J. Appl. Math. & Informatics Vol. 33(2015), No. 5 - 6, pp. 723 - 738
http://dx.doi.org/10.14317/jami.2015.723

HOMOCLINIC SOLUTIONS FOR A PRESCRIBED MEAN CURVATURE RAYLEIGH p-LAPLACIAN EQUATION WITH A DEVIATING ARGUMENT†

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Abstract. In this paper, the prescribed mean curvature Rayleigh p-Laplacian equation with a deviating argument

$$\Big(\varphi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})\Big)' + f(u'(t)) + g(t, u(t-\tau(t))) = e(t)$$

is studied. By using Mawhin's continuation theorem and some analysis methods, we obtain the existence of a set with $2kT$-periodic solutions for this equation and then a homoclinic solution is obtained as a limit of a certain subsequence of the above set.

AMS Mathematics Subject Classification : 34C37.
Key words and phrases : Homoclinic solution, Continuation theorem, Prescribed mean curvature.

1. Introduction

In resent years, The existence of homoclinic solutions have been studied widely, especially for the Hamiltonian systems and the p-Laplacian systems(see [1-4]). For example, in [1], Lzydorek, M and Janczewska, J studied the homoclinic solutions for a class of the second order Hamiltonian systems as the following form

$$\ddot{q} + V_q(t, q) = f(t)$$

where $q \in R^n$ and $V \in C^1(R \times R^n, R)$, $V(t, q) = -K(t, q) + W(t, q)$ is T-periodic in t. And in [4], Lu, SP studied the homoclinic solutions for a class of second-order p-Laplacian differential systems with delay of the form

$$\frac{d}{dt}[\varphi_p(u'(t))] + \frac{d}{dt}\nabla F(u(t)) + \nabla G(u(t)) + \nabla H(u(t-\gamma(t))) = e(t)$$

Received January 10, 2015. Revised January 26, 2015. Accepted January 30, 2015. †This work was supported by the graduate innovation project of Anhui normal university. (No. 2014yks059).

Nowadays, the prescribed mean curvature equation and its modified forms, which arises from some problems associated with differential geometry and physics such as combustible gas dynamics [5-7] have been studied widely. As researchers continue to study the prescribed mean curvature equation, the existence of the periodic solutions for the prescribed curvature mean equation attracts researchers' attention and there are many papers about the existence of the periodic solutions for the prescribed curvature mean equation. For example, in [11], Feng discussed the existence of periodic solutions of a delay prescribed mean curvature Liénard equation of the form

$$(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})' + f(u(t))u'(t) + g(t, u(t-\tau(t))) = e(t)$$

and in [12], Jin Li discussed the existence of periodic solutions for a prescribed mean curvature Rayleigh equation of the form

$$(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})' + f(t, u'(t)) + g(t, u(t-\tau(t))) = e(t)$$

As is well known, a solution $u(t)$ of Eq.(1.1) is named homoclinic (to 0) if $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to +\infty$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

In [13], Liang and Lu studied the homoclinic solution for the prescribed mean curvature Duffing-type equation of the form

$$(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})' + cu'(t) + f(u(t)) = p(t)$$

where $f \in C^1(R, R)$, $p \in C(R, R)$, $c > 0$ is a given constant.

Recently, in [14], Wang studied the periodic solution for the following prescribed mean curvature Rayleigh equation with a deviating argument of the form:

$$\begin{cases} \left(\varphi_p(\frac{x'(t)}{\sqrt{1+(x'(t))^2}})\right)' + f(t, x'(t)) + g(t, x(t-\tau(t))) = e(t) \\ x_1(0) = x_1(\omega),\ x_2(0) = x_2(\omega) \end{cases} \quad (1)$$

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $g \in C(R^2, R)$, $e,\ \tau \in C(R, R)$, $g(t+\omega, x) = g(t, x)$, $f(t+\omega, x) = f(t, x)$, $f(t, 0) = 0$, $e(t+\omega) = e(t)$ and $\tau(t+\omega) = \tau(t)$. Under the assumptions:

$$f(t, x) \geq a|x|^r,\ \forall (t, x) \in R^2$$

and

$$g(t, x) - e(t) \geq -m_1|x| - m_2,\ \forall t \in R,\ x > d.$$

where a, $r \geq 1$; m_1 and m_2 are positive constants. Through the transformation, (1) is equivalent to the system

$$\begin{cases} x_1'(t) = \phi(v(t)) = \frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}} \\ x_2'(t) = -f(t, \frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}) - g(t, u(t-\tau(t))) + e(t), \\ x_1(0) = x_1(\omega), \ x_2(0) = x_2(\omega) \end{cases} \tag{2}$$

By using Mawhin's continuation theorem and given some sufficient conditions, the authors obtained that Eq.(1) has at least one periodic solution.

However, to the best of our knowledge, there are no papers about the studying of the homoclinic solutions for the prescribed mean curvature Rayleigh p-Laplacian equation. In order to solve this problem, in this paper, we consider the following the prescribed mean curvature Rayleigh p-Laplacian equation with a deviating argument

$$\Big(\varphi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})\Big)' + f(u'(t)) + g(t, u(t-\tau(t))) = e(t) \tag{3}$$

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $f \in C(R, R)$, $g \in C(R^2, R)$ and g is T-periodic in the first argument. $e(t)$, $\tau(t)$ are continuous T-periodic function and $T > 0$ is a given constant.

In order to study the homoclinic solution for Eq.(3), firstly, like in the work of Lzydorek and Janczewska in [1], Rabinowitz in [2], X. H. Tang and Li Xiao in [3] and Lu in [4], the existence of a homoclinic solution for Eq.(3) is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following equation:

$$\Big(\varphi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})\Big)' + f(u'(t)) + g(t, u(t-\tau(t))) = e_k(t) \tag{4}$$

where $k \in N$, $e_k : R \to R$ is a $2kT$-periodic function such that

$$e_k(t) = \begin{cases} e(t) & t \in [-kT, kT-\varepsilon_0) \\ e(kT-\varepsilon_0) + \frac{e(-kT)-e(kT-\varepsilon_0)}{\varepsilon_0}(t-kT+\varepsilon_0), & t \in [kT-\varepsilon_0, kT] \end{cases} \tag{5}$$

where $\varepsilon_0 \in (0, T)$ is a constant independent of k. In our approach,the existence of $2kT$-periodic solutions to Eq.(4) is obtained by applying Mawhin's continuation theorem [16].

The structure of the rest of this paper is as follows: Section 2, we state some necessary definitions and lemmas. Section 3, we prove the main result.

2. Preliminary

Throughout this paper, $|\cdot|$ will denote the absolute value and the Euclidean norm on R. For each $k \in N$, let $C_{2kT} = \{u | u \in C(R, R), u(t+2kT) = u(t)\}$, $C^1_{2kT} = \{u | u \in C^1(R, R), u(t+2kT) = u(t)\}$ and $||u||_0 = \max\limits_{t\in[0,2kT]} |u(t)|$. If the norms of C_{2kT} and C^1_{2kT} are defined by $||\cdot||_{C_k} = |\cdot|_0$ and $||x||_{C^1_{2kT}} =$

$\max\{|x|_0, |x'|_0\}$, respectively, then C_{2kT} and C^1_{2kT} are all Banach spaces. Furthermore, for $\phi \in C_{2kT}$, $||\phi||_r = (\int_{-kT}^{kT} |\phi(t)|^r dt)^{\frac{1}{r}}$, where $r \in (1, +\infty)$.

In order to use Mawhin's continuation theorem, we first recall it.
Let X and Y be two Banach spaces, a linear operator $L : D(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that
(a) ImL is a closed subset of Y,
(b) $dimKerL = codimImL < \infty$.
Let X and Y be two Banach spaces, $\Omega \subset X$ be an open and bounded set, and $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero, and continuous operator $N : \Omega \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$ provided that
(c) $K_p(I - Q)N(\bar{\Omega})$ is a relative compact set of X,
(d) $QN(\bar{\Omega})$ is a bounded set of Y,
where we denote $X_1 = KerL, Y_2 = ImL$, then we have the decompositions $X = X_1 \bigoplus X_2$, $Y = Y_1 \bigoplus Y_2$, let $P : X \to X_1$, $Q : Y \to Y_1$ are continuous linear projectors(meaning $P^2 = P$ and $Q^2 = Q$), and $K_p = L \mid^{-1}_{KerP \cap D(L)}$.

Lemma 2.1 (16). *Let X and Y be two real Banach spaces, and Ω is an open and bounded set of X, and $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero and the operator $N : \bar{\Omega} \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$. In addition, if the following conditions hold:*
(h_1) $Lx \neq \lambda Nx, \forall (x, \lambda) \in \partial\Omega \times (0, 1)$;
(h_2) $QNx \neq 0, \forall x \in KerL \cap \partial\Omega$;
(h_3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$, where $J : ImQ \to KerL$ is a homeomorphism, then $Lx = Nx$ has at least one solution in $D(L) \cap \bar{\Omega}$.

Lemma 2.2 ([4]). *Let $s \in C(R, R)$ with $s(t+\omega) \equiv s(t)$ and $s(t) \in [0, \omega]$, $\forall t \in R$. Suppose $p \in (1, +\infty)$, $\alpha = \max_{t\in[0,\omega]} s(t)$ and $u \in C^1(R, R)$ with $u(t + \omega) = u(t)$. Then*

$$\int_0^\omega |u(t) - u(t - s(t))|^p dt \leq \alpha^p \int_0^\omega |u'(t)|^p dt$$

Lemma 2.3. *If $u : R \to R$ is continuously differentiable on R, $a > 0$, $\mu > 1$ and $p > 1$ are constants, then for every $t \in R$, the following inequality holds*

$$|u(t)| \leq (2a)^{-\frac{1}{\mu}} (\int_{t-a}^{t+a} |u(s)|^\mu ds)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} (\int_{t-a}^{t+a} |u'(s)|^p ds)^{\frac{1}{p}}$$

In order to study the existence of $2kT$-periodic solutions for Eq.(1.2), for each $k \in N$, from (1.3) we observe that $e_k \in C_{2kT}$. Let $X_k = C^1_{2kT}$.

Lemma 2.4 ([18]). *Suppose $\tau \in C^1(R, R)$ with $\tau(t + \omega) \equiv \tau(t)$ and $\tau'(t) < 1$, $\forall t \in [0, \omega]$. Then the function $t - \tau(t)$ has an inverse $\mu(t)$ satisfying $\mu \in C(R, R)$ with $\mu(t + \omega) \equiv \mu(t) + \omega$, $\forall t \in [0, \omega]$.*

Throughout this paper, besides τ being a periodic function with period T, we suppose in addition that $\tau \in C^1(R, R)$ with $\tau'(t) < 1$, $\forall t \in [0, T]$.

Remark 2.1. From the above assumption, one can find from Lemma 2.4 that the function $(t-\tau(t))$ has an inverse denoted by $\mu(t)$. Define $\sigma_0 = -\min_{t\in[0,T]} \tau'(t)$, $\sigma_1 = \max_{t\in[0,T]} \tau'(t)$ and $\| \tau \|_0 = \max_{t\in[0,T]} |\tau(t)|$. Clearly, $\sigma_0 \geq 0$ and $0 \leq \sigma_1 < 1$.

Lemma 2.5 ([3]). *Let $u_k \in C^2_{2kT}$ be a $2kT$-periodic function for each $k \in N$ with*

$$|u_k|_0 \leq A_0,\ |u_k'|_0 \leq A_1,\ |u_k''|_0 \leq A_2$$

where A_0, A_1 and A_2 are constants independent of $k \in N$. Then there exists a function $u \in C^1(R, R^n)$ such that for each interval $[c,d] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in N}$ with $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c,d]$.

The system (4) is equivalent to the system

$$\begin{cases} u'(t) = \phi(v(t)) = \frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}} \\ v'(t) = -f(\phi(v(t))) - g(t, u(t-\tau(t))) + e_k(t) \end{cases} \tag{6}$$

where $\varphi_q(s) = |s^{q-2}|s$, $\frac{1}{p} + \frac{1}{q} = 1$, $v(t) = \varphi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}) = \phi^{-1}(u'(t))$.

Let $X_k = \{\omega = (u(t), v(t))^\top \in C(R, R^2), \omega(t) = \omega(t+2kT)\}$ and $Y_k = \{\omega = (u(t), v(t))^\top \in C(R, R^2), \omega(t) = \omega(t+2kT)\}$, where the norm $||\omega|| = \max\{|u|_0, |v|_0\}$ with $||u||_0 = \max_{t\in[0,2kT]} |u(t)|$ and $||v||_0 = \max_{t\in[0,2kT]} |v(t)|$. It is obvious that X_k and Y_k are Banach spaces.

Now we define the operator

$$L : D(L) \subset X_k \to Y_k, L\omega = \omega' = (u'(t), v'(t))^\top$$

where $D(L) = \{\omega | \omega = (u(t), v(t))^\top \in C^1(R, R^2), \omega(t) = \omega(t+2kT)\}$.

Let $Z_k = \{\omega = (u(t), v(t))^\top \in C^1(R, R \times B_k), \omega(t) = \omega(t+2kT)\}$, where $B_k = \{x \in R, |x| < 1, x(t) = x(t+2kT)\}$. Define a nonlinear operator $N : \overline{\Omega} \subset (X_k \cap Z_k) \subset X_k \to Y_k$ as follows:

$$N\omega = (\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}, -f(\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}) - g(t, u(t-\tau(t))) + e_k(t))^\top$$

where $\overline{\Omega} \subset Z_k \subset X_k$ and Ω is an open and bounded set. Then problem (6) can be written as $L\omega = N\omega$ in $\overline{\Omega}$.

we know

$$KerL = \{\omega | \omega \in X_k, \omega' = (u'(t), v'(t))^\top = 0\}$$

then $\forall t \in R$ we have $u'(t) = 0$, $v'(t) = 0$, obviously $u = a_1 \in R$, $v = a_2 \in R$, thus $KerL = R^2$, and it is also easy to prove that $ImL = \{z \in Y_k, \int_0^{2kT} z(s)ds = 0\}$. Therefore, L is a Fredholm operator of index zero.

Let

$$P : X_k \to KerL, P\omega = \frac{1}{2kT}\int_0^{2kT} \omega(s)ds$$

$$Q: Y_k \to ImQ, Qz = \frac{1}{2kT}\int_0^{2kT} z(s)ds$$

Let $K_p = L|^{-1}_{KerL\cap D(L)}$, then it is easy to see that:

$$(K_p z)(t) = \int_0^{2kT} G_k(t,s)z(s)ds$$

where

$$G_k(t) = \begin{cases} \frac{s-2kT}{2kT}, & 0 \le t \le s; \\ \frac{s}{2kT}, & s \le t \le 2kT. \end{cases}$$

For all $\overline{\Omega}$ such that $\overline{\Omega} \subset (X_k \cap Z_k) \subset X_k$, we have $K_p(I-Q)N(\overline{\Omega})$ is a relative compact set of X_k, $QN(\overline{\Omega})$ is a bounded set of Y_k, so the operator N is L-compact in $\overline{\Omega}$.

For the sake of convenience, we list the following assumption which will be used by us in studying the existence of homoclince solutions to the Eq.(3) in Section 3.

$[H_1]$ There exists constants α and $\beta > 0$ such that

$$|xf(x)| \ge \alpha|x|^p \text{ and } |f(x)| \le \beta|x|^{p-1},\ \forall x \in R$$

$[H_2]$ There exists constants m_0 and $m_1 > 0$ such that

$$|xg(t,x)| \ge m_0|x|^p \text{ and } |g(t,x)| \le m_1|x|^{p-1},\ \forall (t,x) \in R^2$$

$[H_3]$ $e \in C(R,R)$ is a bounded function with $e(t) \ne 0$ and

$$A := \max\{(\int_R |e(t)|^{\frac{p}{p-1}}dt)^{\frac{p-1}{p}}, \int_R |e(t)|^2 dt\} + \sup_{t\in R}|e(t)| < +\infty$$

Remark 2.2. From (5), we can see that $|e_k(t)| \le \sup_{t\in R}|e(t)|$. So if $[H_3]$ holds, then for each $k \in N$, $(\int_{-kT}^{kT} |e(t)|^{\frac{p}{p-1}}dt)^{\frac{p-1}{p}} < A$.

3. Main results

In order to study the existence of $2kT$-periodic solutions to system (6), we firstly study some properties of all possible $2kT$-periodic solutions to the following system:

$$\begin{cases} u'(t) = \lambda\phi(v(t)) = \lambda\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}} \\ v'(t) = -\lambda f(\phi(v(t))) - \lambda g(t, u(t-\tau(t))) + \lambda e_k(t), \lambda \in (0,1] \end{cases} \tag{7}$$

where $(u_k, v_k)^\top \in Z_k \subset X_k$. For each $k \in N$ and all $\lambda \in (0,1]$, let Δ represent the set of all the $2kT$-periodic solutions to the above system.

Theorem 3.1. *Assume that conditions [H_1]-[H_3] hold,*

$$\frac{\frac{\beta m_1}{\alpha}(\frac{1}{1-\sigma_1})^{-\frac{1}{p}} + m_1 \parallel \tau \parallel_0 (\frac{m_1}{\alpha})^{\frac{1}{p-1}}}{1-\sigma_1} < \frac{m_0}{1+\sigma_0}$$

and there exists a positive constant d_0 such that

$$(2T)^{-\frac{1}{2}}\sqrt{C_1} + T(2T)^{-\frac{1}{2}}C_2 < 1$$

where

$$C_1 := \beta d_1^{p-1}d_0 + m_1 d_0^{p-1} d_1 \parallel \tau \parallel_0 (\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} + Ad_0$$

$$C_2 := \frac{m_1 d_0^{\frac{p}{2}}}{\sqrt{1-\sigma_1}}[(2T)^{-\frac{1}{p}}d_0 + T(2T)^{-\frac{1}{p}}d_1]^{\frac{p-2}{2}} + A$$

$$d_1 := \Big(\frac{m_1}{\alpha}(\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} d_0^{p-1} + \frac{A}{\alpha}\Big)^{\frac{1}{p-1}}$$

then for each $k \in N$, if $(u,v)^\top \in \Delta$, there are positive constants ρ_1, ρ_2, ρ_3 and ρ_4 which are independent of k and λ, such that

$$||u||_0 \le \rho_1, ||v||_0 \le \rho_2 < 1, ||u'||_0 \le \rho_3, ||v'||_0 \le \rho_4$$

Proof. For each $k \in N$, if $(u,v)^\top \in \Delta$, it must satisfy the system (7). Multiplying the second equation of (7) by $u'(t)$ and integrating from $-kT$ to kT, we have

$$0 = \int_{-kT}^{kT} v'(t)u'(t)dt = -\lambda\int_{-kT}^{kT} f(u'(t))u'(t)dt - \lambda\int_{-kT}^{kT} g(t,u(t-\tau(t)))u'(t)dt + \lambda\int_{-kT}^{kT} e_k(t)u'(t)dt$$

In view of [H_1] and [H_2] and by Hölder inequality, we get

$$\begin{aligned}\alpha\int_{-kT}^{kT}|u'(t)|^p dt &\le m_1\int_{-kT}^{kT}|u(t-\tau(t))|^{p-1}|u'(t)|dt + \int_{-kT}^{kT}|e_k(t)||u'(t)|dt\\ &\le m_1(\int_{-kT}^{kT}|u(t-\tau(t))|^p dt)^{\frac{p-1}{p}}(\int_{-kT}^{kT}|u'(t)|^p dt)^{\frac{1}{p}}\\ &\quad + (\int_{-kT}^{kT}|e_k(t)|^{\frac{p}{p-1}}dt)^{\frac{p-1}{p}}(\int_{-kT}^{kT}|u'(t)|^p dt)^{\frac{1}{p}} \qquad (8)\end{aligned}$$

Furthermore,

$$\int_{-kT}^{kT}|u(t-\tau(t)|^p dt = \int_{-kT-\tau(-kT)}^{kT-\tau(kT)}\frac{1}{1-\tau'(\mu(s))}|u(s)|^p ds$$

and by Lemma 2.4,

$$\int_{-kT-\tau(-kT)}^{kT-\tau(kT)}\frac{1}{1-\tau'(\mu(s))}|u(s)|^p ds = \int_{-kT}^{kT}\frac{1}{1-\tau'(\mu(s))}|u(s)|^p ds$$

It follows from Remark 2.1 that

$$\frac{1}{1+\sigma_0} \parallel u \parallel_p^p \leq \int_{-kT}^{kT} \frac{1}{1-\tau'(\mu(s))}|u(s)|^p ds \leq \frac{1}{1-\sigma_1} \parallel u \parallel_p^p \tag{9}$$

Substituting (9) into (8) and combining with Remark 2.2, we can obtain

$$\alpha \parallel u' \parallel_p^p \leq m_1\Big(\frac{1}{1-\sigma_1}\Big)^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} \parallel u' \parallel_p + A \parallel u' \parallel_p$$

which yields

$$\parallel u' \parallel_p \leq \Big(\frac{m_1}{\alpha}\Big(\frac{1}{1-\sigma_1}\Big)^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} + \frac{A}{\alpha}\Big)^{\frac{1}{p-1}} \tag{10}$$

Multiplying the second equation of (7) by $u(t)$ and integrating from $-kT$ to kT, we have

$$\begin{aligned}\int_{-kT}^{kT} v'(t)u(t)dt &= -\int_{-kT}^{kT} v(t)u'(t)dt \\ &= -\lambda \int_{-kT}^{kT} f(u'(t))u(t)dt \\ &\quad - \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))[u(t) - u(t-\tau(t))]dt \\ &\quad - \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))u(t-\tau(t))dt + \lambda \int_{-kT}^{kT} e_k(t)u(t)dt\end{aligned}$$

From the equality above, we have

$$\begin{aligned}\lambda \int_{-kT}^{kT} \frac{v^2}{\sqrt{1-v^2}}dt &+ \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))u(t-\tau(t))dt \\ &= -\lambda \int_{-kT}^{kT} f(u'(t))u(t)dt \\ &\quad - \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))[u(t) - u(t-\tau(t))]dt \\ &\quad + \lambda \int_{-kT}^{kT} e_k(t)u(t)dt\end{aligned}$$

Since $\frac{|v(t)|^2}{\sqrt{1-v^2(t)}} \geq |v(t)|^2$ and in view of $[H_1]$, $[H_2]$ and Lemma 2.2, we can get

$$\begin{aligned}&\int_{-kT}^{kT} |v(t)|^2 dt + m_0 \int_{-kT}^{kT} |u(t-\tau(t))|^p dt \\ &\leq \int_{-kT}^{kT} |f(u'(t))||u(t)|dt + \int_{-kT}^{kT} |g(t, u(t-\tau(t)))||u(t) - u(t-\tau(t))|dt \\ &\quad + \int_{-kT}^{kT} |e_k(t)||u(t)|dt\end{aligned}$$

$$\le \beta \int_{-kT}^{kT} |u'(t)|^{p-1}|u(t)|dt + m_1 \int_{-kT}^{kT} |u(t-\tau(t))|^{p-1}|u(t)-u(t-\tau(t))|dt$$
$$+ \int_{-kT}^{kT} |e_k(t)||u(t)|dt$$
$$\le \beta(\int_{-kT}^{kT} |u'(t)|^p dt)^{\frac{p-1}{p}} (\int_{-kT}^{kT} |u(t)|^p dt)^{\frac{1}{p}} + m_1 \parallel \tau \parallel_0 (\int_{-kT}^{kT} |u(t-\tau(t))|^p dt)^{\frac{p-1}{p}}$$
$$(\int_{-kT}^{kT} |u'(t)|^p dt)^{\frac{1}{p}} + (\int_{-kT}^{kT} |e_k(t)|^{\frac{p}{p-1}} dt)^{\frac{p-1}{p}} (\int_{-kT}^{kT} |u(t)|^p dt)^{\frac{1}{p}} \tag{11}$$

By applying (9) to (11), we have

$$\parallel v \parallel_2^2 + \frac{m_0}{1+\sigma_0} \parallel u \parallel_p^p \le \beta \parallel u' \parallel_p^{p-1} \parallel u \parallel_p + m_1 \parallel \tau \parallel_0 (\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} \parallel u' \parallel_p$$
$$+ A \parallel u \parallel_p$$

From the inequality above, we can see that

$$\parallel v \parallel_2^2 \le \beta \parallel u' \parallel_p^{p-1} \parallel u \parallel_p + m_1 \parallel \tau \parallel_0 (\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} \parallel u' \parallel_p + A \parallel u \parallel_p \tag{12}$$

and

$$\frac{m_0}{1+\sigma_0} \parallel u \parallel_p^p \le \beta \parallel u' \parallel_p^{p-1} \parallel u \parallel_p + m_1 \parallel \tau \parallel_0 (\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} \parallel u' \parallel_p + A \parallel u \parallel_p \tag{13}$$

Substituting (10) into (13), we get

$$\frac{m_0}{1+\sigma_0} \parallel u \parallel_p^p \le \beta\Big(\frac{m_1}{\alpha}(\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} + \frac{A}{\alpha}\Big) \parallel u \parallel_p + A \parallel u \parallel_p$$
$$+ m_1 \parallel \tau \parallel_0 (\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} \Big(\frac{m_1}{\alpha}(\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} \parallel u \parallel_p^{p-1} + \frac{A}{\alpha}\Big)^{\frac{1}{p-1}}$$

Since $\frac{\frac{\beta m_1}{\alpha}(\frac{1}{1-\sigma_1})^{-\frac{1}{p}} + m_1 \|\tau\|_0 (\frac{m_1}{\alpha})^{\frac{1}{p-1}}}{1-\sigma_1} < \frac{m_0}{1+\sigma_0}$, it is easy to see that there exists a constant d_0 such that

$$\parallel u \parallel_p \le d_0 \tag{14}$$

Substituting (14) into (10), we obtain

$$\parallel u' \parallel_p \le \Big(\frac{m_1}{\alpha}(\frac{1}{1-\sigma_1})^{\frac{p-1}{p}} d_0^{p-1} + \frac{A}{\alpha}\Big)^{\frac{1}{p-1}} := d_1 \tag{15}$$

It follows from Lemma 2.2 that

$$|u(t)| \le (2T)^{-\frac{1}{\mu}}(\int_{t-T}^{t+T} |u(s)|^\mu ds)^{\frac{1}{\mu}} + T(2T)^{-\frac{1}{p}}(\int_{t-T}^{t+T} |u'(s)|^p ds)^{\frac{1}{p}}$$
$$\le (2T)^{-\frac{1}{p}}(\int_{t-kT}^{t+kT} |u(s)|^p ds)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}(\int_{t-kT}^{t+kT} |u'(s)|^p ds)^{\frac{1}{p}}$$
$$= (2T)^{-\frac{1}{p}}(\int_{-kT}^{kT} |u(s)|^p ds)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}(\int_{-kT}^{kT} |u'(s)|^p ds)^{\frac{1}{p}}$$

In view of (14) and (15), we have

$$\begin{aligned}|u(t)| &\le (2T)^{-\frac{1}{p}}d_0 + T(2T)^{-\frac{1}{p}}d_1\\ &:= \rho_1\end{aligned}$$

then we get

$$||u||_0 = \max_{t\in[-kT,kT]} |u(t)| \le \rho_1 \tag{16}$$

Clearly, ρ_1 is independent of k and λ. Furthermore, substituting (14) and (15) into (12), we can see that

$$\begin{aligned}\| v \|_2^2 &\le \beta \| u' \|_p^{p-1} \| u \|_p + m_1 \| \tau \|_0 \left(\frac{1}{1-\sigma_1}\right)^{\frac{p-1}{p}} \| u \|_p^{p-1} \| u' \|_p + A \| u \|_p\\ &\le \beta d_1^{p-1} d_0 + m_1 d_0^{p-1} d_1 \| \tau \|_0 \left(\frac{1}{1-\sigma_1}\right)^{\frac{p-1}{p}} + A d_0\\ &:= C_1\end{aligned} \tag{17}$$

Multiplying the second equation of (7) by $v'(t)$ and integrating from $-kT$ to kT, we have

$$\begin{aligned}\int_{-kT}^{kT} |v'(t)|^2 dt &= -\lambda \int_{-kT}^{kT} f(u'(t))v'(t)dt - \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))v'(t)dt\\ &\quad + \lambda \int_{-kT}^{kT} e_k(t)v'(t)dt\\ &= -\lambda \int_{-kT}^{kT} f(u'(t))d(v(t)) - \lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))v'(t)dt\\ &\quad + \lambda \int_{-kT}^{kT} e_k(t)v'(t)dt\end{aligned} \tag{18}$$

From the first equation of (7), we can see that $v(t) = \varphi_p\Big(\frac{\frac{u'(t)}{\lambda}}{\sqrt{1+(\frac{u'(t)}{\lambda})^2}}\Big)$, thus

$$\begin{aligned}-\lambda \int_{-kT}^{kT} f(u'(t))d(v(t)) &= -\lambda \int_{-kT}^{kT} f(u'(t))d\Big(\varphi_p\Big(\frac{\frac{u'(t)}{\lambda}}{\sqrt{1+(\frac{u'(t)}{\lambda})^2}}\Big)\Big)\\ &= -\int_{-kT}^{kT} f(u'(t))d\Big(\Big|\frac{\frac{u'(t)}{\lambda}}{\sqrt{1+(\frac{u'(t)}{\lambda})^2}}\Big|^{p-2} \cdot \frac{\frac{u'(t)}{\lambda}}{\sqrt{1+(\frac{u'(t)}{\lambda})^2}}\Big)\\ &= 0\end{aligned} \tag{19}$$

Substituting (19) into (18) and in view of $[H_2]$, we get

$$\begin{aligned}\| v' \|_2^2 = \int_{-kT}^{kT} |v'(t)|^2 dt &= -\lambda \int_{-kT}^{kT} g(t, u(t-\tau(t)))v'(t)dt + \lambda \int_{-kT}^{kT} e_k(t)v'(t)dt\\ \le m_1 \int_{-kT}^{kT} |u(t-\tau(t))|^{p-1}|v'(t)|dt &+ \int_{-kT}^{kT} |e_k(t)||v'(t)|dt\end{aligned}$$

$$\leq m_1(\int_{-kT}^{kT}|u(t-\tau(t))|^{2(p-1)}dt)^{\frac{1}{2}}(\int_{-kT}^{kT}|v'(t)|^2dt)^{\frac{1}{2}}$$
$$+(\int_{-kT}^{kT}|e_k(t)|^2dt)^{\frac{1}{2}}(\int_{-kT}^{kT}|v'(t)|^2dt)^{\frac{1}{2}}$$
$$\leq m_1 \parallel u \parallel_0^{\frac{p-2}{2}}(\int_{-kT}^{kT}|u(t-\tau(t))|^pdt)^{\frac{1}{2}}(\int_{-kT}^{kT}|v'(t)|^2dt)^{\frac{1}{2}}$$
$$+(\int_{-kT}^{kT}|e_k(t)|^2dt)^{\frac{1}{2}}(\int_{-kT}^{kT}|v'(t)|^2dt)^{\frac{1}{2}}$$
$$\leq m_1 \parallel u \parallel_0^{\frac{p-2}{2}}\frac{1}{\sqrt{1-\sigma_1}}\parallel u \parallel_p^{\frac{p}{2}}\parallel v' \parallel_2 + A\parallel v' \parallel_2$$

It follows from (14) and (16) that

$$\parallel v' \parallel_2 \leq \frac{m_1 d_0^{\frac{p}{2}}}{\sqrt{1-\sigma_1}}[(2T)^{-\frac{1}{p}}d_0+T(2T)^{-\frac{1}{p}}d_1]^{\frac{p-2}{2}}+A$$
$$:= C_2 \tag{20}$$

Applying the Lemma 2.2 again, we have

$$|v(t)| \leq (2T)^{-\frac{1}{\mu}}(\int_{t-T}^{t+T}|v(s)|^\mu ds)^{\frac{1}{\mu}}+T(2T)^{-\frac{1}{p}}(\int_{t-T}^{t+T}|v'(s)|^p ds)^{\frac{1}{p}}$$
$$\leq (2T)^{-\frac{1}{2}}(\int_{t-kT}^{t+kT}|v(s)|^2 ds)^{\frac{1}{2}}+T(2T)^{-\frac{1}{2}}(\int_{t-kT}^{t+kT}|v'(s)|^2 ds)^{\frac{1}{2}}$$
$$= (2T)^{-\frac{1}{2}}(\int_{-kT}^{kT}|v(s)|^2 ds)^{\frac{1}{2}}+T(2T)^{-\frac{1}{2}}(\int_{-kT}^{kT}|v'(s)|^2 ds)^{\frac{1}{2}} \tag{21}$$

then combining (17) and (20) gives

$$|v(t)| \leq (2T)^{-\frac{1}{2}}\sqrt{C_1}+T(2T)^{-\frac{1}{2}}C_2 := \rho_2$$

It follows from $(2T)^{-\frac{1}{2}}\sqrt{C_1}+T(2T)^{-\frac{1}{2}}C_2 < 1$ that

$$||v||_0 \leq \rho_2 < 1 \tag{22}$$

Clearly, ρ_2 is independent of k and λ.
It follows from (7) that

$$||u'||_0 = \max_{t\in[-kT,kT]}|u'(t)| = \max_{t\in[-kT,kT]}\lambda\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}$$
$$\leq \frac{\rho_2^{q-1}}{\sqrt{1-\rho_2^{2q-2}}} := \rho_3 \tag{23}$$

Clearly, ρ_3 is independent of k and λ. Let define $F_{\rho_3} = \max_{|u'|\leq\rho_3} |f(u'(t))|$, $G_{\rho_1} = \max_{|u|\leq\rho_1} |g(t, u(t))|$, then from the second equation of (7), we can obtain

$$||v'||_0 = \max_{t\in[-kT,kT]} |v'(t)| \leq F_{\rho_3} + G_{\rho_1} + A := \rho_4 \tag{24}$$

and also ρ_4 is independent of k and λ. Therefore, From (16), (22), (23) and (24), we know ρ_1, ρ_2, ρ_3 and ρ_4 are constants independent of k and λ. Hence the conclusion of Theorem 3.1 holds. □

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied . Then, for each $k \in N$, system (7) has at least one $2kT$-periodic solution $(u_k(t), v_k(t))^\top$ in $\Delta \subset X_k$ such that*

$$||u_k||_0 \leq \rho_1, ||v_k||_0 \leq \rho_2 < 1, ||u_k'||_0 \leq \rho_3, ||v_k'||_0 \leq \rho_4$$

where ρ_1, ρ_2, ρ_3 and ρ_4 are constants defined by Theorem 3.1.

Proof. In order to use Lemma 2.1, for each $k \in N$, we consider the following system:

$$\begin{cases} u'(t) = \lambda\varphi(v(t)) = \lambda\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}} \\ v'(t) = -\lambda f(\varphi(v(t))) - \lambda g(t, u(t-\tau(t))) + \lambda e_k(t), \lambda \in (0,1] \end{cases} \tag{25}$$

where $v(t) = \varphi_p\Big(\frac{\frac{u'(t)}{\lambda}}{\sqrt{1+(\frac{u'(t)}{\lambda})^2}}\Big)$. Let $\Omega_1 \subset X_k$ represent the set of all the $2kT$-periodic solutions of system (25). Since $(0,1) \subset (0,1]$, then $\Omega_1 \subset \Delta$, where Δ is defined by Theorem 3.1. If $(u,v)^\top \in \Omega_1$, by using Theorem 3.1, we have

$$||u||_0 \leq \rho_1, ||v||_0 \leq \rho_2 < 1, ||u'||_0 \leq \rho_3, ||v'||_0 \leq \rho_4$$

Let $\Omega_2 = \{\omega = (u,v)^\top \in KerL, QN\omega = 0\}$, if $(u,v)^\top \in \Omega_2$, then $(u,v)^\top = (a_1, a_2)^\top \in R^2$(constant vector) and we can see that

$$\begin{cases} \int_{-kT}^{kT} \frac{\varphi_q(a_2)}{\sqrt{1-\varphi_q^2(a_2)}} dt = 0 \\ \int_{-kT}^{kT} -f(a_1') - g(t, a_1) + e_k(t) dt = 0 \end{cases}$$

i.e.,

$$\begin{cases} a_2 = 0 \\ \int_{-kT}^{kT} -f(a_1') - g(t, a_1) + e_k(t) dt = 0 \end{cases} \tag{26}$$

Multiplying the second equation of (26) by a_1 and combining with $[H_2]$, we have

$$2kTm_0|a_1|^p \leq \int_{-kT}^{kT} |a_1||e_k(t)|dt \leq (2kT)^{\frac{1}{p}}|a_1|A$$

Thus

$$|a_1| \leq A^{\frac{1}{p-1}}(2T)^{\frac{-1}{p}} m_0^{\frac{1}{1-p}} := \gamma$$

Now, if we define $\Omega = \{\omega = (u,v)^\top \in X_k, ||u||_0 < \rho_1 + \gamma, ||v||_0 < \frac{1+\rho_2}{2} < 1\}$, it is easy to see that $\Omega_1 \cup \Omega_2 \subset \Omega$. So, condition (h_1) and condition (h_2) of Lemma 2.1 are satisfied. In order to verify the condition (h_3) of Lemma 2.1, let

$$H(\omega,\mu) : (\Omega \cap R^2) \times [0,1] \to R : H(\omega,\mu) = \mu\omega + (1-\mu)JQN(\omega)$$

where $J : ImQ \to KerL$ is a linear isomorphism, $J(u,v) = (v,u)^\top$. From assumption $[H_1]$ and $[H_2]$, we have

$$\omega^\top H(\omega,\mu) \neq 0, \forall(\omega,\mu) \in \partial\Omega \cap R^2 \times [0,1]$$

Hence,

$$\begin{aligned} deg\{JQN, \Omega \cap R^2, 0\} &= deg\{H(\omega,0), \Omega \cap R^2, 0\} \\ &= deg\{H(\omega,1), \Omega \cap R^2, 0\} \\ &\neq 0 \end{aligned}$$

So, the condition (h_3) of Lemma 2.1 is satisfied. Therefore, by using Lemma 2.1, we see that Eq.(6) has a $2kT$-periodic solution $(u_k,v_k)^\top \in \Omega$. Obviously, $(u_k,v_k)^\top$ is a $2kT$-periodic solution to Eq.(2) for the case of $\lambda = 1$, so $(u_k,v_k)^\top \in \Delta$. Thus, by using Theorem 3.1, we have

$$||u_k||_0 \le \rho_1, ||v_k||_0 \le \rho_2 < 1, ||u_k'||_0 \le \rho_3, ||v_k'||_0 \le \rho_4$$

Hence the conclusion of Theorem 3.2 holds. □

Theorem 3.3. *Suppose that the conditions in Theorem 3.1 hold, then Eq.(1) has a nontrivial homoclinic solution.*

Proof. From Theorem 3.2, we see that for each $k \in N$, there exists a $2kT$-periodic solution $(u_k,v_k)^\top$ to Eq.(2) with

$$||u_k||_0 \le \rho_1, ||v_k||_0 \le \rho_2 < 1, ||u_k'||_0 \le \rho_3, ||v_k'||_0 \le \rho_4 \tag{27}$$

where ρ_1, ρ_2, ρ_3, ρ_4 are constants independent of $k \in N$. And $u_k(t)$ is a solution of (2), so

$$\Big(\varphi_p\big(\frac{u_k'(t)}{\sqrt{1+(u_k'(t))^2}}\big)\Big)' + f(u_k'(t)) + g(t, u_k(t-\tau(t))) = e_k(t) \tag{28}$$

with $v_k(t) = \varphi_p\Big(\frac{u_k'(t)}{\sqrt{1+(u_k'(t))^2}}\Big)$ implies that $v_k(t)$ is continuously differentiable for $t \in R$. Also, from (27), we have $|v_k|_0 \le \rho_2 < 1$. It follows that $u_k'(t) = \phi(v_k(t)) = \frac{\varphi_q(v_k(t))}{\sqrt{1-\varphi_q^2(v_k(t))}}$ is continuously differentiable for $t \in R$, i.e.,

$$u_k''(t) = \frac{\varphi_q'(v_k(t))v_k'(t)}{(1-\varphi_q^2(v_k(t)))^{\frac{3}{2}}}$$

By using (27) again and combining with $\varphi_q(s) = |s|^{q-2}s$ for $s \neq 0$, then we have

$$||u_k''||_0 \le \frac{(q-1)\rho_2^{q-2}\rho_4^2}{\sqrt{1-\rho_2^{q-1}}} := \rho_5$$

Clearly, ρ_5 is a constant independent of $k \in N$. From Lemma 2.5, we can see that there is a function $u_0 \in C^1(R, R^n)$ such that for each interval $[a, b] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_N\}_{k\in N}$ with $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[a, b]$. In the following, we show that $u_0(t)$ is just a homoclinic solution to Eq.(4).
For all $a,\ b \in R$ with $a < b$, there must be a positive integer j such that for $j > j_0$, $[-k_jT, k_jT - \varepsilon_0] \subset [a - \alpha, b + \alpha]$. So, for $j > j_0$, from (3) and (26) we see that

$$\Big(\varphi_p(\frac{u'_{k_j}}{\sqrt{1+(u'_{k_j})^2}})\Big)' + f(u'_{k_j}(t)) + g(t, u_{k_j}(t-\tau(t))) = e(t), t \in [a, b] \tag{29}$$

Then from (29) we can have

$$\begin{aligned}\Big(\varphi_p(\frac{u'_{k_j}}{\sqrt{1+(u'_{k_j})^2}})\Big)' &= -f(u'_{k_j}(t)) - g(t, u_{k_j}(t-\tau(t))) + e(t)\\ &\to -f(u'_0(t)) + g(t, u_0(t-\tau(t))) + e(t) \text{ uniformly on } [a, b].\end{aligned}$$

Since$\Big(\varphi_p(\frac{u'_{k_j}}{\sqrt{1+(u'_{k_j})^2}})\Big)' \to \Big(\varphi_p(\frac{u'_0}{\sqrt{1+(u'_0)^2}})\Big)'$ uniformly for $t \in [a, b]$ and$\Big(\varphi_p(\frac{u'_{k_j}}{\sqrt{1+(u'_{k_j})^2}})\Big)'$ is continuous differentiable for $t \in [a, b]$, we can have

$$\Big(\varphi_p(\frac{u'_0}{\sqrt{1+(u'_0)^2}})\Big)' = -f(u'_0(t)) + g(t, u_0(t-\tau(t))) + e(t), t \in [a, b]$$

Considering that $a,\ b$ are two arbitrary constants with $a < b$, it is easy to see that $u_0(t),\ t \in R$ is a solution to system (1).
Now, we prove $u_0(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$.
Since

$$\begin{aligned}\int_{-\infty}^{+\infty}(|u_0(t)|^p + |u'_0(t)|^p)dt &= \lim_{i\to+\infty}\int_{-iT}^{iT}(|u_0(t)|^p + |u'_0(t)|^p)dt\\ &= \lim_{i\to+\infty}\lim_{j\to+\infty}\int_{-iT}^{iT}(|u_{k_j}(t)|^p + |u'_{k_j}(t)|^p)dt\end{aligned}$$

Clearly, for every $i \in N$ if $k_j > i$, then by (14) and (15), we have

$$\int_{-iT}^{iT}(|u_{k_j}(t)|^p + |u'_{k_j}(t)|^p)dt \le \int_{-k_jT}^{k_jT}(|u_{k_j}(t)|^p + |u'_{k_j}(t)|^p)dt \le d_0^p + d_1^p$$

Let $i \to +\infty$, $j \to +\infty$, we have

$$\int_{-\infty}^{+\infty}(|u_0(t)|^p + |u'_0(t)|^p)dt \le d_0^p + d_1^p \tag{30}$$

and then

$$\int_{|t|\ge r}(|u_0(t)|^p + |u'_0(t)|^p)dt \to 0 \tag{31}$$

as $r \to +\infty$. So, by using Lemma 2.3 as $|t| \to +\infty$, we obtain

$$\begin{aligned}|u_0(t)| &\le (2T)^{-\frac{1}{p}}(\int_{t-T}^{t+T}|u(s)|^p ds)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}(\int_{t-T}^{t+T}|u'(s)|^p ds)^{\frac{1}{p}} \\ &\le [(2T)^{-\frac{1}{p}} + T(2T)^{-\frac{1}{p}}](\int_{t-T}^{t+T}|u(s)|^p ds + \int_{t-T}^{t+T}|u'(s)|^p ds)^{\frac{1}{p}} \\ &\to 0.\end{aligned}$$

Finally, we will proof

$$u_0'(t) \to 0, |t| \to \infty. \tag{32}$$

From (27), we know

$$|u_0(t)| \le \rho_1, |u_0'(t)| \le \rho_3, \forall t \in R$$

Then, we have

$$\begin{aligned}|\Big(\varphi_p(\frac{u_0'}{\sqrt{1+(u_0')^2}})\Big)'| &\le |f(u_0'(t))| + |g(t, u_0(t-\tau(t)))| + |e(t)| \\ &\le \sup_{u\in[-\rho_3,\rho_3]}|f(u'(t))| + \sup_{u\in[-\rho_1,\rho_1]}|g(t,u(t))|] + A \\ &:= M_1, \forall t \in R\end{aligned}$$

If (32) does not hold, then there exist $\varepsilon_1 \in (0, \frac{1}{4})$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < |t_3| < \cdots < |t_k| + 1 < |t_{k+1}|, k = 1, 2, 3, \cdots$$

and

$$|u_0'(t_k)| \ge \frac{2\varepsilon_1}{\sqrt{1-|2\varepsilon_1|^2}}, k = 1, 2, 3, \cdots$$

Then, for $t \in [t_k, t_k + \frac{\varepsilon_1}{1+M_1}]$, we can have

$$\begin{aligned}|u_0'(t)| \ge |\frac{u_0'(t)}{\sqrt{1+|u_0'(t)|^2}}| &= |\frac{u_0'(t_k)}{\sqrt{1+|u_0'(t_k)|^2}} + \int_{t_k}^{t}(\frac{u_0'(s)}{\sqrt{1+|u_0'(s)|^2}})'ds| \\ &\ge |\frac{u_0'(t_k)}{\sqrt{1+|u_0'(t_k)|^2}}| - \int_{t_k}^{t}|(\frac{u_0'(s)}{\sqrt{1+|u_0'(s)|^2}})'|ds \ge \varepsilon_1\end{aligned}$$

It follows that

$$\int_{-\infty}^{+\infty}|u_0'(t)|^p dt \ge \Sigma_{k=1}^{\infty}\int_{t_k}^{t_k+\frac{\varepsilon_1}{1+M_1}}|u_0'(t)|^p dt = \infty$$

which contradicts (30), thus (32) holds. Clearly, $u_0(t) \ne 0$, otherwise $e(t) \equiv 0$, which contradicts assumption (H_3). Hence the conclusion of Theorem 3.3 holds. □

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