

## APPROXIMATIONS OF SOLUTIONS FOR A NONLOCAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH DEVIATED ARGUMENT

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**ABSTRACT.** This paper investigates the existence of mild solution for a fractional integro-differential equations with a deviating argument and nonlocal initial condition in an arbitrary separable Hilbert space  $H$  via technique of approximations. We obtain an associated integral equation and then consider a sequence of approximate integral equations obtained by the projection of considered associated nonlocal fractional integral equation onto finite dimensional space. The existence and uniqueness of solutions to each approximate integral equation is obtained by virtue of the analytic semigroup theory via Banach fixed point theorem. Next we demonstrate the convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. We consider the Faedo-Galerkin approximation of the solution and demonstrate some convergence results. An example is also given to illustrate the abstract theory.

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### 1. Introduction

In recent few decades, researchers have developed great interest in fractional calculus due to its broad applicability in science and engineering. The tool of fractional calculus has been available and applicable to deal with many physical and real world problems such as anomalous diffusion process, traffic flow, non-linear oscillation of earthquake, real system characterized by power laws, critical phenomena, scale free process, describe viscoelastic materials and many others. The details on the theory and its applications can be found in [1]-[4]. The existence of the solution for the differential equations with nonlocal conditions has

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been investigated widely by many authors as nonlocal conditions are more realistic than the classical initial conditions such as in dealing with many physical problems. Concerning the developments in the study of nonlocal problems, we refer to [6]-[16] and references given therein.

To the solvability of evolution problems in the time domain, we have various approaches, namely, the evolution family approach and an approach using finite-dimensional approximations known as Faedo-Galerkin approximations. The Faedo-Galerkin approach may be used for the study of more regular solutions, imposing higher regularity on the data. In [20], author has extended the results of the [19] and considered the Faedo-Galerkin approximations of the solutions for functional Cauchy problem in a separable Hilbert space with the help of analytic semigroup theory and Banach fixed point theorem. In [21], authors have studied the Faedo-Galerkin approximations of the solutions to a class of functional integro-differential equation extended the results of [20]. In [8], the Faedo-Galerkin approximations of the mild solution to non-local history-valued retarded differential equations have been obtained by authors. In [9], authors have established the existence of the mild solution and approximations of mild solutions via technique of Faedo-Galerkin approximations and analytic semigroup theory. In [28], authors have considered an fractional differential equation and studied the Faedo-Galerkin approximations of the solutions for fractional differential equation. In [26], the existence and approximations of the mild solution to fractional differential equation with deviated argument via technique of Faedo-Galerkin approximations have been obtained by authors. The Faedo-Galerkin approximations of solutions for fractional integro-differential equation have been considered by authors in [30]. For the Faedo-Galerkin approximation of solutions, we refer to papers [8]-[9], [21]-[31].

The purpose of this work is to establish the approximation of the solution for following nonlocal integro-differential equation with a deviating argument in a separable Hilbert space  $(H, \|\cdot\|_H, (\cdot, \cdot)_H)$

$$\begin{aligned} {}^c D_{0,t}^q x(t) + Ax(t) &= f(t, x_t, x(a(x(t), t))) + \int_0^t b(t, s)g(s, x(s), x_s)ds, \\ 0 \leq t \leq T_0 < \infty, \quad 0 < T_0 < \infty, & \quad (1) \\ h(x) &= \phi, \quad \text{on } [-r, 0], \quad r > 0, & \quad (2) \end{aligned}$$

where  $0 < q < 1$ ,  ${}^c D_{0,t}^q$  is the generalized fractional derivative of order  $q$  in Caputo sense with lower limit 0. For  $x \in C([0, T_0]; H)$ ,  $x_t : [-r, 0] \rightarrow H$  is defined as  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-r, 0]$ . In (1),  $A : D(A) \subset H \rightarrow H$  is a closed, positive definite and self adjoint linear operator with dense domain  $D(A)$ . We assume that  $-A$  generates an analytic semigroup of bounded linear operators on  $H$ . The functions  $f : [0, T_0] \times C([-r, 0], H) \times H \rightarrow H$ ,  $a : H \times [0, T_0] \rightarrow [0, T_0]$ ,  $b : [0, T_0] \times [0, T_0] \rightarrow \mathbb{R}$ ,  $g : [0, T_0] \times H \times C([-r, 0], H) \rightarrow H$  and  $h : C([-r, 0], H) \rightarrow C([-r, 0], H)$  are appropriate functions to be mentioned later. For more details

on differential equation with deviated argument, we refer to papers [17]-[18], [26] and references cited therein.

The organization of the article is as follows: Section 2 provides some basic definitions, lemmas and theorems as preliminaries as these are useful for proving our results. We firstly obtain an integral equation associated with (1). A mild solution of equation (1) is defined as a solution of associated integral equation. We consider a sequence of approximate integral equations. Section 3 proves the existence and uniqueness of the approximate solutions by using analytic semigroup and fixed point theorem. In section 4, we show the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and the convergence of the approximate Faedo-Galerkin solutions will be shown in section 5. Section 5 gives an example.

### 2. Preliminaries and Assumptions

Some basic definitions, theorems, lemmas and assumptions which will be used to prove existence result, are stated in this section.

Throughout the work, we assume that  $(H, \|\cdot\|_H, (\cdot, \cdot)_H)$  is a separable Hilbert space. The symbol  $C([0, T_0], H)$  stands for the Banach space of all the continuous functions from  $[0, T_0]$  into  $H$  equipped with the norm  $\|z(t)\|_C := \sup_{t \in [0, T_0]} \|z(t)\|_H$  and  $L^p((0, T_0), H)$  stands for Banach space of all Bochner-measurable functions from  $(0, T_0)$  to  $H$  with the norm

$$\|z\|_{L^p} := \left( \int_{(0, T_0)} \|z(s)\|_H^p ds \right)^{1/p}.$$

Since  $-A$  is the infinitesimal generator of an analytic semigroup of bounded linear operators  $\{\mathcal{T}(t); t \geq 0\}$ . Therefore, there exist constants  $C > 0$  and  $\omega \geq 0$  such that  $\|\mathcal{T}(t)\| \leq Ce^{\omega t}$ , for  $t \geq 0$ . In addition, we note that

$$\left\| \frac{d^j}{dt^j} \mathcal{T}(t) \right\| \leq M_j, \text{ for } t > t_0 \text{ and } t_0 > 0, \quad j = 1, 2, \tag{3}$$

where  $M_j$  are some positive constants. Henceforth, without loss of generality, we may assume that  $\mathcal{T}(t)$  is uniformly bounded by  $M$  i.e.,  $\|\mathcal{T}(t)\| \leq M$  and  $0 \in \rho(-A)$  which means that  $-A$  is invertible. This permits us to define the positive fractional power  $A^\alpha$  as closed linear operator with domain  $D(A^\alpha) \subseteq H$  for  $\alpha \in (0, 1]$ . Moreover,  $D(A^\alpha)$  is dense in  $H$  with the norm

$$\|y\|_\alpha := \|A^\alpha y\|, \quad \forall y \in D(A^\alpha). \tag{4}$$

Hence, we signify the space  $D(A^\alpha)$  by  $H_\alpha$  endowed with the  $\alpha$ -norm  $(\|\cdot\|_\alpha)$ . It is easy to show that  $H_\alpha$  is a Banach space with norm  $\|\cdot\|_\alpha$  [35]. Also, we have that  $H_\kappa \hookrightarrow H_\alpha$  for  $0 < \alpha < \kappa$  and therefore, the embedding is continuous. Then, we define  $H_{-\alpha} = (H_\alpha)^*$ , for each  $\alpha > 0$ . The space  $H_{-\alpha}$  stands for the dual space of  $H_\alpha$ , is a Banach space with the norm  $\|z\|_{-\alpha} = \|A^{-\alpha}z\|$ . For

additional parts on the fractional powers of closed linear operators, we refer to book by Pazy [35].

**Lemma 2.1** ([35]). *Let  $-A$  be the infinitesimal generator of an analytic semi-group  $\{\mathcal{T}(t)\}_{t \geq 0}$  such that  $\|\mathcal{T}(t)\| \leq M$ , for  $t \geq 0$  and  $0 \in \rho(-A)$ . Then,*

- (i) *For  $0 < \alpha \leq 1$ ,  $H_\alpha$  is a Hilbert space.*
- (ii) *The operator  $A^\alpha \mathcal{T}(t)$  is bounded for every  $t > 0$  and*

$$\|A\mathcal{T}(t)\| \leq Mt^{-1}, \quad (5)$$

$$\|A^\alpha \mathcal{T}(t)\| \leq M_\alpha t^{-\alpha}. \quad (6)$$

Now, we state some basic definitions and properties of fractional calculus.

**Definition 2.2** ([3]). The Riemann-Liouville fractional integral operator  $J$  of order  $q > 0$  with lower point 0, is defined by

$$J_{0,t}^q F(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s) ds, \quad (7)$$

where  $F \in L^1((0, T_0), H)$ .

**Definition 2.3** ([3]). The Riemann-Liouville fractional derivative is given by

$${}^{RL}D_{0,t}^q F(t) = D_t^\delta J_{0,t}^{\delta-q} F(t), \quad \delta - 1 < q < \delta, \quad \delta \in \mathbb{N}, \quad (8)$$

where  $D_t^\delta = \frac{d^\delta}{dt^\delta}$ ,  $F \in L^1((0, T_0), H)$ ,  $J_{0,t}^{\delta-q} \in W^{\delta,1}((0, T_0), H)$ . Here the notation  $W^{\delta,1}((0, T_0), H)$  stands for the Sobolev space defined as

$$\begin{aligned} W^{\delta,1}((0, T_0), H) = \{ & y \in H : \exists z \in L^1((0, T_0), H) : y(t) = \sum_{j=0}^{\delta-1} d_j \frac{t^j}{j!} \\ & + \frac{t^{\delta-1}}{(\delta-1)!} * z(t), t \in (0, T_0)\}. \end{aligned} \quad (9)$$

Note that  $z(t) = y^\delta(t)$ ,  $d_j = y^j(0)$ .

**Definition 2.4** ([3]). The Caputo fractional derivative is given by

$${}^cD_{0,t}^q F(t) = \frac{1}{\Gamma(\delta-q)} \int_0^t (t-s)^{\delta-q-1} F^\delta(s) ds, \quad (10)$$

for  $\delta - 1 < q < \delta$ ,  $\delta \in \mathbb{N}$ , where  $F \in L^1((0, T_0), H) \cap C^{\delta-1}((0, T_0), H)$ .

Let  $C_0 := C([-r, 0], H)$  be the collection of continuous mappings from  $[-r, 0]$  into  $H$  equipped with the supremum norm  $\|y\|_0 := \sup_{t \in [-r, 0]} \|y(t)\|$  for  $y \in C_0$ . In addition,  $C_t := C([-r, t], H)$  be the Banach space of all  $H$ -valued continuous functions on  $[-r, t]$  endowed with the supremum norm  $\|y\|_t := \sup_{s \in [-r, t]} \|y(s)\|$  for each  $y \in C_t$  and  $t \in (0, T_0]$  and the space of all continuous functions from  $[-r, t]$  into  $H_\alpha$  denoted by  $C_t^\alpha$ , is a Banach space with the supremum norm

$\|y\|_{t,\alpha} := \sup_{s \in [-r,t]} \|A^\alpha y(s)\|$ , for each  $y \in \mathcal{C}_t^\alpha$ .

For  $0 \leq \alpha < 1$ , we define

$$\mathcal{C}_t^{\alpha-1} := \{x \in \mathcal{C}_t^\alpha : \|x(\tau) - x(s)\| \leq \mathcal{L}|\tau - s|, \text{ for all } \tau, s \in [-r, t]\}, \quad (11)$$

where  $\mathcal{L} > 0$  is a appropriate constant to be defined later.

Now, we turn to the following fractional differential equations with nonlocal conditions as

$$\begin{aligned} {}^c D_t^\alpha x(t) &= -Ax(t) + F(t, x_t, x(a(x(t), t))) \\ &+ \int_0^t b(t, s)g(s, x(s), x_s)ds, \quad t \in [0, T_0], \quad h(x) = \phi, \text{ on } [-r, 0]. \end{aligned} \quad (12)$$

We give few examples of function  $h$  as

(1) Let  $w \in L^1(0, r)$  be such that  $\mathcal{W} = \int_0^r w(s)ds \neq 0$ . Let

$$h(\varsigma) = \int_{-r}^0 w(-s)\varsigma(s)ds, \quad \varsigma \in \mathcal{C}_0^\alpha. \quad (13)$$

(ii) Let  $-r \leq t_1 < t_2 < \dots < t_p \leq 0, k_i \geq 0$  and  $K = \sum_{i=1}^p k_i \neq 0$ . Let

$$\begin{aligned} h(\varsigma) &= \sum_{i=1}^p k_i \varsigma(t_i), \quad \varsigma \in \mathcal{C}_0^\alpha, \\ h(\varsigma) &= \sum_{i=1}^p \frac{k_i}{c_i} \int_{t_i - c_i}^{t_i} \varsigma(s)ds, \quad \varsigma \in \mathcal{C}_0^\alpha, \end{aligned} \quad (14)$$

where  $c_i \geq 0$ .

If we take  $\psi \in \mathcal{C}_0^\alpha$  defined as  $\psi(\theta) \equiv \phi$  for all  $\theta \in [-r, 0]$  and  $\mathcal{G} : \mathcal{C}_0^\alpha \rightarrow \mathcal{C}_0^\alpha$  given by  $\mathcal{G}(y)(\theta) \equiv h(y)$  for all  $\theta \in [-r, 0]$  and  $y \in \mathcal{C}_0^\alpha$ . Then the condition  $h(y) = \phi$  is equivalent to the condition  $\mathcal{G}(y) = \psi$ . Thus, the functional differential equation with a more general nonlocal history condition may be considered which is illustrated as follows,

$$\begin{aligned} {}^c D_t^\alpha x(t) &= -Ax(t) + F(t, x_t, x(a(x(t), t))) + \int_0^t b(t, s)g(s, x(s), x_s)ds, \quad t \in (0, T_0], \\ \mathcal{G}(x) &= \psi, \text{ on } [-r, 0]. \end{aligned} \quad (15)$$

which includes (12). For example,

$$\begin{aligned} {}^c D_t^\alpha x(t) &= -Ax(t) + F(t, x_t, x(a(x(t), t))) \\ &+ \int_0^t b(t, s)g(s, x(s), x_s)ds, \quad t \in [0, T_0], \quad x(t) = \psi(t), \text{ on } [-r, 0]. \end{aligned}$$

is a particular case of (15). Thus, the problem (12) and (15) are equivalent.

Next, we make the following assumptions:

- (A1)  $A$  is a closed, densely defined, positive definite and self-adjoint linear operator from  $D(A) \subset H$  into  $H$ . We assume that operator  $A$  has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots, \quad (16)$$

with  $\lambda_m \rightarrow \infty$  as  $m \rightarrow \infty$  and corresponding complete orthonormal system of eigenfunctions  $\{\chi_j\}$ , i.e.,

$$A\chi_j = \lambda_j\chi_j, \quad \text{and} \quad \langle \chi_l, \chi_j \rangle = \delta_{lj}, \quad (17)$$

where

$$\delta_{lj} = \begin{cases} 1, & j = l, \\ 0, & \text{otherwise.} \end{cases}$$

- (A2) (i) There exists a function  $k \in C_t^\alpha$  such that  $h(k) = \phi$ , for all  $t \in [-r, 0]$ . We assume that  $h$  is Lipschitz continuous function on  $C([-r, 0], D(A^\alpha))$ .  
(ii) The function  $k(t) \in D(A^\alpha)$  for each  $t \in [-r, 0]$  and  $k$  is locally Hölder continuous with exponent 1 on  $[-r, 0]$  i.e. there exists a constant  $L_k > 0$  such that

$$\|k(t_1) - k(t_2)\| \leq L_k |t_1 - t_2|, \quad \forall t_1, t_2 \in [-r, 0]. \quad (18)$$

- (A3) The nonlinear function  $f : [0, T_0] \times C_0^\alpha \times H_{\alpha-1} \rightarrow H$  is Lipschitz continuous and there exist constants  $\mathcal{L}_f > 0$  and  $\mu_1 \in (0, 1]$  such that

$$\|f(t, x_1, y_1) - f(s, x_2, y_2)\| \leq \mathcal{L}_f [|t - s|^{\mu_1} + \|x_1 - x_2\|_{0,\alpha} + \|y_1 - y_2\|_{\alpha-1}], \quad (19)$$

for all  $(t, x_1, y_1), (s, x_2, y_2) \in [0, T_0] \times \mathcal{B}_R(C_0^\alpha, \tilde{k}) \times \mathcal{B}_R(H_{\alpha-1}, \tilde{k})$ . Here,  $\mathcal{B}_R(Z, \tilde{k}) = \{z \in Z, \|z - \tilde{k}\|_Z \leq R\}$  and  $Z$  is a Banach space and  $R > 0$  is a constant to be defined later. The function  $\tilde{k}$  is defined by

$$\tilde{k}(t) = \begin{cases} k(t), & t \in [-r, 0], \\ k(0), & t \in [0, T_0]. \end{cases} \quad (20)$$

- (A4) The function  $a : H_\alpha \times [0, T_0] \rightarrow [0, T_0]$  is continuous function and there exist constants  $\mathcal{L}_a > 0$  and  $\mu_2 \in (0, 1]$  such that

$$|a(x_1, t_1) - a(x_2, t_2)| \leq \mathcal{L}_a [\|x_1 - x_2\|_\alpha + |t_1 - t_2|^{\mu_2}], \quad (21)$$

for all  $(x_1, t_1), (x_2, t_2) \in \mathcal{B}_R(H_\alpha, \tilde{k}) \times [0, T_0]$  and  $a(\cdot, 0) = 0$ .

- (A5)  $g : [0, T_0] \times H_\alpha \times C_0^\alpha \rightarrow H$  is continuous function and there exists a positive constant  $\mathcal{L}_g$  such that

$$\|g(t, x_1, y_1) - g(t, x_2, y_2)\| \leq \mathcal{L}_g [\|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_{0,\alpha}], \quad (22)$$

for all  $(x_1, y_1), (x_2, y_2) \in \mathcal{B}_R(H_\alpha, \tilde{k}) \times \mathcal{B}_R(C_0^\alpha, \tilde{k})$  and  $t \in [0, T_0]$ .

Now, we provide the definition of mild solution for the nonlocal system (1)-(2).

**Definition 2.5.** A continuous function  $x : [0, T_0] \rightarrow H$  is said to be a mild solution for the system (1)-(2) if  $x \in C_{T_0}^\alpha \cap C_{T_0}^{\alpha-1}$  and the following integral equation

$$x(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s)f(s, x_s, x(a(x(s), s)))ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) \int_0^s b(s, \tau)g(\tau, x(\tau), x_\tau)d\tau ds & t \in [0, T_0], \end{cases} \tag{23}$$

is verified.

The operator  $\mathcal{S}_q(t)$  and  $\mathcal{T}_q(t)$  are defined as follows:

$$\begin{aligned} \mathcal{S}_q(t)x &= \int_0^\infty \zeta_q(\xi)\mathcal{T}(t^q\xi)xd\xi, \\ \mathcal{T}_q(t)x &= q \int_0^\infty \xi\zeta_q(\xi)T(t^q\xi)xd\xi, \end{aligned}$$

where  $\zeta_q(\xi) = \frac{1}{q}\xi^{1-1/q} \times \psi_q(\xi^{-\frac{1}{q}})$  is a probability density function defined on  $(0, \infty)$  i.e.,  $\zeta_q(\xi) \geq 0, \int_0^\infty \zeta_q(\xi)d\xi = 1$  and

$$\psi_q(\xi) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \xi^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \xi \in (0, \infty).$$

**Lemma 2.6** ([11]). *If  $-A$  is the infinitesimal generator of analytic semigroup of uniformly continuous bounded operators. Then,*

- (1) *The operator  $\mathcal{S}_q(t), t \geq 0$  and  $\mathcal{T}_q(t), t \geq 0$  are bounded linear operators.*
- (2)  *$\|\mathcal{S}_q(t)y\| \leq M\|y\|, \|\mathcal{T}_q(t)y\| \leq \frac{qM}{\Gamma(1+q)}\|y\|$  and  $\|A^\alpha\mathcal{T}_q(t)y\| \leq \frac{qM_\alpha\Gamma(2-\alpha)t^{-q\alpha}}{\Gamma(1+q(1-\alpha))}\|y\|$ , for any  $y \in H$ .*
- (3) *The families  $\{\mathcal{S}_q(t) : t \geq 0\}$  and  $\{\mathcal{T}_q(t) : t \geq 0\}$  are strongly continuous.*
- (4) *If  $\mathcal{T}(t)$  is compact, then  $\mathcal{S}_q(t)$  and  $\mathcal{T}_q(t)$  are compact operators for any  $t > 0$ .*

### 3. Approximate Solutions and Convergence

In this section, we study the existence of approximate solutions for the system (1)-(2).

Let  $\mathcal{H}_n$  be the finite dimensional subspace of  $H$  spanned by  $\{\chi_0, \chi_1, \dots, \chi_n\}$  and  $P^n : H \rightarrow \mathcal{H}_n$  be the corresponding projection operator for  $n = 0, 1, 2, \dots$ . We define

$$f_n : [0, T_0] \times C([-r, 0], H) \times H \rightarrow H, \tag{24}$$

by

$$f_n(t, x_t, x(a(x(t), t))) = f(t, P^n x_t, P^n x(a(x(t), t))), \tag{25}$$

and

$$g_n : [0, T_0] \times H \times C([-r, 0], H) \rightarrow H, \tag{26}$$

by

$$g_n(t, x(t), x_t) = g(t, P^n x(t), P^n x_t). \tag{27}$$

We choose  $T, 0 < T \leq T_0$  sufficiently small such that

$$T < \left\{ \frac{2R}{3} \left[ \frac{(1-\alpha)\Gamma(1+q(1-\alpha))}{M_\alpha(N_f+G_g)\Gamma(2-\alpha)} \right] \right\}^{\frac{1}{q(1-\alpha)}}, \tag{28}$$

$$\| [\mathcal{S}_q(t) - I]\tilde{k}(0) \|_\alpha < \frac{R}{3}, \tag{29}$$

$$\Theta = \frac{M_\alpha\Gamma(2-\alpha)T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} [\mathcal{L}_f(2 + \mathcal{L}\mathcal{L}_a) + 2b_T\mathcal{L}_g] < 1. \tag{30}$$

Now, we consider

$$\mathcal{B}_R = \mathcal{B}_R(\mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1}, \tilde{k}) = \{y \in \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1} : \|y - \tilde{k}\|_{T,\alpha} \leq R\}. \tag{31}$$

By the assumptions (A3) – (A4), we have that  $f$  is continuous on  $[0, T]$ . Therefore, there exist a positive constant  $N_f$  such that

$$N_f = \mathcal{L}_f[T^{\mu_1} + R(1 + \mathcal{L}\mathcal{L}_a) + \mathcal{L}\mathcal{L}_aT^{\mu_2}] + N, \text{ where } N = \|f(0, \tilde{k}_0, \tilde{k}(0))\|, \tag{32}$$

with

$$\|f(t, x_t, x(a(x(t), t)))\| \leq N_f, \quad x \in H, \quad t \in [0, T]. \tag{33}$$

Similarly with the help of the assumption (A5), we can show that  $\|g(t, x, x_t)\| \leq 2\mathcal{L}_gR + \|g(t, \tilde{k}(0), \tilde{k}_0)\| = N_g$ . Therefore, we can indicate effectively that  $G_g = b_TN_g$ , where  $b_T = \sup_{t \in [0, T]} \int_0^T |b(t, \tau)| d\tau$ .

Let us consider the operator  $A^\alpha : \mathcal{C}_t^\alpha \rightarrow \mathcal{C}_t$  defined by  $(A^\alpha y)(t) = A^\alpha(y(t))$  and  $(P^n x_t)(s) = P^n(x(t+s))$ , for all  $s \in [-r, 0]$  and  $t \in [0, T]$ . We consider the operator  $\mathcal{Q}_n : \mathcal{B}_R \rightarrow \mathcal{B}_R$  defined by

$$(\mathcal{Q}_n x)(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, x_s, x(a(x(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x)(s) ds, & t \in [0, T], \end{cases} \tag{34}$$

for each  $x \in \mathcal{B}_R$ , where  $G_n x(t) = \int_0^t b(t, s) g_n(s, x(s), x_s) ds$ .

**Theorem 3.1.** *Suppose (A1)-(A5) holds and  $k(t) \in D(A)$  for all  $t \in [-r, 0]$ . Then, there exists a unique  $x_n \in \mathcal{B}_R$  such that  $\mathcal{Q}_n x_n = x_n$  for each  $n = 0, 1, 2, \dots$ , and  $x_n$  satisfies the following approximate integral equation*

$$x_n(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x_n)(s) ds, & t \in [0, T]. \end{cases} \tag{35}$$

*Proof.* To demonstrate the theorem, we first need to show that  $\mathcal{Q}_n x \in \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1}$ . It is easy to show that  $\mathcal{Q}_n : \mathcal{C}_T^\alpha \rightarrow \mathcal{C}_T^\alpha$  by using the fact that  $f$  and  $g$  are continuous function. Now, it remains to show that  $\mathcal{Q}_n x \in \mathcal{C}_T^{\alpha-1}$ . For  $t, \tau \in [-r, 0]$  with  $t > \tau$ , we have  $\mathcal{Q}_n x \in \mathcal{C}_T^{\alpha-1}$  for  $x \in \mathcal{C}_T^{\alpha-1}$  by using fact that  $k$  is Hölder continuous with exponent 1 i. e., Lipschitz continuous on  $[-r, 0]$ .



For  $x \in C_T^{\alpha-1}$ ,  $0 < \tau < t < T$ , then we have

$$\begin{aligned} & \| \mathcal{Q}_n x(t) - \mathcal{Q}_n x(\tau) \|_{\alpha-1} \\ & \leq \| [\mathcal{S}_q(t) - \mathcal{S}_q(\tau)] \tilde{k}(0) \|_{\alpha-1} \\ & \quad + \int_0^\tau \| (t-s)^{q-1} \mathcal{T}_q(t-s) - (\tau-s)^{q-1} \mathcal{T}_q(\tau-s) \|_{\alpha-1} \| f_n(s, x_s, x(a(x(s), s))) \| ds \\ & \quad + \int_\tau^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) \|_{\alpha-1} \| f_n(s, x_s, x(a(x(s), s))) \| ds, \\ & \quad + \int_0^\tau \| (t-s)^{q-1} \mathcal{T}_q(t-s) - (\tau-s)^{q-1} \mathcal{T}_q(\tau-s) \|_{\alpha-1} \| (G_n x)(s) \| ds \\ & \quad + \int_\tau^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) \|_{\alpha-1} \| (G_n x)(s) \| ds. \end{aligned}$$

From the first term of above inequality, we have

$$[\mathcal{S}_q(t) - \mathcal{S}_q(\tau)] A^{\alpha-1} \tilde{k}(0) = \int_0^\infty \zeta_q(\xi) [\mathcal{T}(t^q \xi) - \mathcal{T}(\tau^q \xi)] A^{\alpha-1} \tilde{k}(0) d\xi, \tag{36}$$

Also, we have that for each  $x \in H$

$$[\mathcal{T}(t^q \xi) - \mathcal{T}(\tau^q \xi)] x = \int_\tau^t \frac{d}{ds} \mathcal{T}(s^q \xi) x ds = \int_\tau^t q \xi s^{q-1} A \mathcal{T}(s^q \xi) x ds. \tag{37}$$

Therefore, we estimate the first term as

$$\begin{aligned} & \int_0^\infty \zeta_q(\xi) \| \mathcal{T}(t^q \xi) - \mathcal{T}(\tau^q \xi) \| \| A^{\alpha-1} \tilde{k}(0) \| d\xi \\ & \leq \int_0^\infty \zeta_q(\xi) \left[ \int_\tau^t \left\| \frac{d}{ds} \mathcal{T}(s^q \xi) \right\| ds \right] \| u_0 \|_{\alpha-1} d\xi, \\ & \leq \int_0^\infty \zeta_q(\xi) [M_1(t-\tau)] \| \tilde{k}(0) \|_{\alpha-1} d\xi, \\ & \leq K_1(t-\tau) \int_0^\infty \zeta_q(\xi) d\xi, \\ & = K_1(t-\tau), \end{aligned} \tag{38}$$

where  $K_1 = M_1 \| \tilde{k}(0) \|_{\alpha-1}$ . The second integrals is estimated as

$$\begin{aligned} & \int_0^\tau \| (t-s)^{q-1} \mathcal{T}_q(t-s) - (\tau-s)^{q-1} \mathcal{T}_q(\tau-s) \|_{\alpha-1} \| f_n(s, x_s, x(a(x(s), s))) \| ds \\ & \leq \int_0^\tau \int_0^\infty \zeta_q(\xi) \left\| \left[ \frac{d}{d\varsigma} \mathcal{T}((\varsigma-s)^q \xi) \right]_{\varsigma=t} - \left[ \frac{d}{d\varsigma} \mathcal{T}((\varsigma-s)^q \xi) \right]_{\varsigma=\tau} \right\| A^{\alpha-2} \| \\ & \quad \times \| f_n(s, x_s, x(a(x(s), s))) \| d\xi ds, \\ & \leq \int_0^\tau \int_0^\infty \zeta_q(\xi) \left[ \int_\tau^t \| A^{\alpha-2} \frac{d^2}{d\varsigma^2} \mathcal{T}((\varsigma-s)^q \xi) \| d\varsigma \right] N_f d\xi ds, \\ & \leq \int_0^\tau \int_0^\infty \zeta_q(\xi) [ \| A^{\alpha-2} \| M_2(t-\tau) ] N_f d\xi ds, \\ & \leq K_2(t-\tau), \end{aligned} \tag{39}$$

where  $K_2 = \| A^{\alpha-2} \| M_2 N_f T$ . The third integrals is estimated as

$$\int_\tau^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) \|_{\alpha-1} \| f_n(s, x_s, x(a(x(s), s))) \| ds$$

$$\begin{aligned}
 &\leq \int_{\tau}^t \int_0^{\infty} \zeta_q(\xi) \| [q(t-s)^{q-1} \xi A \mathcal{T}((t-s)^q \xi)] A^{\alpha-2} \| \| f_n(s, x_s, x(a(x(s), s))) \| ds, \\
 &\leq \int_{\tau}^t \int_0^{\infty} \zeta_q(\xi) \left\| \frac{d}{d\xi} \mathcal{T}((\varsigma-s)^q \xi) \Big|_{\varsigma=t} A^{\alpha-2} \right\| N_f d\xi ds, \\
 &\leq K_3(t-\tau),
 \end{aligned} \tag{40}$$

where  $K_3 = M_1 \| A^{\alpha-2} \| N_f$ . Similarly, we estimate forth integral as

$$\begin{aligned}
 &\int_0^{\tau} \| (t-s)^{q-1} \mathcal{T}_q(t-s) - (\tau-s)^{q-1} \mathcal{T}_q(\tau-s) \|_{\alpha-1} \| (G_n x)(s) \| ds \\
 &\leq K_4(t-\tau),
 \end{aligned} \tag{41}$$

where  $K_4 = \| A^{\alpha-2} \| M_2 T G_g$  and

$$\int_{\tau}^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) \|_{\alpha-1} \| (G_n x)(s) \| ds \leq K_5(t-\tau), \tag{42}$$

where  $K_5 = M_1 \| A^{\alpha-2} \| G_g$ .

Thus, from the inequality (38) to (42), we obtain that

$$\| (\mathcal{Q}_n x)(t) - (\mathcal{Q}_n x)(\tau) \|_{\alpha-1} \leq \mathcal{L}(t-\tau), \tag{43}$$

for a positive suitable constant  $\mathcal{L} = \sum_{l=1}^5 K_l$ . Therefore, we conclude that  $(\mathcal{Q}_n x) \in \mathcal{C}_T^{\alpha-1}$ . Hence, we deduce that the operator  $\mathcal{Q}_n : \mathcal{C}_T^{\alpha-1} \rightarrow \mathcal{C}_T^{\alpha-1}$  is well defined map.

Next, we prove that  $\mathcal{Q}_n : \mathcal{B}_R \rightarrow \mathcal{B}_R$ . For  $0 \leq t \leq T$  and  $x \in \mathcal{B}_R$ , we get that

$$\begin{aligned}
 &\| (\mathcal{Q}_n x)(t) - \tilde{k}(0) \|_{\alpha} \\
 &\leq \| [\mathcal{S}_q(t) - I] \tilde{k}(0) \|_{\alpha} + \int_0^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, x_s, x(a(x(s), s))) \|_{\alpha} ds \\
 &\quad + \int_0^t \| (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x)(s) \|_{\alpha} ds, \\
 &\leq \| [\mathcal{S}_q(t) - I] \tilde{k}(0) \|_{\alpha} + \frac{q M_{\alpha} N_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_0^t (t-s)^{q(1-\alpha)-1} ds \\
 &\quad + \frac{q M_{\alpha} G_g \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_0^t (t-s)^{q(1-\alpha)-1} ds, \\
 &\leq \| [\mathcal{S}_q(t) - I] \tilde{k}(0) \|_{\alpha} + \frac{M_{\alpha} N_f \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} + \frac{M_{\alpha} G_g \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}.
 \end{aligned} \tag{44}$$

From the inequalities (28) and (44), we conclude that  $\mathcal{Q}_n(\mathcal{B}_R) \subset \mathcal{B}_R$ . Finally, we will show that  $\mathcal{Q}_n$  is a contraction map. For  $x, y \in \mathcal{B}_R$  and  $0 \leq t \leq T$ , we have

$$\begin{aligned}
 &\| (\mathcal{Q}_n x)(t) - (\mathcal{Q}_n y)(t) \|_{\alpha} \\
 &\leq \int_0^t \| (t-s)^{q-1} A^{\alpha} \mathcal{T}_q(t-s) \| \| f_n(s, x_s, x(a(x(s), s))) - f_n(s, y_s, y(a(y(s), s))) \| ds
 \end{aligned}$$

$$+ \int_0^t \|(t-s)^{q-1} A^\alpha \mathcal{T}_q(t-s)\| \| (G_n x)(s) - (G_n y)(s) \| ds. \tag{45}$$

We have the following inequalities:

$$\|f_n(s, x_s, x(a(x(s), s))) - f_n(s, y_s, y(a(y(s), s)))\| \leq \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \|x - y\|_{T,\alpha}, \tag{46}$$

and

$$\begin{aligned} \| (G_n x)(s) - (G_n y)(s) \| &\leq \int_0^s |b(s, \tau)| \| g(\tau, x(\tau), x_\tau) - g(\tau, y(\tau), y_\tau) \| d\tau, \\ &\leq 2b_T \mathcal{L}_g \|x - y\|_{T,\alpha}. \end{aligned} \tag{47}$$

Using (46)-(47) in (45), we get,

$$\begin{aligned} &\| (\mathcal{Q}_n x)(t) - (\mathcal{Q}_n y)(t) \|_\alpha \\ &\leq \frac{qM_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \|x - y\|_{T,\alpha} \int_0^t (t-s)^{q(1-\alpha)-1} ds \\ &\quad + \frac{qM_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} (2b_T \mathcal{L}_g) \|x - y\|_{T,\alpha} \int_0^t (t-s)^{q(1-\alpha)-1} ds, \\ &\leq \left[ \frac{M_\alpha \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] + \frac{M_\alpha \Gamma(2-\alpha) T^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} (2b_T \mathcal{L}_g) \right] \\ &\quad \times \|x - y\|_{T,\alpha}. \end{aligned} \tag{48}$$

From the inequality (30), we get

$$\| (\mathcal{Q}_n x)(t) - (\mathcal{Q}_n y)(t) \|_\alpha < \Theta \|x - y\|_{T,\alpha}, \tag{49}$$

with  $\Theta < 1$ . Therefore, it implies that the map  $\mathcal{Q}_n$  is a contraction map i.e.  $\mathcal{Q}_n$  has a unique fixed point  $x_n \in \mathcal{B}_R$  i.e.,  $\mathcal{Q}_n x_n = x_n$  and  $x_n$  satisfies the approximate integral equation

$$x_n(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t) \tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds & \\ \quad + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x_n)(s) ds, & t \in [0, T]. \end{cases} \tag{50}$$

Hence, the proof of the theorem is completed. □

**Lemma 3.2.** *Assume that hypotheses (A1)-(A5) are satisfied. If  $k(t) \in D(A)$  for each  $t \in [-r, 0]$ , then  $x_n(t) \in D(A^\nu)$  for all  $t \in [-r, T]$  with  $0 \leq \nu < 1$ .*

*Proof.* If  $t \in [-r, 0]$ , then results are obvious. Thus, it remains to show results for  $t \in [0, T]$ . From Theorem (3.1), we have that there exists a unique  $x_n \in \mathcal{B} \subset \mathcal{C}_T^\alpha \cap \mathcal{C}_T^{\alpha-1}$  such that  $x_n$  satisfy the integral equation (35). Theorem 2.6.13 in Pazy [35] implies that  $\mathcal{T}(t) : H \rightarrow D(A^\nu)$  for  $t > 0$  and  $0 \leq \nu < 1$  and for  $0 \leq \nu \leq \eta < 1$ ,  $D(A^\eta) \subseteq D(A^\nu)$ . It is easy to see that Hölder continuity of  $x_n$  can be established using the similar arguments from equation (38)-(42). Also from Theorem 1.2.4 in Pazy [35], we have that  $\mathcal{T}(t)x \in D(A)$  if  $x \in D(A)$ . The result follows from these facts and  $D(A) \subseteq D(A^\nu)$  for  $0 \leq \nu \leq 1$ . This completes the proof of Lemma. □

**Corollary 3.3.** *Suppose that (A1)-(A5) are satisfied. If  $k(t) \in D(A), \forall t \in [-r, 0]$ , then for any  $t \in [-r, T]$ , there exists a constant  $U_0$  independent of  $n$  such that*

$$\|A^v x_n(t)\| \leq U_0, \quad n = 1, 2, 3, \dots,$$

with  $0 < \alpha < v < 1$ .

*Proof.* Let  $k(t) \in D(A)$  for every  $t \in [-r, 0]$ . For  $t \in [-r, 0]$ , applying  $A^v$  on the both the sides of (35) and obtaining,

$$\|A^v x_n(t)\| \leq \|\tilde{k}(t)\|_v \leq \|\tilde{k}\|_{0,v}, \quad \forall t \in [-r, 0].$$

For  $t \in (0, T]$ , we apply  $A^v$  on the both the sides of (35) and get  $\|A^v x_n(t)\|$

$$\begin{aligned} &\leq \|A^v \mathcal{S}_q(t)\tilde{k}(0)\| + \int_0^t (t-s)^{q-1} \|A^v \mathcal{T}_q(t-s)\| \\ &\quad \|f_n(s, (x_n)_s, x_n(a(x_n(s), s)))\| ds + \int_0^t (t-s)^{q-1} \|A^v \mathcal{T}_q(t-s)\| \|G_n x_n(s)\| ds, \\ &\leq M \|\tilde{k}(0)\|_v + \frac{qM_v N_f \Gamma(2-v)}{\Gamma(1+q(1-v))} \int_0^t (t-s)^{q(1-v)-1} ds + \frac{qM_v G_g \Gamma(2-v)}{\Gamma(1+q(1-v))} \\ &\quad \times \int_0^t (t-s)^{q(1-v)-1} ds, \\ &\leq M \|\tilde{k}(0)\|_v + \frac{M_v N_f \Gamma(2-v) T^{q(1-v)}}{(1-v)\Gamma(1+q(1-v))} + \frac{M_v G_g \Gamma(2-v) T^{q(1-v)}}{(1-v)\Gamma(1+q(1-v))}, \\ &\leq U_0. \end{aligned} \tag{51}$$

This finishes the proof of lemma. □

### 4. Convergence of Solutions

The convergence of the solution  $x_n \in H_\alpha$  of the approximate integral equations (35) to a unique solution  $x(\cdot)$  of the equation (23) on  $[0, T]$  is discussed in this section.

**Theorem 4.1.** *Let us assume that the conditions (A1)-(A5) are satisfied. If  $k(0) \in D(A)$ , for each  $t \in [-r, 0]$ , then*

$$\lim_{p \rightarrow \infty} \sup_{\{n \geq p, -r \leq t \leq T\}} \|x_n(t) - x_p(t)\|_\alpha = 0. \tag{52}$$

*Proof.* For  $0 < \alpha < v < 1, n \geq p$ . Let  $t \in [-r, 0]$ , we conclude

$$\|A^\alpha x_n(t) - A^\alpha x_p(t)\| = 0. \tag{53}$$

For  $t \in (0, T]$ , we obtain,

$$\begin{aligned} &\|f_n(t, (x_n)_t, x_n(a(x_n(t), t))) - f_p(t, (x_p)_t, x_p(a(x_p(t), t)))\| \\ &\leq \|f_n(t, (x_n)_t, x_n(a(x_n(t), t))) - f_n(t, (x_p)_t, x_p(a(x_p(t), t)))\| \\ &\quad + \|f_n(t, (x_p)_t, x_p(a(x_p(t), t))) - f_p(t, (x_p)_t, x_p(a(x_p(t), t)))\|, \\ &\leq \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \|x_n - x_p\|_{t,\alpha} + \mathcal{L}_f [\| (P^n - P^p)x_p \|_{t,\alpha} \end{aligned}$$

$$+ \| A^{-1} \| \cdot \| (P^n - P^p)x_p(a(x_p(t), t)) \|_\alpha. \tag{54}$$

We also have the following estimation:

$$\| (P^n - P^p)x_p \|_{T,\alpha} \leq \| A^{\alpha-v}(P^n - P^p)A^v x_p \|_t \leq \frac{1}{\lambda_p^{v-\alpha}} \| A^v x_p \|_t.$$

Thus, we obtain

$$\begin{aligned} & \| f_n(t, (x_n)_t, x_n(a(x_n(t), t))) - f_p(t, (x_p)_t, x_p(a(x_p(t), t))) \| \\ & \leq \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \| x_n - x_p \|_{t,\alpha} + \mathcal{L}_f \left[ \frac{1}{\lambda_p^{v-\alpha}} \| x_p \|_{t,v} \right. \\ & \quad \left. + \frac{\| A^{-1} \|}{\lambda_p^{v-\alpha}} \cdot \| A^v x_p(a(x_p(t), t)) \| \right]. \end{aligned} \tag{55}$$

And

$$\| g_n(s, x_n(s), (x_n)_s) - g_p(s, x_p(s), (x_p)_s) \| \leq 2\mathcal{L}_g [\| x_n - x_p \|_{t,\alpha} + \frac{1}{\lambda_p^{v-\alpha}} \| x_p \|_{t,v}].$$

Therefore, we estimate

$$\begin{aligned} & \| (G_n x_n)(t) - (G_p x_p)(t) \| \\ & = \| \int_0^t b(t, s) [g_n(s, x_n(s), (x_n)_s) - g_p(s, x_p(s), (x_p)_s)] ds \|, \\ & \leq \int_0^t |b(t, s)| \| g_n(s, x_n(s), (x_n)_s) - g_p(s, x_p(s), (x_p)_s) \| ds, \\ & \leq 2b_T \mathcal{L}_g [\| x_n - x_p \|_{s,\alpha} + \frac{1}{\lambda_p^{v-\alpha}} \| x_p \|_{s,v}]. \end{aligned}$$

We choose  $t'_0$  such that  $0 < t'_0 < t < T$ , we have

$$\begin{aligned} & \| x_n(t) - x_p(t) \|_\alpha \\ & \leq \left( \int_0^{t'_0} + \int_{t'_0}^t \right) (t-s)^{q-1} \| A^\alpha \mathcal{T}_q(t-s) \| \\ & \quad \times \| f_n(s, (x_n)_s, x_n(a(x_n(s), s))) - f_p(s, (x_p)_s, x_p(a(x_p(s), s))) \| ds \\ & \quad + \left( \int_0^{t'_0} + \int_{t'_0}^t \right) (t-s)^{q-1} \| A^\alpha \mathcal{T}_q(t-s) \| \| (G_n x_n)(s) - (G_p x_p)(s) \| ds. \end{aligned} \tag{56}$$

We estimate the first integral as

$$\begin{aligned} & \int_0^{t'_0} (t-s)^{q-1} \| A^\alpha \mathcal{T}_q(t-s) \| \| f_n(s, x_n(s), x_n(a(x_n(s), s))) \\ & \quad - f_p(s, (x_p)_s, x_p(a(x_p(s), s))) \| ds \\ & \leq \int_0^{t'_0} (t-s)^{q-1} \| A^\alpha \mathcal{T}_q(t-s) \| 2N_f ds, \\ & \leq \frac{2N_f M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} [t^{q(1-\alpha)} - (t-t'_0)^{q(1-\alpha)}], \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2N_f M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} (t-b_1 t'_0)^{q(1-\alpha)-1} t'_0, \quad 0 < b_1 < 1, \\
&\leq \frac{2N_f M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} (t_0-t'_0)^{q(1-\alpha)-1} t'_0. \tag{57}
\end{aligned}$$

By using Corollary 3.3, the second integral is estimated as

$$\begin{aligned}
&\int_{t'_0}^t (t-s)^{q-1} \|A^\alpha \mathcal{T}_q(t-s)\| \|f_n(s, (x_n)_s, x_n(a(x_n(s), s))) \\
&\quad - f_p(s, (x_p)_s, x_p(a(x_p(s), s)))\| ds \\
&\leq \frac{qM_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_{t'_0}^t (t-s)^{q-1} \{\mathcal{L}_f[2+\mathcal{L}\mathcal{L}_a] \|x_n-x_p\|_{s,\alpha} \\
&\quad + \mathcal{L}_f[\frac{1}{\lambda_p^{v-\alpha}} \|x_p\|_{s,v} + \frac{\|A^{-1}\|}{\lambda_p^{v-\alpha}} \|A^v x_p(a(x_p(s), s))\|]\} \\
&\leq \frac{qM_\alpha \mathcal{L}_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} [(1+\|A^{-1}\|) \frac{U_0 T^{q(1-\alpha)}}{q(1-\alpha)\lambda_p^{v-\alpha}} \\
&\quad + (2+\mathcal{L}\mathcal{L}_a) \int_{t'_0}^t (t-s)^{q(1-\alpha)-1} \times \|x_n-x_p\|_{s,\alpha} ds]. \tag{58}
\end{aligned}$$

Third and fourth term are estimated as

$$\begin{aligned}
&\int_0^{t'_0} (t-s)^{q-1} \|A^\alpha \mathcal{T}_q(t-s)\| \|(G_n x_n)(s) - (G_p x_p)(s)\| ds \\
&\leq \frac{2G_g M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} (T-t'_0)^{q(1-\alpha)-1} t'_0, \tag{59}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t'_0}^t (t-s)^{q-1} \|A^\alpha \mathcal{T}_q(t-s)\| \|(G_n x_n)(s) - (G_p x_p)(s)\| ds \\
&\leq \frac{qM_\alpha \mathcal{L}_g b_T \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} 2[\frac{U_0 T^{q(1-\alpha)}}{q(1-\alpha)\lambda_p^{v-\alpha}} + \int_{t'_0}^t (t-s)^{q(1-\alpha)-1} \|x_n-x_p\|_{s,\alpha} ds]. \tag{60}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|x_n(t) - x_p(t)\|_\alpha \\
&\leq D_1 t'_0 + \frac{D_2}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^t (t-s)^{q(1-\alpha)-1} \|x_n-x_p\|_{s,\alpha} ds, \tag{61}
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \frac{2N_f M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} (T-t'_0)^{q(1-\alpha)-1} + \frac{2G_g M_\alpha \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+q(1-\alpha))} \\
&\quad \times (T-t'_0)^{q(1-\alpha)-1}, \\
D_2 &= \frac{qM_\alpha \mathcal{L}_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \times (1+\|A^{-1}\|) \frac{U_0 T^{q(1-\alpha)}}{q(1-\alpha)} + \frac{qM_\alpha \mathcal{L}_g b_T \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \\
&\quad \times 2 \frac{U_0 T^{q(1-\alpha)}}{q(1-\alpha)},
\end{aligned}$$

$$D_3 = \frac{qM_\alpha \mathcal{L}_f \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} (2 + \mathcal{L}\mathcal{L}_a) + 2 \frac{qM_\alpha \mathcal{L}_g b_T \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}.$$

We now put  $t = t + \theta$  in the above inequality, where  $\theta \in [t'_0 - t, 0]$  and get  $\|x_n(t + \theta) - x_p(t + \theta)\|_\alpha$

$$\leq D_1 t'_0 + \frac{D_2}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^{t+\theta} (t + \theta - s)^{q(1-\alpha)-1} \|x_n - x_p\|_{s,\alpha} ds. \tag{62}$$

Taking  $s - \theta = \nu$  in above inequality and obtaining,  $\|x_n(t + \theta) - x_p(t + \theta)\|_\alpha$

$$\begin{aligned} &\leq D_1 t'_0 + \frac{D_2}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0-\theta}^t (t - \nu)^{q(1-\alpha)-1} \|x_n - x_p\|_{\nu+\theta,\alpha} d\nu, \\ &\leq D_1 t'_0 + \frac{D_2}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^t (t - \nu)^{q(1-\alpha)-1} \|x_n - x_p\|_{\nu,\alpha} d\nu. \end{aligned} \tag{63}$$

Thus, we have

$$\begin{aligned} &\sup_{t'_0-t \leq \theta \leq 0} \|x_n(t + \theta) - x_p(t + \theta)\|_\alpha \\ &\leq D_1 t'_0 + \frac{D_2}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^t (t - \nu)^{q(1-\alpha)-1} \|x_n - x_p\|_{\nu,\alpha} d\nu. \end{aligned} \tag{64}$$

Since, for  $t + \theta \leq 0$ , we have  $x_n(t + \theta) = k(t + \theta)$  for all  $n \geq n_0$ . Thus, we obtain  $\sup_{-r-t \leq \theta \leq 0} \|x_n(t + \theta) - x_p(t + \theta)\|_\alpha$

$$\leq \sup_{0 \leq \theta+t \leq t'_0} \|x_n(t + \theta) - x_p(t + \theta)\|_\alpha + \sup_{t'_0-t \leq \theta \leq 0} \|x_n(t + \theta) - x_p(t + \theta)\|_\alpha \tag{65}$$

Thus, for each  $t \in (0, t'_0]$ , we have

$$\|x_n(t + \theta) - x_p(t + \theta)\|_\alpha \leq D_5 t'_0 + \frac{D_6}{\lambda_p^{v-\alpha}}, \tag{66}$$

where  $D_5$  and  $D_6$  are arbitrary positive constants. Using (64) and (66) in (65) and thus getting

$$\begin{aligned} &\sup_{-r \leq t+\theta \leq t} \|x_n(t + \theta) - x_p(t + \theta)\|_\alpha \\ &\leq (D_1 + D_5) t'_0 + \frac{D_2 + D_6}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^t (t - \nu)^{q(1-\alpha)-1} \|x_n - x_p\|_{\nu,\alpha} d\nu. \end{aligned} \tag{67}$$

Thus, we get

$$\|x_n - x_p\|_{t,\alpha} \leq (D_1 + D_5) t'_0 + \frac{D_2 + D_6}{\lambda_p^{v-\alpha}} + D_3 \int_{t'_0}^t (t - \nu)^{q(1-\alpha)-1} \|x_n - x_p\|_{\nu,\alpha} d\nu. \tag{68}$$

By Lemma 5.6.7 in [35], we have that there exists a constant  $\mathcal{K}$  such that

$$\|x_n(t) - x_p(t)\|_\alpha \leq [(D_1 + D_5) t'_0 + \frac{D_2 + D_6}{\lambda_p^{v-\alpha}}] \mathcal{K}, \tag{69}$$

Since  $t'_0$  is arbitrary and letting  $p \rightarrow \infty$ , therefore the right hand side may be made as small as desired by taking  $t'_0$  sufficiently small. This complete the proof of the Theorem.  $\square$

By the Theorem 4.1, we conclude that  $\{x_n\}$  is a Cauchy sequence in  $\mathcal{B}_R$ . Now, we show the convergence of the solution for the approximate integral equation  $x_n(\cdot)$  to the solution of associated integral equation  $x(\cdot)$ .

**Theorem 4.2.** *Suppose that conditions (A1)-(A5) are satisfied and  $k(t) \in D(A)$  for each  $t \in [-r, 0]$ . Then, there exists a unique  $x_n \in \mathcal{B}_R$ , satisfying*

$$x_n(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x_n)(s) ds, & t \in [0, T], \end{cases} \quad (70)$$

and  $x \in \mathcal{B}_R$ , satisfying

$$x(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s, x(a(x(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) (Gx)(s) ds, & t \in [0, T], \end{cases} \quad (71)$$

such that  $x_n$  converges to  $x$  in  $\mathcal{B}_R$  i.e.,  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

*Proof.* Let  $k(t) \in D(A)$  for all  $t \in [-r, 0]$ . For  $0 < t \leq T$ , it follows that there exists  $x_n \in \mathcal{B}_R$  such that  $A^\alpha x_n(t) \rightarrow A^\alpha x(t) \in \mathcal{B}_R$  as  $n \rightarrow \infty$  and  $x(t) = x_n(t) = k(t)$ , for each  $t \in [-r, 0]$  and for all  $n$ . Also, for  $t \in [-r, T]$ , we have  $A^\alpha x_n(t) \rightarrow A^\alpha x(t)$  as  $n \rightarrow \infty$  in  $H$ . Since  $\mathcal{B}_R$  is a closed subspace of  $C_T^{\alpha-1}$  and  $x_n \in \mathcal{B}_R$ , therefore it follows that  $x \in \mathcal{B}_R$  and

$$\lim_{n \rightarrow \infty} \sup_{t_0 \leq t \leq T} \|x_n(t) - x(t)\|_\alpha = 0, \quad \text{for any } t_0 \in (0, T]. \quad (72)$$

Also, we have

$$\begin{aligned} \sup_{t \in [t_0, T]} \|f_n(t, (x_n)_t, x_n(a(x_n(t), t))) - f(t, x_t, x(a(x(t), t)))\| \\ \leq \mathcal{L}_f [2 + \mathcal{L}\mathcal{L}_a] \|x_n - x\|_{t, \alpha} + \mathcal{L}_f [\| (P^n - I)x(t)\|_\alpha \\ + \|A^{-1}\| \| (P^n - I)x(a(x(t), t))\|_\alpha] \rightarrow 0, \end{aligned} \quad (73)$$

as  $n \rightarrow \infty$  and

$$\begin{aligned} \sup_{t \in [t_0, T]} \|g_n(t, x_n(t), (x_n)_t) - g(t, x(t), x_t)\| \\ \leq 2\mathcal{L}_g [\|x_n - x\|_{t, \alpha} + \| (P^n - I)x\|_{t, \alpha}] \rightarrow 0, \end{aligned} \quad (74)$$

as  $n \rightarrow \infty$ . For  $0 < t_0 < t$ , we rewrite (35) as

$$\begin{aligned} x_n(t) &= \mathcal{S}_q(t)\tilde{k}(0) + (\int_0^{t_0} + \int_{t_0}^t) (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \\ &\quad + (\int_0^{t_0} + \int_{t_0}^t) (t-s)^{q-1} \mathcal{T}_q(t-s) (G_n x_n)(s) ds. \end{aligned}$$

We may estimate the first and third integral as

$$\| \int_0^{t_0} (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \| \leq \frac{qMN_f}{\Gamma(1+q)} T^{q-1} t_0,$$



$$\| \int_0^{t_0} (t-s)^{q-1} \mathcal{T}_q(t-s)(G_n x_n)(s) ds \| \leq \frac{qMG_g}{\Gamma(1+q)} T^{q-1} t_0. \tag{75}$$

Thus, we deduce that

$$\begin{aligned} & \| x_n(t) - \mathcal{S}_q(t)\tilde{k}(0) - \int_{t_0}^t (t-s)^{q-1} \mathcal{T}_q(t-s) \\ & \times f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds - \int_{t_0}^t (t-s)^{q-1} \mathcal{T}_q(t-s)(G_n x_n)(s) ds \| \\ & \leq \left[ \frac{qMN_f}{\Gamma(1+q)} T^{q-1} + \frac{qMG_g}{\Gamma(1+q)} T^{q-1} \right] t_0. \end{aligned} \tag{76}$$

Letting  $n \rightarrow \infty$  in the above inequality, we obtain

$$\begin{aligned} & \| x(t) - \mathcal{S}_q(t)k(0) - \int_{t_0}^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s, x(a(x(s), s))) ds \\ & - \int_{t_0}^t (t-s)^{q-1} \mathcal{T}_q(t-s)(Gx)(s) ds \| \leq \left[ \frac{qMN_f}{\Gamma(1+q)} T^{q-1} + \frac{qMG_g}{\Gamma(1+q)} T^{q-1} \right] t_0. \end{aligned}$$

Since  $t_0$  is arbitrary and hence, we conclude that  $x(\cdot)$  satisfies the integral equation (23). □

### 5. Faedo-Galerkin Approximations

In this section, we consider the Faedo-Galerkin Approximation of a solution and show the convergence results for such an approximation.

We know that for any  $0 < T < T_0$ , there exists a unique  $x \in \mathcal{C}_T^\alpha$  satisfying the following integral equation

$$x(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s, x(a(x(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s)(Gx)(s) ds, & t \in [0, T], \end{cases} \tag{77}$$

with  $0 < T < T_0$ .

Also, we have a unique solution  $x_n \in \mathcal{C}_T^\alpha$  of the approximate integral equation

$$x_n(t) = \begin{cases} \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s)(G_n x_n)(s) ds, & t \in [0, T]. \end{cases} \tag{78}$$

Applying the projection on above equation, then Faedo-Galerkin approximation is given by  $v_n(t) = P^n x_n(t)$  satisfying

$$P^n x_n(t) = v_n(t)$$

$$= \begin{cases} P^n \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)P^n \tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) \\ \times P^n f_n(s, (x_n)_s, x_n(a(x_n(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) P^n(G_n x_n)(s) ds, & t \in [0, T]. \end{cases} \tag{79}$$

or

$$v_n(t) = \begin{cases} P^n \tilde{k}(t), & t \in [-r, 0], \\ \mathcal{S}_q(t)P^n \tilde{k}(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) P^n f(s, (v_n)_s, v_n(a(v_n(s), s))) ds \\ + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) P^n(G v_n)(s) ds, & t \in [0, T]. \end{cases} \tag{80}$$

Let solution  $x(\cdot)$  of (77) and  $v_n(\cdot)$  of (79), have the following representation

$$x(t) = \sum_{i=0}^{\infty} \alpha_i(t) \chi_i, \quad \alpha_i(t) = (x(t), \chi_i), \quad i = 0, 1, 2 \dots, \tag{81}$$

$$v_n(t) = \sum_{i=0}^n \alpha_i^n(t) \chi_i, \quad \alpha_i^n(t) = (v_n(t), \chi_i), \quad i = 0, 1, 2 \dots, \tag{82}$$

Using (82) in (79), we obtain a system of fractional order integro-differential equation of the form

$$\frac{d^q}{dt^q} \alpha_i^n(t) + \lambda_i \alpha_i^n(t) = F_i^n(t, \alpha_0^n(t), \alpha_1^n(t), \dots, \alpha_n^n(t)) + \int_0^t b(t, s) G_i^n(t, \alpha_0^n(s), \alpha_1^n(s), \dots, \alpha_n^n(s)) ds, \tag{83}$$

$$\alpha_i^n(0) = k(t), \quad \text{on } [-r, 0], \tag{84}$$

where

$$F_i^n = (f(t, \sum_{i=0}^n (\alpha_i^n)_t \chi_i, \sum_{i=0}^n \tau_i^n \chi_i), \chi_i), \tag{85}$$

$$\tau_i^n = \alpha_i^n(a(\alpha_0^n, \alpha_1^n, \dots, \alpha_n^n, t)), \tag{86}$$

$$G_i^n = (g(t, \sum_{i=0}^n \alpha_i^n \chi_i, \sum_{i=0}^n (\alpha_i^n)_t \chi_i), \chi_i) \tag{87}$$

$$\varphi_i = (k(t), \phi_i), \quad \text{for } i = 1, 2, \dots, n. \tag{88}$$

For the convergence of  $\alpha_i^n$  to  $\alpha_i$ , we have the following convergence theorem.

**Corollary 5.1.** *Assume that (A1)-(A5) are satisfied. If  $k(t) \in D(A)$  for each  $t \in [-r, 0]$ , then*

$$\sup_{t \in [-r, T]} \|v_n(t) - v_p(t)\|_{\alpha} \rightarrow 0, \quad \text{as } p, n \rightarrow \infty. \tag{89}$$

*Proof.* For  $n \geq p$  and  $0 \leq \alpha < \nu$ , we get

$$\|v_n(t) - v_p(t)\|_{\alpha} = \|P^n x_n(t) - P^p x_p(t)\|_{\alpha},$$

$$\begin{aligned} &\leq \| P^n[x_n(t) - x_p(t)] \|_\alpha + \| (P^n - P^p)x_p(t) \|_\alpha, \\ &\leq \| x_n(t) - x_p(t) \|_\alpha + \frac{1}{\lambda_p^{v-\alpha}} \| A^v x_p(t) \|. \end{aligned} \tag{90}$$

Since  $x_n \rightarrow x_p$  and  $\lambda_p \rightarrow \infty$  as  $p \rightarrow \infty$ , thus, for  $t \in [-r, 0]$  and  $k(t) \in D(A)$ , the result follows from Theorem 4.1.  $\square$

**Theorem 5.2.** *Let us assume that (A1)-(A5) are satisfied and  $k(t) \in D(A)$  for all  $t \in [-r, 0]$ . Then there exist a unique function  $v_n \in \mathcal{B}_R$  given as*

$$\begin{aligned} v_n(t) &= \mathcal{S}_q(t)P^n\tilde{k}(0) + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)P^n f(s, (v_n)_s, v_n(a(v_n(s), s)))ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)P^n(Gv_n)(s)ds, \end{aligned} \tag{91}$$

for all  $t \in [0, T]$  and  $x \in \mathcal{B}_R$  satisfying

$$\begin{aligned} x(t) &= \mathcal{S}_q(t)\tilde{k}(0) + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s, x_s, x(a(x(s), s)))ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)(Gx)(s)ds, \end{aligned} \tag{92}$$

for  $t \in [0, T]$ , such that  $v_n \rightarrow x$  as  $n \rightarrow \infty$  in  $\mathcal{B}_R$  and  $x$  satisfies the equation (23) on  $[0, T]$ .

*Proof.* By the Theorem 4.2, we have that

$$\lim_{n \rightarrow \infty} \sup_{t \in [-r, T]} \| x_n(t) - x(t) \|_\alpha = 0. \tag{93}$$

Thus, we conclude that

$$\begin{aligned} \| v_n(t) - x(t) \|_\alpha &= \| P^n x_n(t) - P^n x(t) + P^n x(t) - x(t) \|_\alpha, \\ &\leq \| P^n(x_n(t) - x(t)) \|_\alpha + \| (P^n - I)x(t) \|_\alpha. \end{aligned} \tag{94}$$

Since  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , then, for  $t \in [-r, 0]$  and  $k(t) \in D(A)$ , the result follows from Theorem 4.2.  $\square$

The system (83)-(84) determines the  $\alpha_i^n$ 's. Thus, we have following theorem.

**Theorem 5.3.** *Let us assume that (A1)-(A5) are satisfied. If  $k(t) \in D(A)$  for each  $t \in [-r, 0]$ , then*

$$\lim_{n \rightarrow \infty} \sup_{t \in [-r, T]} \left[ \sum_{i=0}^n \lambda_i^{2\alpha} (\alpha_i(t) - \alpha_i^n(t))^2 \right] = 0. \tag{95}$$

*Proof.* It can easily be determined that

$$A^\alpha[x(t) - v_n(t)] = A^\alpha \left[ \sum_{i=0}^n (\alpha_i(t) - \alpha_i^n(t))\chi_i \right] + A^\alpha \sum_{i=n+1}^\infty \alpha_i(t)\chi_i$$

$$= \sum_{i=0}^n \lambda_i^\alpha (\alpha_i(t) - \alpha_i^n) \chi_i + \sum_{i=n+1}^\infty \lambda_i^\alpha \alpha_i(t) \chi_i. \tag{96}$$

Thus, we conclude that

$$\| A^\alpha [x(t) - v_n(t)] \|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha} (\alpha_i(t) - \alpha_i^n(t))^2. \tag{97}$$

From the Theorem 4.2, we have  $v_n \rightarrow x$  as  $n \rightarrow \infty$ . Thus, we conclude that  $\alpha_i^n \rightarrow \alpha_i$  as  $n \rightarrow \infty$ . This gives the proof of the theorem.  $\square$

### 6. Example

Let us consider the following integro-differential equation with deviated argument of the form

$$\begin{aligned} \frac{\partial^q w(t, x)}{\partial t^q} &= \frac{\partial^2 w(t, x)}{\partial x^2} + H(x, w(t, x)) + G(t, x, w(t + \theta, x)) \\ &\quad + \int_0^t b(t, \tau) \int_{-r}^0 \frac{\gamma_1(\theta)}{1 + w(\tau + \theta, x)} d\theta d\tau, \quad t > 0, x \in (0, 1), \end{aligned} \tag{98}$$

$$w(t, 0) = w(t, 1) = 0, \quad t > 0, \tag{99}$$

$$w(\theta, x) = \frac{1}{p^2} \cdot \frac{|w(\theta, x)|}{1 + |w(\theta, x)|}, \quad -r \leq \theta \leq 0, \tag{100}$$

where  $t \in [0, 1]$ ,  $x \in [0, 1]$ ,  $q \in (0, 1)$ ,  $p \in \mathbb{N}$ ,  $r > 0$ ,  $b$  is real valued,  $\gamma_1 : [-r, 0] \rightarrow \mathbb{R}$  are continuous functions with  $\int_{-r}^0 |\gamma_1(\theta)| d\theta < 1$ ,  $H$  is given by

$$H(x, w(t, x)) := \int_0^x \mathcal{K}(x, y) w(g(t) |w(t, y)|, y) dy, \tag{101}$$

and the function  $G : \mathbb{R}^+ \times [0, 1] \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is measurable in  $x$ , locally Lipschitz continuous in  $w$ , uniformly in  $x$  and locally Hölder continuous in  $t$ . Here, we assume that  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is locally Hölder continuous in  $t$  such that  $g(0) = 0$  and  $\mathcal{K} \in C^1([0, 1] \times [0, 1], \mathbb{R})$ .

Let  $H = L^2((0, 1), \mathbb{R})$ . Now, we define operator by  $Aw = -d^2w/dx^2$  with domain  $D(A) = H^2(0, 1) \cap H_0^2(0, 1)$ . We also have  $X_{1/2} = D((A)^{1/2}) = H_0^1(0, 1)$ , and  $X_{-1/2} = (H_0^1(0, 1))^* = H^{-1}(0, 1) \equiv H^1(0, 1)$ . For each  $w \in D(A)$  and  $\lambda \in \mathbb{R}$  with  $-Aw = \lambda w$ , we get

$$\begin{aligned} \langle -Aw, w \rangle &= \langle \lambda w, w \rangle, \\ \|w'\|_{L^2} &= \lambda \|w\|_{L^2}, \quad \text{for some } \lambda > 0. \end{aligned} \tag{102}$$

The  $w(x) = C \sin(\sqrt{\lambda}x) + D \cos(\sqrt{\lambda}x)$  is the solution of the problem  $Aw = -\lambda w$ . By utilizing the boundary conditions, we get  $D = 0$  and  $\lambda_n = n^2\pi^2$  for  $n \in \mathbb{N}$ . Thus,  $w_n(x) = C \sin(\sqrt{\lambda}x)$  is the eigenvector corresponding to eigenvalue  $\lambda_n$ . We also have  $\langle w_n, w_m \rangle = 0$  for  $n \neq m$  and  $\langle w_n, w_n \rangle = 1$ . Thus, we have that for  $w \in D(A)$ , there exists a sequence  $\beta_n$  of real numbers such that

$w(x) = \sum_{n \in \mathbb{N}} \beta_n w_n(x)$  with  $\sum_{n \in \mathbb{N}} (\beta_n)^2$ , and  $\sum_{n \in \mathbb{N}} (\beta_n)^2 (\lambda_n)^2 < \infty$ . The, we have following representation of the semigroup

$$\mathcal{T}(t)w = \sum_{n \in \mathbb{N}} \exp(-n^2 t) \langle w, w_n \rangle w_n. \tag{103}$$

Now, for  $x \in (0, 1)$ , we define  $f : [0, 1] \times C([-r, 0], H_0^1(0, 1)) \times H^1(0, 1) \rightarrow L^2(0, 1)$  by

$$f(t, \phi, \psi) = H(x, \psi) + G(t, x, \phi), \tag{104}$$

where

$$H(x, \psi(x, t)) = \int_0^x \mathcal{K}(x, y) \psi(y, t) dy. \tag{105}$$

Thus, it can be verified that  $f$  satisfies the hypotheses (A3).

Similarly, for  $x \in (0, 1)$ , we define  $g : [0, 1] \times H_0^1(0, 1) \times C([-r, 0], H_0^1(0, 1)) \rightarrow L^2(0, 1)$  by

$$g(t, \varphi, \phi)(x) = \int_{-r}^0 \frac{\gamma_1(\theta)}{1 + |\phi(\theta)(x)|} d\theta, \quad \theta \in [-r, 0]. \tag{106}$$

Then, it can be seen that  $g$  fulfills hypotheses (A5). Thus, we can apply the results of previous sections to study the existence and convergence of the mild solution to system (98)-(100).

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