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SYMMETRIC IDENTITIES FOR TWISTED q-EULER ZETA FUNCTIONS

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Abstract. In this paper we investigate some symmetric property of the twisted q-Euler zeta functions and twisted q-Euler polynomials.

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1. Introduction

The Euler polynomials and numbers possess many interesting properties in many areas of mathematics and physics. Many mathematicians have studied in the area of the q-extension of the Euler numbers and polynomials (see [3-10]).

Recently, Y. Hu studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field (see [2]). D. Kim *et al.* [3] derived some identities of symmetry for Carlitz's q-Euler numbers and polynomials in complex field. J.Y. Kang and C.S. Ryoo investigated some identities of symmetry for q-Genocchi polynomials (see [1]). In [8], we obtained some identities of symmetry for Carlitz's twisted q-Euler polynomials associated with p-adic q-integral on $\mathbb{Z}_p$. In this paper, we establish some interesting symmetric identities for twisted q-Euler zeta functions and twisted q-Euler polynomials in complex field. If we take $\varepsilon = 1$ in all equations of this article, then [3] are the special case of our results. Throughout this paper we use the following notations. By $\mathbb{N}$ we denote the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{(see } [1,2,3,4]).$$

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Note that $\lim_{q\to 1}[x] = x$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Let ε be the p^N-th root of unity. Then the twisted q-Euler polynomials $E_{n,q,\varepsilon}$ are defined by the generating function to be

$$F_{q,\varepsilon}(t,x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \varepsilon^n e^{[x+n]_q t} = \sum_{n=0}^{\infty} E_{n,q,\varepsilon}(x) \frac{t^n}{n!}. \tag{1.1}$$

When $x = 0$, $E_{n,q,\varepsilon} = E_{n,q,\varepsilon}(0)$ are called the twisted q-Euler numbers. By (1.1) and Cauchy product, we have

$$\begin{aligned} E_{n,q,\varepsilon}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,q,\varepsilon} [x]_q^{n-l} \\ &= (q^x E_{q,\varepsilon} + [x]_q)^n \end{aligned} \tag{1.2}$$

with the usual convention about replacing $(E_{q,\varepsilon})^n$ by $E_{n,q,\varepsilon}$.

By using (1.1), we note that

$$\left. \frac{d^k}{dt^k} F_{q,\varepsilon}(t,x) \right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n \varepsilon^n q^n [n+x]_q^k, (k \in \mathbb{N}). \tag{1.3}$$

By (1.3), we are now ready to define the Hurwitz type of the twisted q-Euler zeta functions.

Definition 1.1. Let $s \in \mathbb{C}$ and $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \ldots$. We define

$$\zeta_{q,\varepsilon}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n q^n}{[n+x]_q^s}. \tag{1.4}$$

Note that $\zeta_{q,\zeta}(s,x)$ is a meromorphic function on $\mathbb{C}$. A relation between $\zeta_{q,\varepsilon}(s,x)$ and $E_{k,q,\varepsilon}(x)$ is given by the following theorem.

Theorem 1.2. *For $k \in \mathbb{N}$, we get*

$$\zeta_{q,\varepsilon}(-k,x) = E_{k,q,\varepsilon}(x). \tag{1.5}$$

Observe that $\zeta_{q,\varepsilon}(-k,x)$ function interpolates $E_{k,q,\varepsilon}(x)$ polynomials at non-negative integers.

2. Symmetric property of twisted q-Euler zeta functions

In this section, by using the similar method of [1, 2, 3], expect for obvious modifications, we investigate some symmetric identities for twisted q-Euler polynomials and twisted q-Euler zeta functions. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$.

Theorem 2.1. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\sum_{i=0}^{w_2-1}[2]_{q^{w_1}}[w_1]_q^s(-1)^i\varepsilon^{w_1 i}q^{w_1 i}\zeta_{q^{w_2},\varepsilon^{w_2}}\left(s, w_1x+\frac{w_1 i}{w_2}\right)$$
$$=\sum_{j=0}^{w_1-1}[2]_{q^{w_2}}[w_2]_q^s(-1)^j\varepsilon^{w_2 j}q^{w_2 j}\zeta_{q^{w_1},\varepsilon^{w_1}}\left(s, w_2x+\frac{w_1 j}{w_2}\right).$$

Proof. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In Definition 1.1, we derive next result by substitute $w_1x + \frac{w_1 i}{w_2}$ for x in and replace q and ε by q^{w_2} and ε^{w_2}, respectively.

$$\begin{aligned}\zeta_{q^{w_2},\varepsilon^{w_2}}(s, w_1x+\frac{w_1 i}{w_2}) &= [2]_{q^{w_2}}\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon^{w_2 n}q^{w_2 n}}{[n+w_1x+\frac{w_1 i}{w_2}]_{q^{w_2}}^s}\\ &= [2]_{q^{w_2}}[w_2]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon^{w_2 n}q^{w_2 n}}{[w_1w_2x+w_1 i+w_2 n]_q^s}.\end{aligned} \tag{2.1}$$

Since for any non-negative integer n and odd positive integer w_1, there exist unique non-negative integer r, j such that $m = w_1r + j$ with $0 \le j \le w_1 - 1$. So, the equation (2.1) can be written as

$$\begin{aligned}&\zeta_{q^{w_2},\zeta^{w_2}}\left(s, w_1x+\frac{w_1 i}{w_2}\right)\\ &= [2]_{q^{w_2}}[w_2]_q^s\sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty}\frac{(-1)^{w_1r+j}\varepsilon^{w_2(w_1r+j)}q^{w_2(w_1r+j)}}{[w_1w_2r+w_1w_2x+w_1 i+w_2 j]_q^s}\\ &= [2]_{q^{w_2}}[w_2]_q^s\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^j\varepsilon^{w_2(w_1r+j)}q^{w_2(w_1r+j)}}{[w_1w_2(r+x)+w_1 i+w_2 j]_q^s}.\end{aligned} \tag{2.2}$$

In similarly, we can see that

$$\begin{aligned}\zeta_{q^{w_1},\varepsilon^{w_1}}\left(s, w_2x+\frac{w_2 j}{w_1}\right) &= [2]_{q^{w_1}}\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon^{w_1 n}q^{w_1 n}}{[n+w_2x+\frac{w_2 j}{w_1}]_{q^{w_1}}^s}\\ &= [2]_{q^{w_1}}[w_1]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon^{w_1 n}q^{w_1 n}}{[w_1w_2x+w_1 n+w_2 j]_q^s}.\end{aligned} \tag{2.3}$$

Using the method in (2.2), we obtain

$$\begin{aligned}&\zeta_{q^{w_1},\varepsilon^{w_1}}\left(s, w_2x+\frac{w_2j}{w_1}\right)\\&=[2]_{q^{w_1}}[w_1]_q^s\sum_{\substack{w_2r+i=0\\0\le i\le w_2-1}}^{\infty}\frac{(-1)^{w_2r+i}\varepsilon^{w_1(w_2r+i)}q^{w_1(w_2r+i)}}{[w_1w_2r+w_1w_2x+w_1i+w_2j]_q^s}\\&=[2]_{q^{w_1}}[w_1]_q^s\sum_{i=0}^{w_2-1}\sum_{r=0}^{\infty}\frac{(-1)^{i}\varepsilon^{w_1(w_2r+i)}q^{w_1(w_2r+i)}}{[w_1w_2(r+x)+w_1i+w_2j]_q^s}.\end{aligned}\tag{2.4}$$

From (2.2) and (2.4), we have

$$\begin{aligned}&\sum_{i=0}^{w_2-1}[2]_{q^{w_1}}[w_1]_q^s(-1)^i\varepsilon^{w_1i}q^{w_1i}\zeta_{q^{w_2},\varepsilon^{w_2}}\left(s,w_1x+\frac{w_1i}{w_2}\right)\\&=\sum_{j=0}^{w_1-1}[2]_{q^{w_2}}[w_2]_q^s(-1)^j\varepsilon^{w_2j}q^{w_2j}\zeta_{q^{w_1},\varepsilon^{w_1}}\left(s,w_2x+\frac{w_2j}{w_1}\right).\end{aligned}\tag{2.5}$$

□

Next, we derive the symmetric results by using definition and theorem of the twisted q-Euler polynomials.

Theorem 2.2. *Let i, j and n be non-negative integers. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned}&\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1i}q^{w_1i}E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1i}{w_2}\right)\\&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1-1}(-1)^j\varepsilon^{w_2j}q^{w_2j}E_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2j}{w_1}\right).\end{aligned}$$

Proof. By substitute $w_1x+\frac{w_1i}{w_2}$ for x in Theorem 1.2 and replace q and ε by q^{w_2} and ε^{w_2}, respectively, we derive

$$\begin{aligned}&E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1i}{w_2}\right)\\&=[2]_{q^{w_2}}\sum_{m=0}^{\infty}(-1)^m\varepsilon^{w_2m}q^{w_2m}\left[w_1x+\frac{w_1i}{w_2}+m\right]_{q^{w_2}}^n\\&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{m=0}^{\infty}(-1)^m\varepsilon^{w_2m}q^{w_2m}[w_1w_2x+w_1i+w_2m]_q^n.\end{aligned}\tag{2.6}$$

Since for any non-negative integer m and odd positive integer w_1, there exist unique non-negative integer r, j such that $m = w_1r + j$ with $0 \le j \le w_1 - 1$.

Hence, the equation (2.6) is written as

$$\begin{aligned}
&E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty}(-1)^{w_1r+j}\varepsilon^{w_2(w_1r+j)}q^{w_2(w_1r+j)}\\
&\qquad\qquad\times[w_1w_2x+w_1i+w_2(w_1r+j)]_q^n\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{i=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^{w_1r+j}\varepsilon^{w_2(w_1r+j)}q^{w_2(w_1r+j)}\\
&\qquad\qquad\times[w_1w_2(x+r)+w_1i+w_2j]_q^n.
\end{aligned}\tag{2.7}$$

In similar, we have

$$\begin{aligned}
&E_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right)\\
&=[2]_{q^{w_1}}\sum_{m=0}^{\infty}(-1)^m\varepsilon^{w_1m}q^{w_1m}\left[w_2x+\frac{w_2j}{w_1}+m\right]_{q^{w_1}}^n\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{m=0}^{\infty}(-1)^m\varepsilon^{w_1m}q^{w_1m}[w_1w_2x+w_2j+w_1m]_q^n
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&E_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right)\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{\substack{w_2r+i=0\\0\le i\le w_2-1}}^{\infty}(-1)^{w_2r+i}\varepsilon^{w_1(w_2r+i)}q^{w_1(w_2r+j)}\\
&\qquad\qquad\times[w_1w_2x+w_1i+w_2(w_1r+j)]_q^n\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2-1}\sum_{r=0}^{\infty}(-1)^{w_2r+i}\varepsilon^{w_1(w_2r+i)}q^{w_1(w_2r+i)}\\
&\qquad\qquad\times[w_1w_2(x+r)+w_1i+w_2j]_q^n.
\end{aligned}\tag{2.9}$$

It follows from the above equation that

$$\begin{aligned} &\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1 i}q^{w_1 i}E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\ &=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{r=0}^{\infty}(-1)^{i+j}\varepsilon^{w_1w_2r+w_1i+w_2j}\\ &\qquad\times q^{w_1w_2r+w_1i+w_2j}[w_1w_2(x+r)+w_1i+w_2j)]_q^n \qquad (2.10)\\ &=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1-1}(-1)^j\varepsilon^{w_2 j}q^{w_2 j}E_{n,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right). \end{aligned}$$

From (2.8) and (2.9), the proof of the Theorem 2.2 is completed. □

By (1.2) and Theorem 2.2, we have the following theorem.

Theorem 2.3. *Let i, j and n be non-negative integers. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned} &[2]_{q^{w_1}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^k[w_2]_q^{n-k}E_{n-k,q^{w_2},\varepsilon^{w_2}}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k\\ &=[2]_{q^{w_2}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^{n-k}[w_2]_q^{k}E_{n-k,q^{w_1},\varepsilon^{w_1}}(w_2x)\sum_{j=0}^{w_1-1}(-1)^j\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k. \end{aligned}$$

Proof. After some calculations, we obtain

$$\begin{aligned} &\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}q^{w_1 i}E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\ &=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1 i}q^{w_1 i}\sum_{l=0}^{n}\binom{n}{l}q^{(n-l)w_1 i}E_{n-l,q^{w_2},\varepsilon^{w_2}}(w_1x)\left[\frac{w_1 i}{w_2}\right]_{q^{w_2}}^l\\ &=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{k=0}^{n}\binom{n}{k}E_{n-k,q^{w_2},\varepsilon^{w_2}}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}\left[\frac{w_1}{w_2}\right]_{q^{w_2}}^k[i]_{q^{w_1}}^k\\ &=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{w_1}{w_2}\right]_{q^{w_2}}^k E_{n-k,q^{w_2},\varepsilon^{w_2}}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k, \end{aligned} \qquad (2.11)$$

and

$$\begin{aligned} &\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1-1}(-1)^j\varepsilon^{w_2 j}q^{w_2 j}E_{n,q^{w_2},\varepsilon^{w_2}}\left(w_2x+\frac{w_2 j}{w_1}\right)\\ &=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{w_1}{w_2}\right]_{q^{w_1}}^k E_{n-k,q^{w_1},\varepsilon^{w_1}}(w_2x) \qquad (2.12)\\ &\qquad\times\sum_{j=0}^{w_1-1}(-1)^j\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k. \end{aligned}$$

From (2.11), (2.12) and Theorem 2.2, we obtain that

$$\begin{aligned}
&[2]_{q^{w_1}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[w_1]_q^{n-k}[w_2]_q^k} E_{n-k,q^{w_2},\varepsilon^{w_2}}(w_1 x) \\
&\qquad \times \sum_{i=0}^{w_2-1} (-1)^i \varepsilon^{w_1 i} q^{(1+n-k)w_1 i} [i]_{q^{w_1}}^k \\
&= [2]_{q^{w_2}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[w_1]_q^{k}[w_2]_q^{n-k}} E_{n-k,q^{w_1},\varepsilon^{w_1}}(w_2 x) \\
&\qquad \times \sum_{j=0}^{w_1-1} (-1)^j \varepsilon^{w_2 j} q^{(1+n-k)w_2 j} [j]_{q^{w_2}}^k .
\end{aligned}$$

Hence, we have above theorem. □

By Theorem 2.3, we obtain the interesting symmetric identity for twisted q-Euler numbers in complex field.

Corollary 2.4. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned}
&[2]_{q^{w_1}} \sum_{k=0}^{n} \binom{n}{k} [w_1]_q^k [w_2]_q^{n-k} E_{n-k,q^{w_2},\varepsilon^{w_2}} \sum_{i=0}^{w_2-1} (-1)^i \varepsilon^{w_1 i} q^{(1+n-k)w_1 i} [i]_{q^{w_1}}^k \\
&= [2]_{q^{w_2}} \sum_{k=0}^{n} \binom{n}{k} [w_1]_q^{n-k} [w_2]_q^{k} E_{n-k,q^{w_1},\varepsilon^{w_1}} \sum_{j=0}^{w_1-1} (-1)^j \varepsilon^{w_2 j} q^{(1+n-k)w_2 j} [j]_{q^{w_2}}^k .
\end{aligned}$$

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