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RENEWAL AND RENEWAL REWARD THEORIES FOR T-INDEPENDENT FUZZY RANDOM VARIABLES†

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Abstract. Recently, Wang et al. [Computers and Mathematics with Applications 57 (2009) 1232–1248.] and Wang and Watada [Information Sciences 179 (2009) 4057–4069.] studied the renewal process and renewal reward process with fuzzy random inter-arrival times and rewards under the T-independence associated with any continuous Archimedean t-norm. But, their main results do not cover the classical theory of the random elementary renewal theorem and random renewal reward theorem when fuzzy random variables degenerate to random variables, and some given assumptions relate to the membership function of the fuzzy variable and the Archimedean t-norm of the results are restrictive. This paper improves the results of Wang and Watada and Wang et al. from a mathematical perspective. We release some assumptions of the results of Wang and Watada and Wang et al. and completely generalize the classical stochastic renewal theorem and renewal rewards theorem.

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1. Introduction

The theory of fuzzy sets, introduced by Zadeh [27, 28], has been widely examined and applied to statistics and the possibility theory in recent years. Since Puri and Ralescu's [20] introduction of the concept of fuzzy random variables, there has been growing interest in fuzzy variables [14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30]. A number of studies [7, 8, 9, 10, 11, 19] have investigated renewal theory in the fuzzy environment. Hwang [12] considered the stochastic process for fuzzy random variables and proved a theorem for the fuzzy

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rate of a fuzzy renewal process. Popova and Wu [19] proposed a theorem presenting the long-run average fuzzy reward by using the strong law of large numbers. Zhao and Liu [29] and Hong [11] discussed the renewal process by considering the fuzzy inter-arrival time and proved that the expected reward per unit time is the expected value of the ratio of the reward spent in one cycle to the length of the cycle. Zhao and Tang [30] obtained some properties of a fuzzy random renewal process generated by a sequence of independent and identically distributed fuzzy random interval times based on fuzzy random theory, particularly Blackwell's theorem for fuzzy random variables. Hong [10] considered the convergence of fuzzy random elementary renewal variables and total rewards earned by time t in the sense of the extended Hausdorff metric d_∞ and applied this method to a sequence of independent and identically distributed fuzzy random variables to prove the fuzzy random elementary renewal theorem and the fuzzy random renewal reward theorem. All of these studies used min-norm-based fuzzy operations. In general, we can consider the extension principle realized by the means of some t-norm. Such a generalized extension principle yields different operations for fuzzy numbers, in accordance with different t-norms. Recently, many studies have focused on such t-norm-based operations regarding fuzzy numbers and fuzzy random variables [3, 4, 5, 6, 7, 22, 23, 24, 25].

Recentlt, Wang et al. [25] studied a fuzzy random renewal process in which the interarrival times are assumed to be independent and identically distributed fuzzy random variables, using the extension principle associated with a class of continuous Archimedean triangular norms. They discussed the fuzzy random renewal process based on the obtained limit theorems, and derived a fuzzy random elementary renewal theorem for the long-run expected renewal rate. Wang and Watada [24] also studied a renewal reward process with fuzzy random interarrival times and rewards under the T-independence associated with any continuous Archimedean t-norm. The interarrival times and rewards of the renewal reward process are assumed to be positive fuzzy random variables whose fuzzy realizations are T-independent fuzzy variables. They derived some limit theorems in mean chance measures for fuzzy random renewal rewards and proved a fuzzy random renewal reward theorem for the long-run expected reward per unit time of the renewal reward process. In this paper, we consider the results of Wang et al. [25] and Wang and Watada [24] again. There are two main points to be improved. The first point is that when the renewal theorem and renewal rewards theorem obtained in these papers degenerate to the corresponding the classical result in stochastic renewal process, the results do not cover those of classical stochastic renewal and renewal reward theories. The second point is that some conditions relate to the membership function of the fuzzy variable and the Archimedean t-norm of the results are restrictive. This paper improves the results of Wang and Watada [24] and Wang et al. [25] from a mathematical perspective. As a consequence, our results cover the classical theory of the random elementary renewal theorem and random renewal reward theorem. We also release some conditions relating to the membership function of fuzzy variable and Archimedean t-norm of the results of Wang and Watada [24] and Wang et al. [25] and completely generalize the classical stochastic renewal theorem and renewal rewards theorem.

2. Preliminary

Let $(\Gamma, P(\Gamma), Pos)$ be a possibility space. As defined in [27], a normal fuzzy variable on the real line, $\mathcal{R}$ is defined as a function $Y : \Gamma \to \mathcal{R}$ which has a unimodal, upper semi-continuous membership function μ_Y on the real line such that there exists a unique real number m satisfying $\mu_Y(m) = \sup_x \mu_Y(x) = 1$. The number $m = m(Y)$ is called the modal value of Y.

An L-R fuzzy variable $Y = (a, \alpha, \beta)_{LR}$ has a membership function from the reals into interval $[0, 1]$ satisfying

$$\mu_Y(t) = \begin{cases} R\left(\frac{t-a}{\beta}\right) & \text{for} \quad a \le t \le a+\beta, \\ L\left(\frac{a-t}{\alpha}\right) & \text{for} \quad a-\alpha \le t \le a, \\ 0, & \text{otherwise.} \end{cases}$$

where L and R are non-increasing and continuous functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

Let $[Y]_\alpha$ be the α-level sets with $[Y]_\alpha = \{x \in \mathcal{R} | \mu_Y \ge \alpha\}$ for $\alpha \in (0, 1]$, and $Y_0 = \mathrm{cl}\{x \in \mathcal{R} | \mu_Y > 0\}$. Let $\mathcal{K}(\mathcal{R})$ denote the class of nonempty compact convex subsets of $\mathcal{R}$ The linear structure induced by the scalar product and the Minkowski addition is that

$$\lambda A = \{\lambda a | a \in A\}, A + B = \{a + b | a \in A, b \in B\},$$

for all $A, B \in \mathcal{K}(\mathcal{R})$, and $\lambda \in \mathcal{R}$. If d_H is the Hausdorff metric on $\mathcal{K}(\mathcal{R})$, which for $A, B \in \mathcal{K}(\mathcal{R})$ is given by

$$d_H(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |a - b|\right\},$$

then $(\mathcal{K}(\mathcal{R}), d_H)$ is a complete and separable metric space [diamond1]. The norm of an element of $\mathcal{K}(\mathcal{R})$ is denoted by

$$\|A\| = d_H(A, \{0\}) = \sup\{|x| : x \in A\}.$$

For a fuzzy variable Y and any subset D of the real numbers, the quantity

$$Nes\{Y|D\} := 1 - \sup_{x\notin D} \mu_Y(x) := 1 - Pos\{Y|D^c\}$$

is considered to measure the necessity of Y belonging to D (see [3]). If D is an interval (a, b), we may also write $Nes\{a < Y < b\}$ instead of $Nes\{Y|D\}$.

The credibility of Y belonging to D and the expected value $E[Y]$ (see [18]) are defined as

$$Cr\{Y|D\} = (1/2)\{Pos(Y|D\}+Nes\{Y|D\}) = \frac{1}{2}\left(1 + Pos\{Y|D\} - Pos\{Y|D^c\}\right),$$

$$E[Y] = \int_0^\infty Cr\{Y \geq r\}dr - \int_{-\infty}^0 Cr\{Y \leq r\}dr$$

provided that at least one of the two integrals is finite. In particular, if Y is a nonnegative fuzzy variable (i.e., $Cr\{Y < 0\} = 0$), then $E[Y] = \int_0^\infty Cr\{Y \geq r\}dr$.

Recall that a triangular norm (t-norm for short) is a function $T : [0,1]^2 \to [0,1]$ such that for any $x, y, z \in [0,1]$ the following four axioms are satisfied [13]:
(T1) Commutativity: $T(x, y) = T(y, x)$,
(T2) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$,
(T3) Monotonicity: $T(x, y) \leq T(x, z)$ whenever $y \leq z$,
(T4) Boundary condition: $T(x, 1) = x$.

The associativity (T2) allows us to extend each t-norm T in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, \cdots, x_n) \in [0,1]^n$

$$T_{k=1}^n x_k = T(T_{k=1}^{n-1} x_k, x_n) = T(x_1, x_2, \cdots, x_n).$$

A t-norm T is said to be Archimedean if $T(x, x) < x$ for all $x \in (0,1)$. It is easy to check that the minimum t-norm is not Archimedean. Moreover, from [13], every continuous Archimedean t-norm T can be represented by a continuous and strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and

$$T(x_1, x_2, \cdots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n)),$$

for all $x \in (0,1)$, $1 \leq i \leq n$, where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{for} \quad y \in [0, f(0)], \\ 0, & \text{if} \quad y > f(0). \end{cases}$$

The function f is called the additive generator of T.

Example 2.1. Examples of continuous Archimedean t–norms with additive generators are listed as follows [13]:

(a) *Dombi* $t-norm$ D, $p > 0$:

$$D_p(x, y) = \frac{1}{1 + ((\frac{1}{x} - 1)^p + (\frac{1}{y} - 1)^p)^{\frac{1}{p}}}$$

with the additive generator $f(x) = ((1-x)/x)^p$.

(b) *Hamacher* $t-norm$ H, $p \geq 0$:

$$H_p(x, y) = \frac{xy}{p + (1-p)(x + y - xy)}$$

with the additive generator $f(x) = \ln((p + (1-p)x)/x)$.

(c) *Sklar* $t-norm$ S, $p > 0$:

$$S_p(x, y) = (x^{-p} + y^{-p} - 1)^{-\frac{1}{p}}$$

with the additive generator $f(x) = (1/p)(x^{-p} - 1)$.

(d) $Frank\ t-norm\ F,\ p>0,\ p\neq 1$:

$$F_p(x,y)=\log_p\left[1+\frac{(p^x-1)(p^y-1)}{p-1}\right]$$

with the additive generator $f(x)=\log_p((p-1)/(p^x-1))$.

(e) $Yager\ t-norm\ Y,\ p>0$:

$$Y_p(x,y)=\max\{1-((1-x)^p+(1-y)^p)^{1/p},0\}$$

with the additive generator $f(x)=(1-x)^p$.

Definition 2.2 ([2])**.** Let T be a t-norm. A family of fuzzy variables $\{Y_i, i\in I\}$ is called T-independent if for any subset $\{i_1,i_2,\cdots,i_n\}\subset I$ with $n\geq 2$,

$$\mathrm{Pos}\{Y_{i_k}\in B_k, k=1,2,\cdots,n\}=T_{k=1}^n\mathrm{Pos}\{Y_{i_k}\in B_k\},$$

for any subsets $B_1,B_2,\cdots,B_n$ of $\mathcal{R}$. We say two families of fuzzy variables $\{Y_i,i\in I\}$ and $\{Z_j,j\in J\}$ are mutually T-independent if for any $\{i_1,i_2,\cdots,i_n\}\subset I$ and $\{j_1,j_2,\cdots,j_m\}\subset J$ with $n,m\geq 1$, fuzzy vectors $\{Y_{i_1},\cdots,Y_{i_n}\}$ and $\{Z_{j_1},\cdots,Z_{j_m}\}$ are T-independent.

For T-independent fuzzy variables Y_k, $1\leq k\leq m$ with possibility distributions μ_k, $1\leq k\leq m$, and a function $g:\mathcal{R}^m\to\mathcal{R}$, the possibility distribution of $g(Y_1,Y_2,\cdots,Y_m)$ is determined via the possibility distributions $\mu_1,\mu_2,\cdots,\mu_m$ as

$$\begin{aligned}\mu_{g(Y_1,Y_2,\cdots,Y_m)}(x) &= \mathrm{Pos}\{g(Y_1,Y_2,\cdots,Y_m)=x\}\\ &= \sup_{x_1,x_2,\cdots,x_m\in\mathcal{R},\ x=g(x_1,x_2,\cdots,x_m)} T_{k=1}^m\mu_k(x_k),\end{aligned}$$

where T can be any general t-norm. This is the (generalized) extension principle associated with t-norm.

For example, the sum $Y_1+Y_2+\cdots+Y_n$ and corresponding arithmetic mean $(Y_1+Y_2+\cdots+Y_n)/n$ are the fuzzy variables as defined by

$$\mu_{Y_1+Y_2+\cdots+Y_n}(z)=\sup_{x_1+x_2+\cdots+x_n=z} T(\mu_{Y_1}(x_1),\cdots,\mu_{Y_n}(x_n))$$

and

$$\frac{1}{n}(Y_1+Y_2+\cdots+Y_n)(z)=(Y_1+Y_2+\cdots+Y_n)(nz),$$

respectively.

Following Fullér and Keresztfalvi [5][5], if T is an arbitrary t-norm and $\{Y_k\}$ are normal fuzzy variables, then the equality

$$[Y_1+\cdots+Y_n]_\alpha=\bigcup_{T(\alpha_1,\cdots,\alpha_n)\geq\alpha}[Y_1]_{\alpha_1}+\cdots+[Y_n]_{\alpha_n},\ \alpha\in(0,1] \tag{1}$$

holds.

Definition 2.3 ([15]). Suppose that $(\Omega, \Sigma, \mathrm{Pr})$ is a probability space, $\mathcal{F}_v$ is a collection of fuzzy variables defined on the possibility space $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Pos})$. A fuzzy random variable is a map $\xi : \Omega \to \mathcal{F}_v$ such that for any Borel subset B of $\mathcal{R}$, $\mathrm{Pos}\{\xi(\omega) \in B\}$ is a measurable function of ω.

Suppose ξ is a fuzzy random variable on Ω, from the above definition, we know for each $\omega \in \Omega$, $\xi(\omega)$ is a fuzzy variable. Further, a fuzzy random variable ξ is said to be positive if for almost every ω, the fuzzy variable $\xi(\omega)$ is positive almost surely.

To a fuzzy random variable ξ on Ω, for each $\omega \in \Omega$, the expected value of the fuzzy variable $\xi(\omega)$, denoted by $E[\xi(\omega)]$, has been proved to be a random variable. Based on such fact, the expected value of the fuzzy random variable ξ is defined as the mathematical expectation of the random variable $E[\xi(\omega)]$.

Definition 2.4 ([15]). Let ξ be a fuzzy random variable given on a probability space $(\Omega, \Sigma, \mathrm{Pr})$. The expected value of ξ is defined as

$$E[\xi] = \int_\Omega \left[\int_0^\infty \mathrm{Cr}\{\xi(\omega) \geq r\} dr - \int_{-\infty}^0 \mathrm{Cr}\{\xi(\omega) \leq r\} dr \right] \mathrm{Pr}(d\omega).$$

Definition 2.5 ([16]). Let ξ be a fuzzy random variable, and B be a Borel subset of $\mathcal{R}$. The mean chance of an event $\xi \in B$ is defined as

$$\mathrm{Ch}\{\xi \in B\} = \int_\Omega \mathrm{Cr}\{\xi(\omega) \in B\} \mathrm{Pr}(d\omega).$$

The expected value (2) is equivalent to the following form :

$$E[\xi] = \int_0^\infty \mathrm{Ch}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq r\} dr.$$

A sequence of fuzzy random variables $\{\xi_n\}$ is said to be uniformly and essentially bounded if there are two real numbers b_L and b_U such that for each $k = 1, 2, \cdots$, we have $\mathrm{Ch}\{b_L \leq \xi_n \leq b_U\} = 1$. Moreover, we have the following convergence mode for a sequence of fuzzy random variables.

3. Fuzzy random renewal and renewal rewards process

Let ξ_n, $n = 1, 2, \cdots$ be a sequence of fuzzy random variables defined on a probability space (Ω, Σ, Pr). For each n, we denote ξ_n as the interarrival time between the $(n-1)$th and the nth event. Define

$$S_0 = 0, \quad S_n = \sum_{k=1}^n \xi_k, n \geq 1.$$

It is clear that S_n is the time when the nth renewal occurs, and for any given $\omega \in \Omega$ and integer n, $S_n(\omega) = \xi_1(\omega) + \cdots + \xi_n(\omega)$ is a fuzzy variable. Let $N(t)$ denote the total number of the events that have occurred by time t. Then we have

$$N(t) = \max\{n | 0 < S_n \leq t\}.$$

For any $\omega \in \Omega$, $N(t)(\omega) = \max\{n|S_n(\omega) \leq t\}$ is a nonnegative integer-valued fuzzy variable on the possibility space $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Pos})$. That is, $N(t)(\omega)(\gamma)$ is a nonnegative integer for any $\gamma \in \Gamma$, $\omega \in \Omega$ and $t > 0$. We call $N(t)$ a fuzzy random renewal variable, and the process $\{N(t), t > 0\}$ a fuzzy random renewal process.

On the basis of the above renewal process $\{N(t), t > 0\}$ with fuzzy random interarrival times ξ_n, $n \geq 1$, suppose each time a renewal occurs we receive a reward which is a fuzzy random variable. We denote η_n as the reward earned at each time of the nth renewal. Let $C(t)$ represent the total reward earned by time t, then we have

$$C(t) = \sum_{k=1}^{N(t)} \eta_k,$$

where $N(t)$ is the fuzzy random renewal variable.

In the paper of Wang and Watada [24], and Wang et al. [25], they discussed fuzzy stochastic renewal theories within the following conditions:

A1. T can be any continuous Archimedean t-norm with additive generator f. The interarrival times $\{\xi_n\}$ and rewards $\{\eta_n\}$ are positive fuzzy random variables, and for almost every $\omega \in \Omega$, $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ are T-independent fuzzy variable sequences, respectively.

A2. Π is a nonnegative unimodal real function with $\Pi(0) = 1$ and $\Pi(-r) = 0$, where r can be any positive real number. The support of Π is denoted by Ξ.

A3. The convex hull of composition function $f \circ \Pi$ in the support Ξ of Π satisfies: $\mathrm{co}(f \circ \Pi)_\Xi(r) > 0$ for any nonzero $r \in \Xi$.

Their two main results are the following.

Theorem 3.1 (Fuzzy random elementary renewal theorem [25]). *Suppose $\xi_1, \xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ for almost every $\omega \in \Omega$ $k = 1, 2, \cdots$ with $\Pi_I(-a) = 0$, $a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $\{U_k\}$ is i.i.d. random variable sequences with a finite expected value and $U_k \geq a + h$ almost surely, then we have*

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{E[U_1]}.$$

Theorem 3.2 (Fuzzy random renewal reward theorem [24]). *Suppose (ξ_1, η_1), $(\xi_2, \eta_2), \cdots$ is a sequence of pairs of fuzzy random inter-arrival times and rewards, where $\{\eta_k\}$ is uniformly essentially bounded, $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ with $U_k \geq a + h$ almost surely, and $\mu_{\eta_k(\omega)}(x) = \Pi_R(x - V_k(\omega))$, for almost every $\omega \in \Omega$, $k = 1, 2, \cdots, N(t)$ is the fuzzy random renewal variable, and $C(t)$ is the total reward. If $\{U_k\}$ and $\{V_k\}$ are i.i.d. random variable sequences with finite expected values, respectively, and $\{\xi_k(\omega)\}$ and $\{\eta_k(\omega)\}$ are mutually T-independent for almost every $\omega \in \Omega$, then*

$$\lim_{t \to \infty} \frac{E[C(t)]}{t} = \frac{E[V_1]}{E[U_1]}.$$

There are several remarks regarding these results which can be improved.

Remark 3.1. In the assumption A1. the interarrival times $\{\xi_n\}$ and rewards $\{\eta_n\}$ need not be the same T-independent fuzzy variable sequences, respectively.

Remark 3.2. The condition $U_k \geq a+h$ in Theorem 3.1 is strong. If $\{\xi_k, k \geq 1\}$ degenerates to a sequence of i.i.d. random variables, then the result in Theorem 1 does not cover the elementary renewal theorem in the stochastic case (see [21])

Remark 3.3. The condition $U_k \geq a+h$ in Theorem 3.2 is strong. If $\{(\xi_k, \eta_k), k \geq 1\}$ degenerates to a sequence of i.i.d. random variables, then the result in Theorem 3.2 does not cover the renewal reward theorem in the stochastic case (see [21])

We consider the condition A3. $\mathrm{co}(f \circ \Pi)_\Xi(r) > 0$ for any nonzero $r \in \Xi$.

Example 3.3. Let Π be the possibility distribution of a normal fuzzy variable with $\Pi(x) = e^{-x^2}, x \in \mathcal{R}$, $Y_p(x, y)$ is a Yager $t-$norm with the additive generator $f(x) = (1-x)^p$. Then $f \circ \Pi(x) = (1-e^{-x^2})^p$ on $\Xi = \mathcal{R}$ and hence $\mathrm{co}(f \circ \Pi)_\Xi(x) = 0, x \in \Xi$.

Example 3.4. Let Π be the possibility distribution of a normal fuzzy variable with $\Pi(x) = e^{-|x|^\lambda}, 0 < \lambda < 1,\ x \in \mathcal{R}$, $H_1(x, y)$ is a Hamacher $t-$norm with the additive generator $f(x) = -lnx$. Then $f \circ \Pi(x) = |x|^\lambda, 0 < \lambda < 1$ on $\Xi = \mathcal{R}$ and hence $\mathrm{co}(f \circ \Pi)_\Xi(x) = 0, x \in \Xi$.

Remark 3.4. As we can see in the Examples 3.3 and 3.4, the condition A3. is strong. This assumption can be released.

Remark 3.5. The condition that $\{\eta_k\}$ is uniformly essentially bounded in Theorem 2 can be released.

Throughout this paper, we discuss fuzzy stochastic renewal and renewal rewards theories within the following conditions:

H1. $T_i, i = 1, 2$ can be any continuous Archimedean t-norms with additive generators $f_i, i = 1, 2$. The interarrival times $\{\xi_n\}$ and rewards $\{\eta_n\}$ are positive fuzzy random variables, and for almost every $\omega \in \Omega$, $\{\xi_n(\omega)\}$ is T_1-independent fuzzy variable sequences and $\{\eta_n(\omega)\}$ is T_2-independent fuzzy variable sequences, respectively.

H2. Π is a nonnegative unimodal real function with $\Pi(0) = 1, \Pi(x) < 1, x \neq 0$, $lim_{x\to\infty}\Pi(x) = lim_{x\to-\infty}\Pi(x) = 0$.

3.1. Fuzzy random elementary renewal theorem. We first consider the following result.

Proposition 3.5. *et T be a continuous Archimedean t-norms with additive generators f. Let $\{Y_k\}$ be a sequence of T-independent fuzzy variables with membership function $\mu_k(x) = \Pi(x-u_k)$ and let $S_n = Y_1+\cdots+Y_n, m(S_n) = u_1+\cdots+u_n$.*

Then, for all $\epsilon > 0$ we have

$$\lim_{n\to\infty} Nes\left\{\left|\frac{S_n - m(S_n)}{n}\right| < \epsilon\right\} = 1.$$

Proof. Define

$$\bar{\Pi}(x) = \begin{cases} \lim_{z\downarrow x}\Pi(z) & \text{if} \quad x > 0, \\ \lim_{z\uparrow x}\Pi(z) & \text{if} \quad x < 0, \\ 1, & \text{if} \quad x = 0. \end{cases}$$

Then $\bar{\Pi}$ is an upper semi-continuous nonnegative unimodal real function with $\bar{\Pi}(0) = 1, \bar{\Pi}(r) < 1, r \neq 0$ and $\Pi(x) \leq \bar{\Pi}(x), x \in \mathcal{R}$. Let $\bar{Y_k}$ be fuzzy variable with membership function $\bar{\mu_k}(x) = \bar{\Pi}(x)$ and let $\bar{S_n} = \bar{Y_1} + \cdots + \bar{Y_n}, m(\bar{S_n}) = u_1 + \cdots + u_n$. Since $m(\bar{S_n}) = 0$ and, for all $\epsilon > 0$

$$Nes\left\{\left|\frac{S_n - m(S_n)}{n}\right| < \epsilon\right\} \geq Nes\left\{\left|\frac{\bar{S_n}}{n}\right| < \epsilon\right\},$$

it suffices to prove that

$$\lim_{n\to\infty} Nes\left\{\left|\frac{\bar{S_n}}{n}\right| < \epsilon\right\} = 1.$$

And to show this fact, it is also easy to check that, for any $0 < \alpha < 1$

$$\lim_{n\to\infty}\left\|\left[\frac{\bar{S_n}}{n}\right]_\alpha\right\| = 0.$$

Let $0 < \alpha < 1, \delta > 0$ be given and $f(1-\delta) = t_0$. Suppose that $\sum_{i=1}^n f(\alpha_i) \leq f(\alpha)$ and let $H_{\alpha,\delta} = \{k | f(\alpha_k) > t_0\}$. Then, the number of $H_{\alpha,\delta}$ is less than or equal to N where N is the smallest natural number bigger than $f(\alpha)/t_0$. We note that if $k \notin H_{\alpha,\delta}$, then $\alpha_k \geq 1 - \delta$ and if $k \in H_{\alpha,\delta}$, then $\alpha_k \geq \alpha$ Then, we have from (1)

$$\begin{aligned}
[\bar{Y_1} + \cdots + \bar{Y_n}]_\alpha &= \bigcup_{\sum_{i=1}^n f(\alpha_i)\leq f(\alpha)} ([\bar{Y_1}]_{\alpha_1} + \cdots + [\bar{Y_n}]_{\alpha_n}) \\
&\subset \bigcup_{\sum_{i=1}^n f(\alpha_i)\leq f(\alpha)} \left(\sum_{k\notin H_{\alpha,\delta}} [\bar{Y_k}]_{1-\delta} + \sum_{k\in H_{\alpha,\delta}} [\bar{Y_k}]_\alpha\right) \\
&\subset [\bar{Y_1}]_{1-\delta} + \cdots + [\bar{Y_n}]_{1-\delta} + \bigcup_{\sum_{i=1}^n f(x_i)\leq f(\alpha)} \left(\sum_{k\in H_{\alpha,\epsilon}} [\bar{Y_n}]_\alpha\right) \\
&\subset [\bar{Y_1}]_{1-\delta} + \cdots + [\bar{Y_n}]_{1-\delta} + NB(0, t_\alpha) \\
&= \sum_{1\leq i\leq n} [\bar{Y_i}]_{1-\delta} + NB(0, t_\alpha),
\end{aligned}$$

where $B(0, t_\alpha) = \{x : |x| \leq t_\alpha\},\ t_\alpha = \|[\bar{Y_1}]_\alpha\| = max\{-l_\alpha, r_\alpha\}$ such that $\bar{\Pi}(l_\alpha) = \bar{\Pi}(r_\alpha) = \alpha$ and the second equality above comes from the convexity of

$[A]_\alpha$. Hence, we have, for large n,

$$\begin{aligned} \left\| \left[\frac{S_n - m(S_n)}{n} \right]_\alpha \right\| &\le \|[\bar{Y}_1]_{1-\delta}\| + \frac{N}{n} t_\alpha \\ &\le \|[\bar{Y}_1]_{1-\delta}\| + N \frac{t_\alpha}{n}. \end{aligned}$$

The first term goes to 0 because δ is arbitrary and $\lim_{\delta\to 0} \|[\bar{Y}_1]_{1-\delta}\| = 0$ by the assumption H2 and the second term goes to 0 as $n \to \infty$. This completes the proof. □

Following the line of the proof in regarding to the Theorem of Hong [8] and using Proposition 3.5, the following result is obtained.

Proposition 3.6. *Let T be a continuous Archimedean t-norms with additive generators f. Let $\{Y_k\}$ be a sequence of a T-independent fuzzy variable with the membership function $\mu_k(x) = \Pi(x - u_k)$ and let $S_n = Y_1 + \cdots + Y_n, m(S_n) = u_1 + \cdots + u_n$. If $\mu_k \ge a, k = 1, 2, \cdots$ and $(u_1 + \cdots + u_n)/n \to u$ as $n \to \infty$, then we have, for all $\epsilon > 0$*

$$\lim_{n\to\infty} Nes \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| < \epsilon \right\} = 1.$$

Note 1. It is noted that since

$$Cr \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| \ge \epsilon \right\} \le Pos \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| \ge \epsilon \right\} = 1 - Nes \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| < \epsilon \right\},$$

the condition

$$\lim_{n\to\infty} Nes \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| < \epsilon \right\} = 1$$

implies

$$\lim_{n\to\infty} Cr \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| \ge \epsilon \right\} = 0.$$

From Note 1, we have the following result:

Proposition 3.7. *Let T be a continuous Archimedean t-norms with additive generators f. Let $\{Y_k\}$ be a sequence of a T-independent fuzzy variable with the membership function $\mu_k(x) = \Pi(x - u_k)$ and let $S_n = Y_1 + \cdots + Y_n, m(S_n) = u_1 + \cdots + u_n$. If $\mu_k \ge a, k = 1, 2, \cdots$ and $(u_1 + \cdots + u_n)/n \to u$ as $n \to \infty$, then we have, for all $\epsilon > 0$*

$$\lim_{t\to\infty} Cr \left\{ \left| \frac{N(t)}{t} - \frac{1}{u} \right| \ge \epsilon \right\} = 0.$$

The following is a well-known result of probability theory [1].

Lemma 3.8 ([1])**.** *Let $\{X_n\}_{n=1}^\infty$ be independent and identically distributed (i.i.d.) random variables. Then $E|X_1| < \infty$ iff $n^{-1} \max_{1\le k\le n} |X_k| \to 0$ a.s. iff lim $n^{-1} \sum_{k=1}^n X_k = E[X_1]$ a.s.*

The following result is immediate from Proposition 3.7 and Lemma 3.8.

Proposition 3.9. *Suppose ξ_1, $\xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ for almost every $\omega \in \Omega$ $k = 1, 2, \cdots$ with $\Pi_I(-a) = 0$, $a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $\{U_k\}$ is i.i.d. random variable sequences with finite expected value and $U_k \geq a$ almost surely, then we have*

$$\lim_{t\to\infty} Cr\left\{\left|\frac{N(t)(\omega)}{t} - \frac{1}{E[U_1]}\right| \geq \epsilon\right\} = 0 \text{ almost surely.}$$

Lemma 3.10 ([25])**.** *Suppose ξ_1, $\xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ for almost every $\omega \in \Omega$ $k = 1, 2, \cdots$ with $\Pi_I(-a) = 0$, $a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $U_k \geq a$ almost surely, then we have*

$$\begin{aligned} Pos\left\{\frac{N(t)(\omega)}{t} \geq r\right\} &\leq Pos\left\{\xi_1(\omega) - U_1(\omega) \leq \frac{1}{r} - \sum_{k=1}^{M} \frac{U_k(\omega)}{M}\right\} \\ &= Pos\left\{\xi \leq \frac{1}{r} - \sum_{k=1}^{M} \frac{U_k(\omega)}{M}\right\} \end{aligned}$$

where M is the smallest integer such that $M \geq rt$ and ξ is a fuzzy variable with $\mu_\xi(x) = \Pi_I(x)$.

Definition 3.11 ([16])**.** A sequence $\{\xi_n\}$ of fuzzy variables is said to converge in credibility to a fuzzy variable ξ, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} Cr\{|\xi_n - \xi| \geq \varepsilon\} = 0.$$

Theorem 3.12. *Suppose ξ_1, $\xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ for almost every $\omega \in \Omega$ $k = 1, 2, \cdots$ with $\Pi_I(-a) = 0$, $a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $\{U_k\}$ is a sequences of i.i.d. random variables with finite expected value and $U_k \geq a$ almost surely, then we have*

$$\lim_{t\to\infty} \frac{E[N(t)(\omega)]}{t} = \frac{1}{E[U_1]} \text{ almost surely.}$$

Proof. According to Proposition 3.9, we know that $\{N(t)(\omega)/t\}$ converges in credibility to $1/E[U_1]$ almost surely. Then, it follows from Liu [17, Theorem 3.59] that for any $r \in R$,

$$\lim_{t\to\infty} Cr\{N(t)(\omega)/t \geq r\} = Cr\left\{\frac{1}{E[U_1]} \geq r\right\} \text{ almost surely.}$$

Let $E[U_1] - a = \epsilon > 0$. Since $lim_{M\to\infty} \sum_{k=1}^{M} \frac{U_k(\omega)}{M} = E[U_1] > a$ almost surely, we note that, for almost all (ω), there exists $T(\omega) > 0$, depending on ω such that for $M \geq T(\omega)$, $\sum_{k=1}^{M} \frac{U_k(\omega)}{M} \geq E[U_1] - \epsilon/2$. By Lemma 3.10, for $r > 2/\epsilon$

$$Cr\left\{\frac{N(t)(\omega)}{t} \geq r\right\} \leq Pos\left\{\frac{N(t)(\omega)}{t} \geq r\right\}$$

$$\begin{aligned}
&\le Pos\left\{\xi \le \frac{1}{r} - E[U_1] + \frac{\epsilon}{2}\right\}\\
&\le Pos\{\xi \le -E[U_1] + \epsilon\}\\
&= Pos\{\xi \le -a\}\\
&= 0.
\end{aligned}$$

As a consequence, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{E[N(t)]}{t} &= \lim_{t\to\infty} \int_0^\infty Cr\{N(t)/t \ge r\}dr\\
&= \frac{1}{E[U_1]} \text{ almost surely.}
\end{aligned}$$

□

Here, we define an order of two fuzzy variables. Let A, B be two fuzzy variables. We define $A \le B$ iff $Cr\{A \ge r\} \le Cr\{B \ge r\}$ for all real number r.

Lemma 3.13. *Suppose ξ_1, $\xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ for almost every $\omega \in \Omega$ $k = 1, 2, \cdots$ with $\Pi_I(-a) = 0$, $a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $\{U_k\}$ is a sequences of i.i.d. random variables with finite expected value and $U_k \ge a$ almost surely, then we have*

$$E\left[\left(\frac{E[N(t)(\omega)]}{t}\right)^2\right] = O(1)$$

Proof. Since $P\{U_k = a\} < 1$, there exists $\delta > 0$ such that $P\{U_k \ge a+\delta\} = p > 0$ for all $n = 1, 2, 3, \cdots$. Define

$$U'_k(\omega) = \begin{cases} \delta & \text{if } U_k(\omega) \ge a + \delta,\\ 0 & \text{if } U_k(\omega) < a + \delta. \end{cases}$$

and $\xi'_k(\omega) = U'_k(\omega)$. Let S'_k and $N'(t)$ be the corresponding quantities for the sequence $\{\xi'_k, k \ge 1\}$. It is obvious that, considering $\{\xi'_k, k \ge 1\}$ as degenerated fuzzy random variables such that $\xi'_k \le \xi_k, k \ge 1$, $S'_n \le S_n$ and $N'(t) \ge N(t)$ for each t, and hence we have that

$$\begin{aligned}
E[(N(t)(\omega))] &= \int_0^\infty \{(N(t)(\omega)) \ge r\}dr\\
&\le \int_0^\infty Cr\{(N'(t)(\omega)) \ge r\}dr\\
&= N'(t)(\omega).
\end{aligned}$$

Since the random variables $\{\xi'_k/\delta\}$ are independent with a Bernoullian distribution, elementary computations show that

$$E[N'(t)^2] = O\left(\frac{t^2}{\delta^2}\right) \text{ as } t \to \infty.$$

Hence we have, δ being fixed,

$$\begin{aligned} E\left[\left(\frac{E[N(t)(\omega)]}{t}\right)^2\right] &= \int \left(\frac{E[N(t)(\omega)]}{t}\right)^2 Pr(d\omega) \\ &\leq E\left[\left(\frac{N'(t)}{t}\right)^2\right] = O(1) \end{aligned}$$

$\square$

Theorem 3.14 ([1]). *Let $\{X_n\}$ is a sequence of random variables such that $sup_n E|X_n|^p = M < \infty$ for some $p > 0$. If $X_n \to X$ in distribution, then for each $r < p$:*

$$\lim_{n\to\infty} E[|X_n|^r] = E[|X|^r].$$

Our first main result is the following which improves Theorem 3.1.

Theorem 3.15 (Fuzzy random elementary renewal theorem). *Suppose $\xi_1, \xi_2, \cdots$ is a sequence of fuzzy random inter-arrival times where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ $k = 1, 2, \cdots$ for almost every $\omega \in \Omega$, with $\Pi_I(-a) = 0,\ a > 0$, and $N(t)$ is the fuzzy random renewal variable. If $\{U_k\}$ is a sequence of i.i.d. random variables with finite expected value and $U_k \geq a$ almost surely, then we have*

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{E[U_1]}. \tag{2}$$

Proof. By Lemma 3.13, we have

$$E\left[\left(\frac{E[N(t)(\omega)]}{t}\right)^2\right] = O(1).$$

Since Theorem 3.12 implies

$$\lim_{t\to\infty} \frac{E[N(t)(\omega)]}{t} = \frac{1}{E[U_1]} \text{ in distribution,}$$

an application of Theorem 3.14 with $X_n(\omega) = E[N(n)(\omega)]/n$ and $p = 2$ yield (2) with t replaced by n in (2), from which (2) itself follows at once. $\square$

Remark 3.6. If $\{\xi_k, k \geq 1\}$ degenerates to a sequences of i.i.d. random variables, the result in Theorem 3.15 is just the renewal theorem in the stochastic case (see [21]).

3.2. Fuzzy random renewal reward theorem. A slight modification of the result regarding the Theorem of Hong [8] using Lemma 3.8 and Proposition 3.5, gives the following result.

Proposition 3.16. *Suppose (ξ_1, η_1), $(\xi_2, \eta_2), \cdots$ is a sequence of pairs of fuzzy random interarrival times and rewards, where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ with $\Pi_I(-a) = 0, U_k(\omega) \geq a > 0$ almost surely, and $\mu_{\eta_k(\omega)}(x) = \Pi_R(x - V_k(\omega))$, $k = 1, 2, \cdots, N(t)$ is the fuzzy random renewal variable, and $C(t)$ is the total reward.*

If $\{U_k\}$ and $\{V_k\}$ are i.i.d. random variable sequences with finite expected values, respectively, then for any $\epsilon > 0$,

$$\lim_{t\to\infty} Cr\left\{\left|\frac{C(t)(\omega)}{t} - \frac{E[V_1]}{E[U_1]}\right| \geq \epsilon\right\} = 0 \text{ almost surely.}$$

Lemma 3.17. *Suppose (ξ_1, η_1), $(\xi_2, \eta_2), \cdots$ is a sequence of pairs of fuzzy random interarrival times and rewards, where $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ with $\Pi_I(-a) = 0, U_k(\omega) \geq a > 0$ almost surely, and $\mu_{\eta_k(\omega)}(x) = \Pi_R(x - V_k(\omega)), k = 1, 2, \cdots,$, with $\int_{-\infty}^{\infty} \Pi_R(r)dr < \infty$, $N(t)$ is the fuzzy random renewal variable, and $C(t)$ is the total reward. If $\{U_k\}$ and $\{V_k\}$ are i.i.d. random variable sequences with $E[U_1] < \infty$ and $E[|V_1|] < \infty$, then there exists a function $f_\omega^+(r) \geq 0$, $r \geq 0$ and a function $f_\omega^-(r) \geq 0$, $r \leq 0$ almost surely such that for big $t > 0$,*

$$Cr\left\{\frac{C(t)(\omega)}{t} \geq r\right\} \leq f_\omega^+(r) \text{ big } r \geq 0,$$

and

$$Cr\left\{\frac{C(t)(\omega)}{t} \leq r\right\} \leq f_\omega^-(r) \text{ big } r < 0 \text{ almost surely,}$$

and

$$\int_{-\infty}^{0} f_\omega^-(r)dr + \int_0^\infty f_\omega^+(r)dr < \infty \text{ almost surely.}$$

Proof. Let $N'(t)$ be the renewal random variables defined in Lemma 3. We consider that for $0 < l_1 < 1/p\delta = 1/E[\xi_1'] < l_2$, and $r \geq 0$,

$$\begin{aligned}
Pos\left\{\frac{C(t)(\omega)}{t} \geq r\right\} &= Pos\left\{\frac{\sum_{i=1}^{N(t)(\omega)} \eta_i(\omega)}{t} \geq r\right\} \\
&\leq Pos\left\{\frac{\sum_{i=1}^{N'(t)(\omega)} \eta_i(\omega)}{t} \geq r\right\} \\
&= \sup_n Pos\left\{\frac{\sum_{i=1}^{N'(t)(\omega)} \eta_i(\omega)}{t} \geq r,\ N'(t)(\omega) = n\right\} \\
&= \sup_{n\in[l_1 t, l_2 t]} Pos\left\{\frac{\sum_{i=1}^{N'(t)(\omega)} \eta_i(\omega)}{t} \geq r,\ N'(t)(\omega) = n\right\} \\
&\leq \sup_{n\in[l_1 t, l_2 t]} Pos\left\{\frac{\sum_{i=1}^{n} \eta_i(\omega)}{t} \geq r\right\} \\
&\vee \sup_{n\notin[l_1 t, l_2 t]} Pos\left(N'(t)(\omega) \notin [l_1 t, l_2 t]\right) \\
&\equiv A_t(r)(\omega) \vee B_t(r)(\omega)
\end{aligned}$$

where

$$A_t(r)(\omega) = \sup_{n\in[l_1 t, l_2 t]} Pos\left\{\frac{\sum_{i=1}^{n} \eta_i(\omega)}{t} \geq r\right\}$$

and

$$\begin{aligned} B_t(r)(\omega) &= Pos\left\{N'(t)(\omega) \notin [l_1 t, l_2 t]\right\} \\ &= Pos\left\{\frac{N'(t)(\omega)}{t} \notin [l_1, l_2]\right\}. \end{aligned}$$

Then, by the classical renewal theorem, $\lim_{t\to\infty} N'(t)(\omega)/t = 1/E[\xi_1']$ almost surely, we have for big $t > 0$,

$$B_t(r)(\omega) = Pos\left\{\frac{N'(t)(\omega)}{t} \notin [l_1, l_2]\right\} = 0 \text{ almost surely.}$$

Nest, we consider that

$$\begin{aligned} A_t(r)(\omega) &= \sup_{n\in[l_1 t, l_2 t]} Pos\left\{\frac{\sum_{i=1}^n \eta_i(\omega)}{t} \geq r\right\} \\ &= \sup_{\frac{n}{t}\in[l_1, l_2]} Pos\left\{\frac{\sum_{i=1}^n \eta_i(\omega)}{n} \geq \frac{t}{n} r\right\} \\ &\leq Pos\left\{\frac{\sum_{i=1}^n \eta_i(\omega)}{n} \geq l_2^{-1} r\right\} \\ &\leq Pos\left\{\eta + \sum_{k=1}^n \frac{V_k(\omega)}{n} \geq l_2^{-1} r\right\} \end{aligned}$$

where η is a fuzzy variable with $\mu_\eta(x) = \Pi_R(x)$. Then, we have for big $t > 0$, for some $\epsilon > 0$,

$$A_t(r)(\omega) \leq Pos\left\{\eta + \max\{E[V_1] + \epsilon, 0\} \geq l_2^{-1} r\right\} \equiv f_\omega^+(r) \text{ almost surely}$$

and

$$\int_0^\infty f_\omega^+(r)dr = \max\{E[V_1] + \epsilon, 0\} + l_2 \int_0^\infty \Pi_R(r)dr < \infty.$$

By similar argument, we have for big $t > 0$, for some $\epsilon > 0$,

$$A_t(r)(\omega) \leq Pos\left\{\eta + \min\{E[V_1] - \epsilon, 0\} \leq l_1^{-1} r\right\} \equiv f_\omega^-(r) \text{ almost surely}$$

and

$$\int_{-\infty}^0 f_\omega^-(r)dr = \min\{E[V_1] - \epsilon, 0\} + l_1 \int_{-\infty}^0 \Pi_R(r)dr < \infty.$$

□

Theorem 3.18. *Suppose (ξ_1, η_1), $(\xi_2, \eta_2), \cdots$ is a sequence of pairs of fuzzy random interarrival times and rewards, where $\{\eta_k\}$ is $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ with $\Pi_I(-a) = 0, U_k(\omega) \geq a > 0$ almost surely, and $\mu_{\eta_k(\omega)}(x) = \Pi_R(x - V_k(\omega))$, for almost every $\omega \in \Omega$ with $\int_{-\infty}^\infty \Pi_R(x)dx < \infty$, $k = 1, 2, \cdots, N(t)$ is the fuzzy random renewal variable, and $C(t)$ is the total reward. If $\{U_k\}$ and $\{V_k\}$ are i.i.d. random variable sequences with $E[U_1] < \infty$ and $E[|V_1|] < \infty$, then*

$$\lim_{t\to\infty} \frac{E[C(t)(\omega)]}{t} = \frac{E[V_1]}{E[U_1]} \text{ almost surely.}$$

Proof. According to Proposition 3.16, we know that $\{C(t)(\omega)/t\}$ converges in credibility to $E[V_1]/E[U_1]$ almost surely. It follows from Liu [17, Theorem3.59] that for almost all $r \geq 0$,

$$\lim_{t\to\infty} Cr\left\{\frac{C(t)(\omega)}{t} \geq r\right\} = Cr\left\{\frac{E[V_1]}{E[U_1]} \geq r\right\}$$

and for almost all $r \leq 0$,

$$\lim_{t\to\infty} Cr\left\{\frac{C(t)(\omega)}{t} \leq r\right\} = Cr\left\{\frac{E[V_1]}{E[U_1]} \leq r\right\} \text{ almost surely.}$$

As a consequence, by Lemma 3.17 and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{E[C(t)(\omega)]}{t} \\
=\ & \lim_{t\to\infty} \int_{-\infty}^{0} Cr\left\{\frac{C(t)(\omega)}{t} \leq r\right\} dr + \int_{0}^{\infty} Cr\left\{\frac{C(t)(\omega)}{t} \geq r\right\} dr \\
=\ & \frac{E[V_1]}{E[U_1]} \text{ almost surely.}
\end{aligned}$$

□

Theorem 3.19 ([1]). *Let $\{X_n\}$ be a sequence of random variables such that $E|X_n| < \infty, n = 1, 2, 3, \cdots$ and $X_n \to X$ in probability. Then the following two propositions are equivalent:*
i)$\{|X_n|\}$ is uniformly integrable
ii)$E|X_n| \to E|X|$.

Our second main result is the following which improves Theorem 3.2.

Theorem 3.20 (Fuzzy random renewal reward theorem). *Suppose (ξ_1, η_1), $(\xi_2, \eta_2), \cdots$ is a sequence of pairs of fuzzy random interarrival times and rewards, where $\{\eta_k\}$ is $\mu_{\xi_k(\omega)}(x) = \Pi_I(x - U_k(\omega))$ with $\Pi_I(-a) = 0, U_k(\omega) \geq a > 0$ almost surely, and $\mu_{\eta_k(\omega)}(x) = \Pi_R(x - V_k(\omega))$, for almost every $\omega \in \Omega$ with $\int_{-\infty}^{\infty} \Pi_R(x)dx < \infty$, $k = 1, 2, \cdots, N(t)$ is the fuzzy random renewal variable, and $C(t)$ is the total reward. If $\{U_k\}$ and $\{V_k\}$ are i.i.d. random variable sequences with $E[U_1] < \infty$ and $E[|V_1|] < \infty$, then*

$$\lim_{t\to\infty} \frac{E[C(t)]}{t} = \frac{E[V_1]}{E[U_1]}.$$

Proof. Let $N'(t)$ is the random renewal variable defined in Lemma 3.13. Let $\Pi_{R'}$ be a function with $\Pi_{R'}(x) = \Pi_R(x)$ if $x \geq 0$ and 0, otherwise, and let $\{\eta'_k(\omega)\}$ be fuzzy random variables with $\mu_{\eta'_k(\omega)}(x) = \Pi_{R'}(x - V_k(\omega))$, for almost every $\omega \in \Omega$. Similarly, we define $\Pi_{R''}$ to be a function with $\Pi_{R''}(x) = \Pi_R(x)$ if $x \leq 0$ and 0, otherwise, and $\{\eta''_k(\omega)\}$ to be fuzzy random variables with $\mu_{\eta''_k(\omega)}(x) = \Pi_{R''}(x - V_k(\omega))$, for almost every $\omega \in \Omega$. Then we clearly have

$$E[\eta''_k(\omega)] \leq E[\eta_k(\omega)] \leq E[\eta'_k(\omega)]$$

and

$E[\eta''_1(\omega)] + \cdots + E[\eta''_k(\omega)] \le E[\eta_1(\omega) + \cdots + \eta_k(\omega)] \le E[\eta'_1(\omega)] + \cdots + E[\eta'_k(\omega)]$.

It is also noted that $E[\eta'_k(\omega)] = V_k(\omega) + v'$ and $E[\eta''_k(\omega)] = V_k(\omega) - v''$ for some positive real numbers $v', v'' > 0$, respectively. Then we have

$$\begin{aligned} & E[C(t)(\omega)] \\ \le\ & E\left[\sum_{k=1}^{N'(t)(\omega)} \eta'_k(\omega)\right] = \sum_{k=1}^{N'(t)(\omega)} E[\eta'_k(\omega)] \\ =\ & \sum_{k=1}^{N'(t)(\omega)} (V_k(\omega) + v'). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & E[C(t)(\omega)] \\ \ge\ & E\left[\sum_{k=1}^{N'(t)(\omega)} \eta''{}_k(\omega)\right] = \sum_{k=1}^{N'(t)(\omega)} E[\eta''_k(\omega)] \\ =\ & \sum_{k=1}^{N'(t)(\omega)} (V_k(\omega) - v''). \end{aligned}$$

Then, since $(\xi'_k, V_k),\ k = 1, 2, \cdots$ is a sequence of pairs of i.i.d. random variables, by the classical stochastic renewal reward theorem, we have

$$\lim_{t\to\infty} \frac{\sum_{k=1}^{N'(t)(\omega)} V_k(\omega)}{t} = \frac{E[V_1]}{E[\xi'_1]} = \frac{E[V_1]}{p\delta} \text{ almost surely}$$

and

$$\lim_{t\to\infty} \frac{E[\sum_{k=1}^{N'(t)(\omega)} V_k(\omega)]}{t} = \frac{E[V_1]}{p\delta}.$$

Then, by Theorem 3.19, the class of random variables $\{\sum_{k=1}^{N'(t)(\omega)} V_k(\omega)/t\}$ is uniformly integrable, and hence $\{\sum_{k=1}^{N'(t)(\omega)}(V_k(\omega) + v')/t\}$ and $\{\sum_{k=1}^{N'(t)(\omega)}(V_k(\omega) - v'')/t\}$ are uniformly integrable. Since

$$0 < |E[C(t)(\omega)]| \le \max\left\{\left|\sum_{k=1}^{N'(t)(\omega)} (V_k(\omega) + v')\right|, \left|\sum_{k=1}^{N'(t)(\omega)} (V_k(\omega) - v'')\right|\right\},$$

$\{E[C(t)(\omega)]/t\}$ is also uniformly integrable. Since, by Theorem 3.18, we have

$$\lim_{t\to\infty} \frac{E[C(t)(\omega)]}{t} = \frac{E[V_1]}{E[U_1]} \text{ almost surely,}$$

and hence by Theorem 3.19, again we have

$$\lim_{t\to\infty} \frac{E[C(t)]}{t} = \frac{E[V_1]}{E[U_1]}.$$

which completes the proof. □

Remark 3.7. If $\{(\xi_k, \eta_k), k \geq 1\}$ degenerates to a sequences of pair of i.i.d. random variables, the result in Theorem 3.20 is simply the renewal rewards theorem in stochastic case (see [21]).

4. Conclusion

In this paper, we considered the random fuzzy elementary renewal theorem and the random fuzzy renewal reward theorem studied by Wang et al. [25] and Wang and Watada [24] when fuzzy random variables are T-independent and improved the results from a mathematical perspective. If the sequence of fuzzy random variables degenerates to a sequence of random variables, the main results are just the renewal theorem and the renewal rewards theorem in the stochastic case (see [21]).

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