J. Appl. Math. & Informatics Vol. **33**(2015), *No. 5 - 6, pp. 687 - 697*
http://dx.doi.org/10.14317/jami.2015.687

ASYMPTOTIC PROPERTY FOR PERTURBED NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. This paper shows that the solutions to the perturbed nonlinear functional differential system

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds, f(t, 0) = 0, g(t, 0, 0) = 0$$

go to zero as t goes to infinity. To show asymptotic property, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ and the fundamental matrix of the unperturbed system $y' = f(t, y)$.

AMS Mathematics Subject Classification : 34D05.
Key words and phrases : asymptotically stable, exponentially asymptotic stability, exponentially asymptotic stability in variation., nonlinear nonautonomous system.

1. Introduction

Elaydi and Farran [8] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte [13] investigated the stability and asymptotic behavior of solutions of the functional differential equation. Gonzalez and Pinto [9] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al. [6,7] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo [11] and Choi and Goo [2,4] investigated Lipschitz and asymptotic stability for perturbed differential systems.

In this paper we will obtain some results on asymptotic property for nonlinear perturbed differential systems. We will employ the theory of integral inequalities to study asymptotic property for solutions of the nonlinear differential systems. The method incorporating integral inequalities takes an important place among

Received March 29, 2015. Accepted June 29, 2015. *Corresponding author .

the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential system of (1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds, \; y(t_0) = y_0, \tag{2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (3).

Before giving further details, we give some of the main definitions that we need in the sequel [8].

Definition 2.1. The system (1) (the zero solution $x = 0$ of (1)) is called
(S)*stable* if for any $\epsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \ge t_0 \ge 0$,
(AS)*asymptotically stable* if it is stable and if there exists $\delta = \delta(t_0) > 0$ such that if $|x_0| < \delta$, then $|x(t)| \to 0$ as $t \to \infty$,
(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \le K\, |x_0| e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \le K\, e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \infty$.

Remark 2.1 ([9]). The last definition implies that for $|x_0| \le \delta$

$$|x(t)| \le K\,|x_0| e^{-c(t-t_0)}, 0 \le t_0 \le t.$$

We give some related properties that we need in the sequel. We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.1. *Let x and y be a solution of (1) and (5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \ge t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.2 (Bihari-type inequality). *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],$$

where $t_0 \le t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.3 ([10]). *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) w(u(\tau)) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) u(r) dr) d\tau ds.$$

Then

$$u(t) \quad \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) dr) d\tau ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^t \lambda_1(s)\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^\tau \lambda_4(r)dr)d\tau ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.4 ([3])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau)w(u(\tau)) \\ + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)w(u(r))dr)d\tau ds + \int_{t_0}^t \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)u(\tau)d\tau ds. \end{aligned} \tag{6}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)d\tau)ds\Big], \end{aligned} \tag{7}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.2, and*

$$\begin{aligned} b_1 = \quad \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\textstyle\int_{t_0}^s \lambda_7(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Define a function $v(t)$ by the right member of (6). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)u(t) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s)w(u(s)) + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)w(u(\tau))d\tau)ds \\ &+\lambda_6(t)\int_{t_0}^t \lambda_7(s)u(s)ds\Big) \\ &\le \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s) + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau)ds \\ &+\lambda_6(t)\int_{t_0}^t \lambda_7(s)ds\Big)\Big]w(v(t)), \end{aligned}$$

$t \ge t_0$, since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$\begin{aligned} v(t) \le c + \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)d\tau\Big)w(z(s))ds. \end{aligned} \tag{8}$$

It follows from Lemma 2.2 that (8) yields the estimate (7). □

For the proof we need the following corollary from Lemma 2.4.

Corollary 2.5. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$*,* $w \in C((0,\infty))$*, and* $w(u)$ *be nondecreasing in* u*,* $u \le w(u)$*. Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$*,*

$$u(t) \le c + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s \lambda_2(\tau) u(\tau) d\tau ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) w(u(\tau)) \\ + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) w(u(r)) dr) d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \textstyle\int_{t_0}^t (\lambda_1(s) \int_{t_0}^s \lambda_2(\tau) d\tau + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) \\ + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) dr) d\tau) ds\Big],$$

where $t_0 \le t < b_1$*,* W*,* W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^t (\lambda_1(s) \int_{t_0}^s \lambda_2(\tau) d\tau + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) \\ + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) dr) d\tau) ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.6 ([5])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$*,* $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u*,* $u \le w(u)$*. Suppose that for some* $c > 0$*,*

$$u(t) \le c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) u(\tau) d\tau ds \\ + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) w(u(\tau)) d\tau ds, \quad 0 \le t_0 \le t. \tag{9}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds\Big], \tag{10}$$

where $t_0 \le t < b_1$*,* W*,* W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau \\ + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Define a function $z(t)$ by the right member of (9). Then, we have $z(t_0) = c$ and

$$z'(t) = \lambda_1(t) u(t) + \lambda_2(t) w(u(t)) + \lambda_3(t) \textstyle\int_{t_0}^t \lambda_4(s) u(s) ds + \lambda_5(t) \int_{t_0}^t \lambda_6(s) w(u(s)) ds \\ \le (\lambda_1(t) + \lambda_2(t) + \lambda_3(t) \textstyle\int_{t_0}^t \lambda_4(s) ds + \lambda_5(t) \int_{t_0}^t \lambda_6(s) ds) w(z(t)), \ t \ge t_0,$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \le w(u)$, and $u(t) \le z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$z(t) \le c + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau) w(z(s)) + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau) w(z(s))) ds. \quad (11)$$

It follows from Lemma 2.2 that (11) yields the estimate (10). □

We prepare two corollaries from Lemma 2.6 that are used in proving the theorems.

Corollary 2.7. *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) u(\tau) d\tau ds, \quad 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau) ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 2.2, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau) ds \in \mathrm{domW}^{-1}\Big\}.$$

Corollary 2.8. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \le \quad & c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds \\ & + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) w(u(\tau)) d\tau ds, \quad 0 \le t_0 \le t. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau) ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 2.2, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau) ds \in \mathrm{domW}^{-1}\Big\}.$$

3. Main results

In this section, we investigate asymptotic property for solutions of perturbed nonlinear functional differential systems.

Theorem 3.1. *Let the solution $x = 0$ of (1) be EASV. Suppose that the perturbing term $g(t, y, Ty)$ satisfies*

$$|g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t) w(|y(t)|) + |Ty(t)|\Big), \quad (12)$$

and

$$|Ty(t)| \le b(t) \int_{t_0}^{t} k(s)|y(s)| ds, \quad (13)$$

where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$, $w(u)$ *is nondecreasing in* u, *and* $u \le w(u)$. *If*

$$M(t_0) = W^{-1}\Big[W(c) + \int_{t_0}^{\infty} Me^{\alpha s} \int_{t_0}^{s} [a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr]d\tau ds\Big] < \infty, \quad (14)$$

where $t \ge t_0$ *and* $c = |y_0|Me^{\alpha t_0}$, *then all solutions of (2) approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, it is EAS by remark 2.1. Using Lemma 2.1, (12), and (13), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} e^{-\alpha\tau}(a(\tau)w(|y(\tau)|) \\ &+b(\tau) \int_{t_0}^{\tau} k(r)e^{-\alpha r}|y(r)|dr)d\tau ds.\end{aligned}$$

Then, we obtain

$$\begin{aligned}|y(t)| &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} (a(\tau)w(|y(\tau)|e^{\alpha\tau}) \\ &+b(\tau) \int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr)d\tau ds,\end{aligned}$$

since w is nondecreasing. Set $u(t) = |y(t)|e^{\alpha t}$. By Lemma 2.3 and (14) we have

$$\begin{aligned}|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + \int_{t_0}^{t} Me^{\alpha s} \int_{t_0}^{s} [a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr]d\tau ds\Big] \\ &\le e^{-\alpha t}M(t_0), \ t \ge t_0,\end{aligned}$$

where $c = M|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. □

Remark 3.1. Letting $b(t) = 0$ in Theorem 3.1, we obtain the similar result as that of Theorem 3.5 in [4].

Theorem 3.2. *Let the solution* $x = 0$ *of (1) be EASV. Suppose that the perturbing term* $g(t, y, Ty)$ *satisfies*

$$|g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + |Ty(t)|\Big), \quad (15)$$

and

$$|Ty(t)| \le b(t)w(|y(t)|) + c(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds, \quad (16)$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L_1(\mathbb{R}^+)$, $w(u)$ *is nondecreasing in* u, $u \leq w(u)$. *If*

$$M(t_0) = W^{-1}\Big[W(c) + \int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}[a(\tau) + b(\tau) + c(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds\Big] < \infty, \quad (17)$$

where $t \geq t_0$ *and* $c = |y_0|Me^{\alpha t_0}$, *then all solutions of (2) approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, it is EAS by Remark 2.1. Applying Lemma 2.1, (15), and (16), we have

$$|y(t)| \leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s} e^{-\alpha\tau}(a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|)$$
$$+c(\tau)\int_{t_0}^{\tau} k(r)e^{-\alpha r}w(|y(r)|)dr)d\tau ds.$$

Since w is nondecreasing, we obtain

$$|y(t)| \leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}(a(\tau)|y(\tau)|e^{\alpha\tau} + b(\tau)w(|y(\tau)|e^{\alpha\tau})$$
$$+c(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|e^{\alpha r})dr)d\tau ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Corollary 2.5 and (17), we have

$$|y(t)| \leq e^{-\alpha t}W^{-1}\Big[W(c) + \int_{t_0}^{t} Me^{\alpha s}\int_{t_0}^{s}[a(\tau) + b(\tau) + c(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds\Big]$$
$$\leq e^{-\alpha t}M(t_0),\ t \geq t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Remark 3.2. Letting $w(u) = u, b(t) = c(t) = 0$ in Theorem 3.2, we obtain the similar result as that of Corollary 3.6 in [4].

Theorem 3.3. *Let the solution* $x = 0$ *of (1) be EASV. Suppose that the perturbed term* $g(t, y, Ty)$ *satisfies*

$$\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \leq e^{-\alpha t}\Big(a(t)w(|y(t)|) + |Ty(t)|\Big), \quad (18)$$

and

$$|Ty(t)| \leq b(t)\int_{t_0}^{t} k(s)|y(s)|ds, \quad (19)$$

where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$ *and* $w(u)$ *is nondecreasing in* u, $u \leq w(u)$. *If*

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] < \infty, \quad (20)$$

where $b_1 = \infty$ *and* $c = M|y_0|e^{\alpha t_0}$, *then all solutions of (2) approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, it is EAS. By conditions, Lemma 2.1, (18), and (19), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}[e^{-\alpha s}a(s)w(|y(s)|) \\ +e^{-\alpha s}b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau]ds.$$

Since w is nondecreasing, we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}a(s)w(|y(s)|e^{\alpha s})ds \\ + \int_{t_0}^{t} Me^{-\alpha t}b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|e^{\alpha\tau}d\tau ds.$$

Then, it follows from Corollary 2.7 with $u(t) = |y(t)|e^{\alpha t}$ and (20) that

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] \\ \le e^{-\alpha t}M(t_0),\ t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (2) approach zero as $t \to \infty$, and so the proof is complete. □

Remark 3.3. Letting $b(t) = 0$ in Theorem 3.3, we obtain the similar result as that of Theorem 3.7 in [4].

Theorem 3.4. *Let the solution* $x = 0$ *of (1) be EASV. Suppose that the perturbed term* $g(t, y, Ty)$ *satisfies*

$$\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le e^{-\alpha t}\Big(a(t)|y(t)| + |Ty(t)|\Big), \tag{21}$$

and

$$|Ty(t)| \le b(t)w(|y(t)|) + c(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds, \tag{22}$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L_1(\mathbb{R}^+)$ *and* $w(u)$ *is nondecreasing in* u, $u \le w(u)$. *If*

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}(a(s) + b(s) + c(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] < \infty, \tag{23}$$

where $b_1 = \infty$ *and* $c = M|y_0|e^{\alpha t_0}$, *then all solutions of (2) approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, it is EAS. By means of Lemma 2.1, (21), and (22), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}[e^{-\alpha s}(a(s)|y(s)| + b(s)w(|y(s)|) \\ +c(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau)]ds.$$

Since w is nondecreasing, we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}(a(s)|y(s)|e^{\alpha s} + b(s)w(|y(s)|e^{\alpha s})ds \\ + \int_{t_0}^{t} Me^{-\alpha t}c(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, it follows from Corollary 2.8 and (23) that

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}(a(s) + b(s) + c(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] \\ \le e^{-\alpha t}M(t_0),\ t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. From the above inequality, we obtain the desired result. □

Remark 3.4. Letting $w(u) = u, b(t) = c(t) = 0$ in Theorem 3.4, we obtain the similar result as that of Corolary 3.8 in [4].

Acknowledgment

The authors are very grateful for the referee's valuable comments.

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