplus V^0) \longrightarrow C \otimes_R (W^1 \oplus V^1) \longrightarrow \cdots$$

is a complete $\mathcal{W}\mathcal{W}_C^T$-resolution of Y and $Y \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

Next assume that $Y, Z \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. Then Theorem 3.7 implies that X has a coproper $\mathcal{W}_C^T$-coresolution which is $\mathrm{Hom}_R(\mathcal{W}, -)$-exact. Consider the both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact short exact sequence $0 \to Y_1 \to W_0 \to Y \to 0$ with $W_0 \in \mathcal{W}$. We have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_1 & = & Y_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Z' & \to & W_0 & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It is easy to verify that the middle row and the first column in the above diagram are both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact. Consider the both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact short exact sequence $0 \to Z_1 \to V_0 \to Z \to 0$ with $V_0 \in \mathcal{W}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & Z_1 & \to & V_0 & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Z' & \to & W_0 & \to & Z & \to & 0
\end{array}$$

Then we have a both $\mathrm{Hom}_R(\mathcal{W}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact exact sequence $0 \to Z_1 \to Z' \oplus V_0 \to W_0 \to 0$, and so $Z' \oplus V_0$ has a proper $\mathcal{W}$-resolution that is $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact by the preceding proof. Consequently Z' has a proper $\mathcal{W}$-resolution that is $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact. Now by Theorem 3.1, we get that X has a proper $\mathcal{W}$-resolution which is $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact. It follows that $X \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

Finally assume that $X, Y \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. Then Theorem 3.1 implies that Z has a proper $\mathcal{W}$-resolution which is $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact. Consider the both $\mathrm{Hom}_R(-, \mathcal{W}_C^T)$-exact and $\mathrm{Hom}_R(\mathcal{W}, -)$-exact short exact sequence $0 \to Y \to C \otimes_R V^0 \to Y^1 \to 0$ with $V^0 \in \mathcal{W}$. We have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & C \otimes_R V^0 & \to & X' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y^1 & = & Y^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is easy to verify that the middle row and the third column in the above diagram are both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. Consider the both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact short exact sequence $0 \to X \to C\otimes_R W^0 \to X^1 \to 0$ with $W^0 \in \mathcal{W}$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & C\otimes_R W^0 & \longrightarrow & X' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & C\otimes_R V^0 & \longrightarrow & X^1 & \longrightarrow & 0 \end{array}$$

Then we have a both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequence $0 \to C\otimes_R W^0 \to X'\oplus(C\otimes_R V^0) \to X^1 \to 0$, and so $X'\oplus(C\otimes_R V^0)$ has a coproper $\mathcal{W}_C^T$-coresolution that is $\mathrm{Hom}_R(\mathcal{W},-)$-exact by the preceding proof. Consequently X' has a coproper $\mathcal{W}_C^T$-coresolution that is $\mathrm{Hom}_R(\mathcal{W},-)$-exact. Now by Theorem 3.7, we get that Z has a coproper $\mathcal{W}_C^T$-coresolution which is $\mathrm{Hom}_R(\mathcal{W},-)$-exact. It follows that $Z \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. □

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of R-Mod. Write $\mathcal{X}\perp\mathcal{Y}$ if $\mathrm{Ext}_R^{\geq 1}(X,Y) = 0$ for each $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$.

Lemma 4.6. (1) *Let $M \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. If $\mathcal{W}\perp\mathcal{W}_C^T$, then $\mathrm{Ext}_R^{\geq 1}(W,M) = 0 = \mathrm{Ext}_R^{\geq 1}(M, C\otimes_R W)$ for any $W\in\mathcal{W}$.*

(2) *Let $N \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$. If $\mathcal{W}_C^H\perp\mathcal{W}$, then $\mathrm{Ext}_R^{\geq 1}(\mathrm{Hom}_R(C,W),N) = 0 = \mathrm{Ext}_R^{\geq 1}(N,W)$ for any $W\in\mathcal{W}$.*

Proof. It is a standard homological algebra fare. □

Corollary 4.7. *Given a short exact sequence of left R-modules*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0. \qquad (4.2)$$

(1) *Assume that $\mathcal{W}\perp\mathcal{W}_C^T$. If the exact sequence* (4.2) *is $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact and $X \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$, then $Y \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ if and only if $Z \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.*

(2) *Assume that $\mathcal{W}\perp\mathcal{W}_C^T$. If the exact sequence* (4.2) *is $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $Z \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$, then $X \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ if and only if $Y \in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.*

(3) *Assume that $\mathcal{W}_C^H\perp\mathcal{W}$. If the exact sequence* (4.2) *is $\mathrm{Hom}_R(\mathcal{W}_C^H,-)$-exact and $Z \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$, then $X \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ if and only if $Y \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$.*

(4) *Assume that $\mathcal{W}_C^H\perp\mathcal{W}$. If the exact sequence* (4.2) *is $\mathrm{Hom}_R(-,\mathcal{W})$-exact and $X \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$, then $Y \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ if and only if $Z \in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$.*

Lemma 4.8. *Let $W\in\mathcal{W}$.*

(1) *If $\mathcal{W}\subseteq\mathcal{A}_C(R)$, then we have a natural equivalence in $D(R\ltimes C)$:*

$$\mathrm{RHom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R W) \simeq \mathrm{RHom}_R(-,C\otimes_R W).$$

(2) *If $\mathcal{W}\subseteq\mathcal{B}_C(R)$, then we have a natural equivalence in $D(R\ltimes C)$:*

$$\mathrm{RHom}_{R\ltimes C}(\mathrm{Hom}_R(R\ltimes C,W),-) \simeq \mathrm{RHom}_R(\mathrm{Hom}_R(C,W),-).$$

Proof. We just prove (1) since (2) is proved dually.

Let $P_\centerdot \to W$ be a projective resolution of W over R. Then there exists an isomorphism $C \otimes_R P_\centerdot \cong C \otimes_R W$ in $D(R \ltimes C)$ by assumption. Consider the exact sequence $0 \to C \to R \ltimes C \to R \to 0$. If viewed as a sequence of left R-modules, then it is split. So $0 \to C \otimes_R W \to (R \ltimes C) \otimes_R W \to W \to 0$ is exact, which gives that $(R \ltimes C) \otimes_R P_\centerdot \cong (R \ltimes C) \otimes_R W$ in $D(R \ltimes C)$. This is a computation,

$$\begin{aligned}\mathrm{RHom}_R(-, C \otimes_R W) &\overset{a}{\simeq} \mathrm{RHom}_R(-, \mathrm{RHom}_{R\ltimes C}(R, R \ltimes C) \otimes^L_R W)\\ &\overset{b}{\simeq} \mathrm{RHom}_R(-, \mathrm{RHom}_{R\ltimes C}(R, (R \ltimes C) \otimes^L_R W))\\ &\overset{c}{\simeq} \mathrm{RHom}_{R\ltimes C}(-, (R \ltimes C) \otimes^L_R W)\\ &\simeq \mathrm{RHom}_{R\ltimes C}(-, (R \ltimes C) \otimes_R W).\end{aligned}$$

where (a) is by [7, 1.3(4)], (b) is by [2, A.4.23] and (c) is by adjunction. □

Lemma 4.9. (1) *Let M be a left R-module which is in $\mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$. If $\mathcal{W} \subseteq \mathcal{A}_C(R) \cap \mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$, then there exist both* $\mathrm{Hom}_R(\mathcal{W}, -)$*-exact and* $\mathrm{Hom}_R(-, \mathcal{W}^T_C)$*-exact short exact sequences of left R-modules*

(a) $$0 \longrightarrow M \longrightarrow C \otimes_R W \longrightarrow M' \longrightarrow 0,$$

(b) $$0 \longrightarrow M'' \longrightarrow W' \longrightarrow M \longrightarrow 0,$$

where $W, W' \in \mathcal{W}$ and $M', M'' \in \mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$.

(2) *Let N be a left R-module which is in $\mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$. If $\mathcal{W} \subseteq \mathcal{B}_C(R) \cap \mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$, then there exist both* $\mathrm{Hom}_R(\mathcal{W}^H_C, -)$*-exact and* $\mathrm{Hom}_R(-, \mathcal{W})$*-exact short exact sequences of left R-modules*

(a') $$0 \longrightarrow N' \longrightarrow \mathrm{Hom}_R(C, V) \longrightarrow N \longrightarrow 0,$$

(b') $$0 \longrightarrow N \longrightarrow V' \longrightarrow N'' \longrightarrow 0,$$

where $V, V' \in \mathcal{W}$ and $N', N'' \in \mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$.

Proof. We just prove (1) since (2) is proved dually.

(a) By assumption, there is a both $\mathrm{Hom}_{R\ltimes C}((R \ltimes C) \otimes_R \mathcal{W}, -)$-exact and $\mathrm{Hom}_{R\ltimes C}(-, (R \ltimes C) \otimes_R \mathcal{W})$-exact exact sequence $0 \to M \to (R \ltimes C) \otimes_R W \to L \to 0$, where $W \in \mathcal{W}$ and $L \in \mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$. Applying the functor $- \otimes_R W$ to the exact sequence $0 \to C \to R \ltimes C \to R \to 0$, we get an exact sequence of left $(R \ltimes C)$-modules $0 \to C \otimes_R W \to (R \ltimes C) \otimes_R W \to W \to 0$. If viewed as a sequence of left R-modules, then it is split. So we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C\otimes_R W & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & (R\ltimes C)\otimes_R W & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Note that $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)\cong \mathrm{Hom}_R(\mathcal{W},-)$ and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R \mathcal{W})\cong \mathrm{Hom}_R(-,\mathcal{W}_C^T)$ by Lemma 4.8. Since the middle column in the above diagram is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact, it is both $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$-exact and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$-exact, and so the first row and the third column in the above diagram are both $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$-exact and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$-exact. Hence the first row in this diagram is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact by Lemma 4.8. Note that $W,L\in\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$. Then $M'\in\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$ by [9, 4.7]. Thus the first row in the above diagram is the desired exact sequence.

(b) By assumption, there is a both $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$-exact and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$-exact exact sequence $0\to L'\to(R\ltimes C)\otimes_R W'\to M\to 0$, where $W'\in\mathcal{W}$ and $L'\in\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$. So we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C\otimes_R W' & = & C\otimes_R W' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L' & \longrightarrow & (R\ltimes C)\otimes_R W' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M'' & \longrightarrow & W' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since the middle column in the above diagram is both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact, it is both $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$-exact and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$-exact by Lemma 4.8, and so the third row and the first column in the above diagram are both $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)$-exact and $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R \mathcal{W})$-exact. Hence the third row and the first column in this diagram are both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact by Lemma 4.8. Note that $(C\otimes_R W')\oplus M''\cong L'$. Then $M''\in\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$ by [9, 4.6]. Thus the third row in the above diagram is the desired short exact sequence. $\square$

Lemma 4.10. (1) *Let M be a left R-module which is in $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. If $\mathcal{W}\subseteq\mathcal{A}_C(R)\cap\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$, then there is both* $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$*-exact and* $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$*-exact exact sequences of left $(R\ltimes C)$-modules*

(a) $$0\longrightarrow M\longrightarrow (R\ltimes C)\otimes_R W\longrightarrow M'\longrightarrow 0,$$

(b) $$0\longrightarrow M''\longrightarrow (R\ltimes C)\otimes_R W'\longrightarrow M\longrightarrow 0,$$

where $W,W'\in\mathcal{W}$ and $M',M''\in\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

(2) *Let N be a left R-module which is in $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$. If $\mathcal{W}\subseteq\mathcal{B}_C(R)\cap\mathcal{G}(\mathrm{Hom}_R(R\ltimes C,\mathcal{W}))_C$, then there is both* $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R\mathcal{W},-)$*-exact and* $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$*-exact exact sequences of left $(R\ltimes C)$-modules*

(a') $$0\longrightarrow N'\longrightarrow \mathrm{Hom}_R(R\ltimes C,V)\longrightarrow N\longrightarrow 0,$$

(b') $$0\longrightarrow N\longrightarrow \mathrm{Hom}_R(R\ltimes C,V')\longrightarrow N''\longrightarrow 0,$$

where $V,V'\in\mathcal{W}$ and $N',N''\in\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$.

Proof. Again we will just prove (1) since (2) is proved dually.

By Lemma 4.9, we get $(R\ltimes C)\otimes_R\mathcal{W}\subseteq\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. Thus $\mathcal{W}\subseteq\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ and $\mathcal{W}_C^T\subseteq\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

(a) By assumption, there is a both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequence $0\to M\to C\otimes_R W\to L\to 0$ with $W\in\mathcal{W}$. Since $\mathcal{W}_C^T\subseteq\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$, $L\in\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ by Theorem 4.5. Applying the functor $-\otimes_R W$ to the split exact sequence $0\to C\to R\ltimes C\to R\to 0$, we get an exact sequence of left $(R\ltimes C)$-modules $0\to C\otimes_R W\to(R\ltimes C)\otimes_R W\to W\to 0$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C\otimes_R W & \longrightarrow & L & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & (R\ltimes C)\otimes_R W & \longrightarrow & M' & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We will prove that the middle row here has the properties claimed in the lemma. It is easy to verify that the middle row and the third column in the above diagram are $\mathrm{Hom}_R(\mathcal{W},-)$-exact. Note that $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)\cong \mathrm{Hom}_R(\mathcal{W},-)$. Then the middle row is also $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)$-exact and $M'\cong L\oplus W\in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. Finally by construction, the middle row in this diagram is $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. By Lemma 4.8, we see that the middle row is also $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R \mathcal{W})$-exact.

(b) By assumption, there is a both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact exact sequence $0\to L'\to W'\to M\to 0$ with $W'\in\mathcal{W}$. Since $\mathcal{W}\subseteq \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$, $L'\in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ by Theorem 4.5. Applying the functor $-\otimes_R W'$ to the split exact sequence $0\to C\to R\ltimes C\to R\to 0$, we get an exact sequence of left $(R\ltimes C)$-modules $0\to C\otimes_R W'\to (R\ltimes C)\otimes_R W'\to W'\to 0$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C\otimes_R W' & = & C\otimes_R W' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M'' & \to & (R\ltimes C)\otimes_R W' & \to & M & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & L' & \to & W' & \to & M & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We will prove that the middle row here has the properties claimed in the lemma. It is easy to verify that the first column and the middle row in the above diagram are both $\mathrm{Hom}_R(\mathcal{W},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{W}_C^T)$-exact. Note that $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)\cong \mathrm{Hom}_R(\mathcal{W},-)$. Then the middle row is also $\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathcal{W},-)$-exact. By Lemma 4.8, the middle row is also $\mathrm{Hom}_{R\ltimes C}(-,(R\ltimes C)\otimes_R\mathcal{W})$-exact and $M''\cong L'\oplus(C\otimes_R W')\in \mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$. $\square$

By the above arguments, we get the following assertions.

Theorem 4.11. (1) *Let $M\in R$-Mod. If $\mathcal{W}\subseteq \mathcal{A}_C(R)\cap\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$, then $M\in\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$ if and only if $M\in\mathcal{G}((R\ltimes C)\otimes_R\mathcal{W})_C$.*

(2) *Let $N\in R$-Mod. If $\mathcal{W}\subseteq\mathcal{B}_C(R)\cap\mathcal{G}(\mathrm{Hom}_R(R\ltimes C,\mathcal{W}))_C$, then $N\in \mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$ if and only if $N\in\mathcal{G}(\mathrm{Hom}_R(R\ltimes C,\mathcal{W}))_C$.*

Corollary 4.12 ([7, 2.13])**.** *For any left R-module M, we have*

(1) *M is G_C projective if and only if M is Gorenstein projective over $R\ltimes C$.*

(2) *M is G_C injective if and only if M is Gorenstein injective over $R\ltimes C$.*

Proposition 4.13. (1) *Let* $\mathcal{W} \subseteq \mathcal{A}_C(R) \cap \mathcal{G}((R \ltimes C) \otimes_R \mathcal{W})_C$. *If* X *is a complete* $\mathcal{W}\mathcal{W}_C^T$*-resolution, then all the images, the kernels and the cokernels of* X *are in* $\mathcal{G}_C(\mathcal{W}\mathcal{W}_C^T)$.

(2) *Let* $\mathcal{W} \subseteq \mathcal{B}_C(R) \cap \mathcal{G}(\mathrm{Hom}_R(R \ltimes C, \mathcal{W}))_C$. *If* Y *is a complete* $\mathcal{W}_C^H\mathcal{W}$*-resolution. Then all the images, the kernels and the cokernels of* Y *are in* $\mathcal{G}_C(\mathcal{W}_C^H\mathcal{W})$.

Proof. It follows from Theorem 4.5 and Lemma 4.9. □

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