J. Appl. Math. & Informatics Vol. **33***(2015), No. 5 - 6, pp. 593 - 605*
http://dx.doi.org/10.14317/jami.2015.593

ON THE NORMS OF SOME SPECIAL MATRICES WITH GENERALIZED FIBONACCI SEQUENCE

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ABSTRACT. In this study, we define $r-$circulant, circulant, Hankel and Toeplitz matrices involving the integer sequence with recurrence relation $U_n = pU_{n-1} + U_{n-2}$, with $U_0 = a$, $U_1 = b$. Moreover, we obtain special norms of above mentioned matrices. The results presented in this paper are generalizations of some of the results of [1, 10, 11].

AMS Mathematics Subject Classification : 15A45, 15A60, 15A36, 11B39.
Key words and phrases : Circulant, $r-$ circulant, semi-circulant, Hankel, spectral norm, Euclidean norm.

1. Introduction and Preliminaries

A lot of research papers on the norms of some special matrices have been written during the last decade [1, 2, 6, 10, 11]. Akbulak and Bozkurt [1] found lower and upper bounds for the spectral norms of Toeplitz matrices $A = [F_{i-j}]_{i,j=1}^{n}$. Solak found bounds for the special norms of circulant matrices [10]. In [12] the authors determined the upper and lowers bounds for Cauchy-Toeplitz and Cauchy Hankel matrices. In [8] bounds of circulant, $r-$circulant, semi-circulant and Hankel matrices with tribonacci sequence obtained. In [6], the author presented some results about circulant, negacyclic and semi-circulant matrices with the modified Pell, Jacobsthal and Jacobsthal- Lucas numbers. Shen and Cen found the bounds of spectral norm of Fibonacci and Lucas numbers [11]. The generalized Fibonacci sequence is defined as:

$$U_n = pU_{n-1} + U_{n-2}, \tag{1}$$

with initial conditions $U_0 = a$, $U_1 = b$, where a and b are positive integer.
It is clear that (1) can be written as:

$$U_n = aF_{n-1} + bF_n, \tag{2}$$

Received June 8, 2014. Revised June 8, 2015. Accepted June 22, 2015. *Corresponding author.

where F_n is called the nth term of p-Fibonacci sequence and defined by

$$F_n = pF_{n-1} + F_{n-2} \text{ and } F_0 = 0, \quad F_1 = 1. \tag{3}$$

Generally from equation (3), we have

$$F_{-n} = (-1)^{n+1} F_n. \tag{4}$$

Equation (2) can be written as:

$$U_{-n} = (-1)^n (aF_{n+1} - bF_n) \tag{5}$$

A matrix $\mathsf{A} = \mathsf{A}_r = (\mathsf{a}_{ij}) \in M_{n,n}(\mathbb{C})$ is called $r-$circulant on generalized sequence, if it is of the form

$$\mathsf{a}_{ij} = \begin{cases} \mathsf{U}_{j-i} & j \geq i \\ r\mathsf{U}_{n+j-i} & j < i \end{cases} \tag{6}$$

where $r \in \mathbb{C}$. If r=1, then matrix A is called circulant.

A matrix $\mathsf{A} = (\mathsf{a}_{ij}) \in M_{n,n}(\mathbb{C})$ is called semi-circulant on generalized Fibonacci sequence, if it is of the form

$$a_{ij} = \begin{cases} \mathsf{U}_{j-i+1} & i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

A Hankel matrix on generalized Fibonacci sequence is defined as:
$\mathsf{H} = (\mathsf{h}_{ij}) \in M_{n,n}(\mathbb{C})$, where $\mathsf{h}_{ij} = \mathsf{U}_{i+j-1}$. Similarly, a matrix $\mathsf{A} = (\mathsf{a}_{ij}) \in M_{n,n}(\mathbb{C})$ is Toeplitz matrix on generalized Fibonacci sequence (1), if it is of the form $a_{ij} = U_{i-j}$. The ℓ_p norm of a matrix $\mathsf{A} = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is defined by

$$\|\mathsf{A}\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}, \quad (1 \leq p \leq \infty).$$

If $p = \infty$, then $\|\mathsf{A}\|_\infty = \lim_{p\to\infty} \|\mathsf{A}\|_p = \max_{i,j} |\mathsf{a}_{ij}|$. The Euclidean (Frobenius) norm of the matrix A is defined as:

$$\|\mathsf{A}\|_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The spectral norm of the matrix A is given as:

$$\|\mathsf{A}\|_2 = \sqrt{\max_{1\leq i\leq n} |\gamma_i|},$$

where γ_i are the eigenvalues of the matrix $\left(\bar{\mathsf{A}}\right)^t \mathsf{A}$.

The following inequality between Euclidean and spectral norm holds [13]

$$\frac{1}{\sqrt{n}} \|\mathsf{A}\|_E \leq \|\mathsf{A}\|_2 \leq \|\mathsf{A}\|_E \tag{7}$$

Definition 1.1 ([9])**.** Let $\mathsf{A} = (\mathsf{a}_{ij})$ and $\mathsf{B} = (\mathsf{b}_{ij})$ be $m \times n$ matrices. Then, the Hadamard product of A and B is given by

$$\mathsf{A} \circ \mathsf{B} = (\mathsf{a}_{ij}\mathsf{b}_{ij}).$$

Definition 1.2 ([10])**.** The maximum column length norm $c_1(.)$ and maximum row length norm $r_1(.)$ for $m \times n$ matrix $\mathsf{A} = (\mathsf{a}_{ij})$ is defined as

$$c_1(\mathsf{A}) = \sqrt{\max_j \sum_i |a_{ij}|^2} \text{ and } r_1(\mathsf{A}) = \sqrt{\max_i \sum_j |a_{ij}|^2} \text{ respectively.}$$

Theorem 1.3 ([7])**.** *Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be $p \times q$ matrices. If $C = A \circ B$, then $\| C \|_2 \leq r_1(A)c_1(B)$.*

The following lemmas describe the properties of p-Fibonacci sequence.

Lemma 1.4 ([5])**.** *Let F_n be the n-th term of p-Fibonacci sequence then,*

$$\sum_{i=1}^{n} F_i F_{i+1} = R_n = \frac{F_{n+1}^2 + F_n F_{n+2} - 1}{2p}.$$

Lemma 1.5 ([5])**.** *The sum of square of first n terms of p-Fibonacci sequence is given by*

$$\sum_{i=1}^{n} F_i^2 = S_n = \frac{F_n F_{n+1}}{p}.$$

The following lemmas describes the properties of generalized Fibonacci sequence U_n.

Lemma 1.6. *The sum of first n terms of generalized Fibonacci sequence U_n is given as:*

$$\sum_{i=1}^{n} U_i = \frac{U_n + U_{n+1} - a - b}{p}.$$

Lemma 1.7 ([5])**.** *The sum of square of first n terms of the sequence U_n is given by:*

$$\sum_{i=1}^{n} U_i^2 = \frac{U_n U_{n+1} - ab}{p}.$$

Lemma 1.8. *Sum of product of consecutive terms of generalized Fibonacci sequence is given as:*

$$\sum_{i=1}^{n} U_i U_{i+1} = \left(a^2 + pab + b^2\right) R_n - \left(a^2 + pab\right) F_n F_{n+1} + ab \left(2S_n - F_n^2\right),$$

where $R_n = \frac{F_{n+1}^2 + F_n F_{n+2} - 1}{2p}$ and $S_n = \frac{F_n F_{n+1}}{p}$.

Proof. From equation (2), we have.

$$U_i = aF_{i-1} + bF_i.$$

$$U_iU_{i+1} = (aF_{i-1} + bF_i)(aF_i + bF_{i+1}).$$

$$U_iU_{i+1} = a^2F_{i-1}F_i + b^2F_iF_{i+1} + abF_{i-1}F_{i+1} + abF_i^2.$$

$$U_iU_{i+1} = a^2F_{i-1}F_i + b^2F_iF_{i+1} + abF_{i-1}(pF_i + F_{i-1}) + abF_i^2.$$

$$\sum_{i=1}^{n} U_iU_{i+1} = \left(a^2 + pab\right)\sum_{i=1}^{n} F_{i-1}F_i + b^2\sum_{i=1}^{n} F_iF_{i+1} + ab\left(\sum_{i=1}^{n} F_{i-1}^2 + \sum_{i=1}^{n} F_i^2\right).$$

By lemmas (1.4) and (1.5), we get

$$\begin{aligned}\sum_{i=1}^{n} U_iU_{i+1} &= \left(a^2 + pab\right)(R_n - F_nF_{n+1}) + b^2(R_n) + ab\left(2S_n - F_n^2\right)\\ &= \left(a^2 + pab + b^2\right)R_n - \left(a^2 + pab\right)F_nF_{n+1} + ab\left(2S_n - F_n^2\right).\end{aligned}$$

□

Theorem 1.9. *For all $n \geq 1$*

$$\sum_{j=1}^{n}\sum_{k=1}^{j} U_k^2 = \mathrm{T_n} = \frac{\left(a^2 + pab + b^2\right)R_n - \left(a^2 + pab\right)F_nF_{n+1} + ab\left(2S_n - F_n^2\right) - nab}{p},$$

where R_n and S_n are defined in lemma (1.4) and (1.5) respectively.

Proof. From Lemma (1.7) and (1.8), we have

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{j} U_k^2 &= \sum_{j=1}^{n}\left(\frac{U_jU_{j+1} - ab}{p}\right)\\ &= \left(\frac{1}{p}\right)\left(\sum_{j=1}^{n} U_jU_{j+1} - nab\right)\\ &= \frac{\left(a^2 + pab + b^2\right)R_n - \left(a^2 + pab\right)F_nF_{n+1} + ab\left(2S_n - F_n^2\right) - nab}{p}.\end{aligned}$$

□

Lemma 1.10. *For all $n > 1$*

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=1}^{i} U_{-j}^2 = \mathrm{T}_{-(n-1)} = \frac{1}{2p^2}\Big[&2(a^2p - 2ab)(F_nF_{n-1} - p)\\ &+ (a^2 + b^2 - abp)\left(F_n^2 + F_{n-1}F_{n+1} - 1\right) + 2np(ab - pa^2)\Big].\end{aligned}$$

Proof. From equation (5), we obtain

$$\sum_{i=1}^{n-1}\sum_{j=1}^{i} U_{-j}^2 = \sum_{i=1}^{n-1}\left(a^2\sum_{j=1}^{i} F_{j+1}^2 + b^2\sum_{j=1}^{i} F_j^2 - 2ab\sum_{j=1}^{i} F_jF_{j+1}\right)$$

$$= \sum_{i=1}^{n-1}\left[a^2\left(\frac{F_{i+1}F_{i+2}}{p}-1\right)+b^2\left(\frac{F_iF_{i+1}}{p}\right)-2ab\left(\frac{F_{i+1}^2+F_iF_{i+2}-1}{2p}\right)\right]$$
$$= \frac{1}{p}\left[a^2\sum_{i=1}^{n-1}F_{i+1}F_{i+2}+b^2\sum_{i=1}^{n-1}F_iF_{i+1}-ab\sum_{i=1}^{n-1}F_{i+1}^2\right.$$
$$\left.-ab\sum_{i=1}^{n-1}F_iF_{i+2}+(ab-pa^2)(n-1)\right].$$

On the other hand, from equation (3), we have

$$\sum_{i=1}^{n-1}F_iF_{i+2}=p\sum_{i=1}^{n-1}F_iF_{i+1}+\sum_{i=1}^{n-1}F_i^2 \text{ and } \sum_{i=1}^{n-1}F_{i+1}F_{i+2}=p\sum_{i=1}^{n-1}F_{i+1}^2+\sum_{i=1}^{n-1}F_iF_{i+1}.$$

Thus, we have

$$= \frac{1}{p}\left[(a^2p-ab)\sum_{i=1}^{n-1}F_{i+1}^2+(a^2+b^2-abp)\sum_{i=1}^{n-1}F_iF_{i+1}-ab\sum_{i=1}^{n-1}F_i^2+(ab-pa^2)(n-1)\right]$$
$$= \frac{1}{p}\left[(a^2p-2ab)\left(\frac{F_nF_{n+1}-p}{p}\right)+(a^2+b^2-abp)\left(\frac{F_n^2+F_{n-1}F_{n+1}-1}{2p}\right)+(ab-pa^2)(n)\right]$$
$$= \frac{1}{2p^2}\left[2(a^2p-2ab)\left(F_nF_{n-1}-p\right)+(a^2+b^2-abp)\left(F_n^2+F_{n-1}F_{n+1}-1\right)+2np(ab-pa^2)\right].$$

□

2. $r-$circulant, circulant and semi-circulant

In this section, we shall give main results related to $r-$circulant, circulant and sem-circulant on generalized Fibonacci sequence U_n.

Theorem 2.1. *Let $\mathsf{A}=\mathsf{A}_r(U_0,U_1,...,U_{n-1})$ be $r-$circulant matrix.*
If $|r|\geq 1$, then $\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}\leq\|\mathsf{A}\|_2\leq|r|\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)$
If $|r|<1$, then $|r|\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}\leq\|\mathsf{A}\|_2\leq\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)(a^2+n-1)}.$

Proof. The $r-$circulant matrix A on the sequence (1) is given as:

$$\mathsf{A}=\begin{bmatrix} U_0 & U_1 & U_2 & \cdots & U_{n-1}\\ rU_{n-1} & U_0 & U_1 & \cdots & U_{n-2}\\ rU_{n-2} & rU_{n-1} & U_0 & \cdots & U_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ rU_1 & rU_2 & rU_3 & \cdots & U_0\end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|\mathsf{A}\|_E^2=\sum_{k=0}^{n-1}(n-k)\,U_k^2+\sum_{k=1}^{n-1}k|r|^2U_k^2. \tag{8}$$

Here we have two cases depending on r.

Case 1. If $|r| \geq 1$, then from equation (8), we have

$$\|\mathsf{A}\|_E^2 \geq \sum_{k=0}^{n-1}(n-k)U_k^2 + \sum_{k=1}^{n-1} kU_k^2 = n\sum_{k=0}^{n-1} U_k^2,$$

and from lemma (1.7), we get.

$$\|\mathsf{A}\|_E^2 \geq n\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right) \quad \Rightarrow \quad \frac{1}{\sqrt{n}}\|A\|_E \geq \sqrt{\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right)}.$$

By inequality (7), we obtain

$$\|\mathsf{A}\|_2 \geq \sqrt{\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right)}. \tag{9}$$

On the other hand, let us define two new matrices C and D as :

$$\mathsf{C} = \begin{bmatrix} rU_0 & 1 & 1 & \cdots & 1 \\ rU_{n-1} & rU_0 & 1 & \cdots & 1 \\ rU_{n-2} & rU_{n-1} & rU_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rU_1 & rU_2 & rU_3 & \cdots & rU_0 \end{bmatrix} \quad \text{and} \quad \mathsf{D} = \begin{bmatrix} U_0 & U_1 & U_2 & \cdots & U_{n-1} \\ 1 & U_0 & U_1 & \cdots & U_{n-2} \\ 1 & 1 & U_0 & \cdots & U_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & U_0 \end{bmatrix}$$

Then it is easy to see that $\mathsf{A} = \mathsf{C} \circ \mathsf{D}$, so from definition (1.2)

$$r_1(\mathsf{C}) = \max_{i \leq i \leq n}\sqrt{\sum_{j=1}^{n}|\mathsf{c}_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|\mathsf{c}_{nj}|^2} = \sqrt{|r|^2\sum_{k=0}^{n-1}U_k^2} = |r|\sqrt{\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right)},$$

$$\mathsf{c}_1(\mathsf{D}) = \max_{1 \leq j \leq n}\sqrt{\sum_{i=1}^{n}|\mathsf{d}_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|\mathsf{d}_{nj}|^2} = \sqrt{\sum_{k=0}^{n-1}U_k^2} = \sqrt{\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right)}$$

Now using theorem (1.3), we obtain

$$\|\mathsf{A}\|_2 \leq r_1(\mathsf{C})c_1(\mathsf{D}) = |r|\,\frac{(U_nU_{n-1} - ab + a^2)}{p},$$

$$\|\mathsf{A}\|_2 \leq |r|\,\frac{(U_nU_{n-1} - ab + a^2)}{p}. \tag{10}$$

Combine inequalities (9) and (10), we get following inequality

$$\sqrt{\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right)} \leq \|\mathsf{A}\|_2 \leq |r|\left(\frac{U_nU_{n-1} - ab + pa^2}{p}\right).$$

Case 2. If $|r| \leq 1$, then we have

$$\|\mathsf{A}\|_E^2 \geq \sum_{k=0}^{n-1}(n-k)\,|r|^2U_k^2 + \sum_{k=0}^{n-1} k|r|^2U_k^2 = n\sum_{k=0}^{n-1}|r|^2U_k^2$$

$$\frac{1}{\sqrt{n}}\|\mathsf{A}\|_E \geq |r|\sqrt{\left(\frac{U_nU_{n-1}-ab+pU_a^2}{p}\right)}.$$

By inequality (7), we get

$$\|\mathsf{A}\|_2 \geq |r|\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}. \tag{11}$$

On the other hand, let the matrices C' and D' be defined as:

$$\mathsf{C}'=\begin{bmatrix} U_0 & 1 & 1 & \cdots & 1\\ r & U_0 & 1 & \cdots & 1\\ r & r & U_0 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ r & r & r & \cdots & U_0\end{bmatrix} \text{ and } \mathsf{D}'=\begin{bmatrix} U_0 & U_1 & U_2 & \cdots & U_{n-1}\\ U_{n-1} & U_0 & U_1 & \cdots & U_{n-2}\\ U_{n-2} & U_{n-1} & U_0 & \cdots & U_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ U_1 & U_2 & U_3 & \cdots & U_0\end{bmatrix}$$

such that $\mathsf{A}=\mathsf{C}'\circ\mathsf{D}'$, then by definition (1.2), we obtain

$$r_1(\mathsf{C}')=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|\mathsf{c}'_{ij}|^2}=\sqrt{U_0^2+(n-1)}=\sqrt{a^2+(n-1)}$$

and

$$c_1(\mathsf{D}')=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|\mathsf{d}'_{ij}|^2}=\sqrt{\sum_{k=0}^{n-1}U_k^2}=\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}.$$

Again by applying theorem (1.3)

$$\|\mathsf{A}\|_2 \leq r_1(\mathsf{C}')c_1(\mathsf{D}')=\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}\sqrt{a^2+n-1},$$

$$\|\mathsf{A}\|_2 \leq \sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)(a^2+n-1)}, \tag{12}$$

and combing inequality (11) and (12), we obtain the required result.

$$|r|\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)} \leq \|\mathsf{A}\|_2 \leq \sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)(a^2+n-1)}.$$

□

Remark 2.2. The above theorem is the generalization of the result [11]. If put $p=1$, $U_0=0$ and $U_1=1$ then $U_n=U_{n-1}+U_{n-2}$, which is same as $F_n=F_{n-1}+F_{n-2}$ with initial conditions $F_0=0$ and $F_1=1$.

Theorem 2.3. *Let A be the circulant matrix on generalized Fibonacci sequence. Then,* $\|A\|_E = \sqrt{n\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}$ *and*
$\sqrt{\frac{U_nU_{n-1}-ab+pa^2}{p}} \leq \| A\|_2 \leq \sqrt{\left(\frac{U_nU_{n-1}-ab+pa_2}{p}\right)}\sqrt{1+\frac{U_nU_{n-1}-ab}{P}}$.

Proof. Since by definition of circulant matrix, the matrix A is of the form

$$\mathsf{A} = \begin{bmatrix} U_0 & U_1 & U_2 & \cdots & U_{n-1} \\ U_{n-1} & U_0 & U_1 & \cdots & U_{n-2} \\ U_{n-2} & U_{n-1} & U_0 & \cdots & U_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_1 & U_2 & U_3 & \cdots & U_0 \end{bmatrix}$$

and form the definition of Euclidean norm, one can get,

$$\| \mathsf{A}\|_E = \sqrt{n\sum_{i=0}^{n-1} U_i^2} = \sqrt{n}\sqrt{\frac{U_nU_{n-1}-ab+pa^2}{p}}. \tag{13}$$

By inequality (7), we get

$$\sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)} \leq \|\mathsf{A}\|_2. \tag{14}$$

Let matrices B and C be defined as:

$$\mathsf{B} = \begin{cases} \mathsf{b}_{ij} = U_{(\bmod (j-i,n))}, & i \leq j \\ \mathsf{b}_{ij} = 1, & i < j \end{cases} \quad \text{and} \quad \mathsf{C} = \begin{cases} \mathsf{c}_{ij} = U_{(\bmod (j-i,n))}, & i < j \\ \mathsf{c}_{ij} = 1, & i \geq j \end{cases}$$

Then the row norm and column norm of B and C are given as:

$$r_1(\mathsf{B}) = \max_i \sqrt{\sum_{j=1}^{n} |\mathsf{b}_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} U_i^2} = \sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)},$$

$$c_1(\mathsf{C}) = \max_j \sqrt{\sum_{i=1}^{n} |\mathsf{c}_{ij}|^2} = \sqrt{1+\sum_{i=1}^{n-1} U_i^2} = \sqrt{1+\frac{U_nU_{n-1}-ab}{p}}.$$

Using theorem (1.3), we have

$$\|\mathsf{A}\|_2 \leq \sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}\sqrt{1+\frac{U_nU_{n-1}-ab}{p}}. \tag{15}$$

Combine (14) and (15), we get

$$\sqrt{\frac{U_nU_{n-1}-ab+pa^2}{p}} \leq \| \mathsf{A}\|_2 \leq \sqrt{\left(\frac{U_nU_{n-1}-ab+pa^2}{p}\right)}\sqrt{1+\frac{U_nU_{n-1}-ab}{p}}.$$

□

Remark 2.4. Above result is the generalization of Solak 's work [10], in which the author found the upper and lower bounds for the Euclidean and spectral norms of circulant matrices.

Theorem 2.5. *Let A be an $n \times n$ semi-circulant matrix $A = (a_{ij})$ with the generalized Fibonacci numbers then,*

$$\|A\|_E^2 = \frac{\left(a^2 + pab + b^2\right) R_n - \left(a^2 + pab\right) F_n F_{n+1} + ab\left(2S_n - F_n^2\right) - nab}{p}.$$

Proof. For the semi-circulant matrix $A = (a_{ij})$ with the Generalized Fibonacci sequence numbers we have

$$a_{ij} = \begin{cases} U_{j-i+1} & i \leq j \\ 0 & otherwise. \end{cases}$$

From the definition of Euclidean norm, we have

$$\|A\|_E^2 = \sum_{j=1}^{n} \sum_{i=1}^{j} \left(U_{j-i+1}\right)^2 = \sum_{j=1}^{n} \left(\sum_{k=1}^{j} U_k^2 \right).$$

Using lemma (1.9), we get the required result

$$\|A\|_E^2 = \frac{\left(a^2 + pab + b^2\right) R_n - \left(a^2 + pab\right) F_n F_{n+1} + ab\left(2S_n - F_n^2\right) - nab}{p}.$$

□

3. Hankel and Toeplitz matrix norm

In this section, we have calculated the bounds of Hankel and Toeplitz matrix associated with generalized Fibonacci sequence.

Theorem 3.1. *If $A = (a_{ij})$ is an $n \times n$ Hankel matrix with $a_{ij} = U_{i+j-1}$, then*

$$\|A\|_E = \left[\frac{\mathrm{T}_{2n-1} - 2\mathrm{T}_{n-1} - ab}{p} \right]^{\frac{1}{2}},$$

where T_n is defined in lemma (1.9).

Proof. From the definition of Hankel matrix, the matrix A is of the form

$$A = \begin{bmatrix} U_1 & U_2 & U_3 & \cdots & U_{n-1} & U_n \\ U_2 & U_3 & U_4 & \cdots & U_n & U_{n+1} \\ U_3 & U_4 & U_5 & \cdots & U_{n+1} & U_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{n-1} & U_n & U_{n+1} & \cdots & U_{2n-3} & U_{2n-2} \\ U_n & U_{n+1} & U_{n+2} & \cdots & U_{2n-2} & U_{2n-1} \end{bmatrix}.$$

So, we have

$$\|\mathsf{A}\|_E = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|\mathsf{a}_{ij}|^2\right)^{1/2}$$

$$\|\ \mathsf{A}\|_E = \left(\sum_{k=1}^{n}U_k^2 + \sum_{k=2}^{n+1}U_k^2 + ... + \sum_{k=n}^{2n-1}U_k^2\right)^{1/2}$$

$$\|\mathsf{A}\|_E = \left(\left(\sum_{k=1}^{n}U_k^2 + \sum_{k=1}^{n+1}U_k^2 + ... + \sum_{k=1}^{2n-1}U_k^2\right) - \left(\sum_{k=1}^{n-1}\sum_{i=1}^{k}U_i^2\right)\right)^{1/2}$$

$$\|\mathsf{A}\|_E = \Bigg[\left(\frac{U_nU_{n+1}-ab}{p}\right) + (\frac{U_{n+1}U_{n+2}-ab}{p}) + ... + \left(\frac{U_{2n-1}U_{2n}-ab}{p}\right) - \sum_{k=1}^{n-1}\left(\frac{U_kU_{k+1}-ab}{p}\right)\Bigg]^{\frac{1}{2}}$$

$$\|\mathsf{A}\|_E = \left[\sum_{k=1}^{2n-1}\left(\frac{U_kU_{k+1}-ab}{p}\right) - 2\sum_{k=1}^{n-1}\left(\frac{U_kU_{k+1}-ab}{p}\right)\right]^{\frac{1}{2}}$$

$$\|\mathsf{A}\|_E = \left[\frac{1}{p}\sum_{k=1}^{2n-1}U_kU_{k+1} - 2\frac{1}{p}\sum_{k=1}^{n-1}U_kU_{k+1} - \frac{ab}{p}\right]^{\frac{1}{2}}$$

$$\|\mathsf{A}\|_E = \left[\frac{\mathrm{T}_{2n-1} - 2\mathrm{T}_{n-1} - ab}{p}\right]^{\frac{1}{2}}.$$

□

Theorem 3.2. *If $A = (\mathsf{a}_{ij})$ is an $n \times n$ Hankel matrix with $\mathsf{a}_{ij} = U_{i+j-1}$ then, we have*

$$\frac{1}{\sqrt{n}}\|A\|_E \leq \|A\|_2 \leq \frac{\sqrt{(U_nU_{n+1}-ab)\,(U_nU_{n+1}-ab+p(1-b^2))}}{p}.$$

Proof. From theorem (3.1) and inequality (7), we have

$$\frac{1}{\sqrt{n}}\|\mathsf{A}\|_E \leq \|\mathsf{A}\|_2$$

Let us define two new matrices

$$\mathsf{M} = \begin{cases} \mathsf{m}_{ij} = U_{i+j-1} & i \leq j \\ \mathsf{n}_{ij} = 1 & i > j \end{cases} \text{ and } \mathsf{N} = \begin{cases} \mathsf{n}_{ij} = U_{i+j-1} & i > j \\ \mathsf{n}_{ij} = 1 & i \leq j \end{cases}$$

It can be easily seen that $\mathsf{A} = \mathsf{M} \circ \mathsf{N}$. Thus we get

$$r_1(\mathsf{M}) = \max_i \sqrt{\sum_j |\mathsf{m}_{ij}|^2} = \sqrt{\sum_{i=1}^{n} U_i^2} = \sqrt{\left[\frac{U_n U_{n+1} - ab}{p}\right]},$$

$$c_1(\mathsf{N}) = \max_j \sqrt{\sum_i |\mathsf{n}_{ij}|^2} = \sqrt{1 + \sum_{i=2}^{n} U_i^2} = \sqrt{\frac{U_n U_{n+1} - ab + p(1-b^2)}{p}}.$$

Using the theorem (1.3), we have

$$\|\mathsf{A}\|_2 \leq \frac{\sqrt{(U_n U_{n+1} - ab)\,(U_n U_{n+1} - ab + p(1-b^2))}}{p}.$$

□

Theorem 3.3. *If $A = (\mathsf{a}_{ij})$ is an $n \times n$ Hankel matrix with $\mathsf{a}_{ij} = U_{i+j-1}$. Then we have $\|A\|_1 = \|A\|_\infty = U_{2n+1} - U_{n+1}$.*

Proof. From the definition of the matrix A , we can write

$$\|\mathsf{A}\|_1 = \max_{i \leq j \leq n} \sum_{i=1}^{n} |\mathsf{a}_{ij}| = \max_{1 \leq j \leq n} \{|\mathsf{a}_{1j}| + |\mathsf{a}_{2j}| + |\mathsf{a}_{3j}| \dots |\mathsf{a}_{nj}|\}$$

$$\|\ \mathsf{A}\|_1 = U_n + U_{n+1} + U_{n+2} + \cdots + U_{2n-1}$$

$$\|\mathsf{A}\|_1 = \sum_{i=1}^{2n-1} U_i - \sum_{i=1}^{n-1} U_i,$$

by lemma (1.6), we have

$$\|\mathsf{A}\|_1 = \frac{U_{2n-1} + U_{2n} - U_{n-1} - U_n}{p}.$$

Similarly, the row norm of the matrix A can be computed as:

$$\|\ \mathsf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\mathsf{a}_{ij}| = \frac{U_{2n-1} + U_{2n} - U_{n-1} - U_n}{p}.$$

□

Theorem 3.4. *The bounds of spectral norms of the Toeplitz matrix A are given as:*

$$\|A\|_2 \geq \sqrt{\frac{na^2 + \mathrm{T}_{n-1} + \mathrm{T}_{-(n-1)}}{n}},$$

and

$$\|A\|_2 \leq \left[\left(\frac{U_{n-1}U_n - ab}{p} + a^2\right) \times \left\{1 + a^2\left(\frac{F_n F_{n+1}}{p} - 1\right) + b^2\left(\frac{F_n F_{n-1}}{p}\right) - 2ab\left(\frac{F_n^2 + F_{n-1}F_{n+1} - 1}{2p}\right)\right\}\right]^{1/2},$$

where T_{n-1} and $\mathrm{T}_{-(n-1)}$ are defined in lemma (1.9) and (1.10) respectively.

Proof. The Toeplitz matrix A define by the sequence (1) is given as

$$\mathsf{A} = \begin{bmatrix} U_0 & U_{-1} & U_{-2} & \cdots & U_{2-n} & U_{2-n} \\ U_1 & U_0 & U_{-1} & \cdots & U_{3-n} & U_{2-n} \\ U_2 & U_1 & U_0 & \cdots & U_{4-n} & U_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{n-2} & U_{n-3} & U_{n-4} & \cdots & U_0 & U_{-1} \\ U_{n-1} & U_{n-2} & U_{n-3} & \cdots & U_1 & U_0 \end{bmatrix}.$$

From the definition of Euclidean norm , we have

$$\|\mathsf{A}\|_E^2 = nU_0^2 + \sum_{i=1}^{n-1}\sum_{k=1}^{i} U_k^2 + \sum_{i=1}^{n-1}\sum_{k=1}^{i} U_{-k}^2$$

From Lemma (1.9) and (1.10), we have

$$\|\mathsf{A}\|_E^2 = na^2 + \mathrm{T}_{n-1} + \mathrm{T}_{n-1} \tag{16}$$

Using inequality (7), we obtain

$$\|\mathsf{A}\|_2 \geq \sqrt{\frac{na^2 + \mathrm{T}_{n-1} + \mathrm{T}_{n-1}}{n}}. \tag{17}$$

On the other hand , let us consider the matrices.

$$C = (c_{ij}) = \begin{cases} c_{ij} = 1 & j = 1 \\ c_{ij} = U_{i-j} & j \neq 1 \end{cases} \quad \text{and} \quad D = (d_{ij}) = \begin{cases} d_{ij} = 1 & j \neq 1 \\ d_{ij} = U_{i-j} & j = 1 \end{cases}$$

such that, $\mathsf{A} = C \circ D$. Then using definition (1.2)

$$\begin{aligned} r_1(C) &= \max_i \sqrt{\sum_j (c_{ij})^2} = \sqrt{1 + \sum_{k=1}^{n-1} U_{-k}^2} \\ &= \sqrt{1 + a^2 \sum_{k=1}^{n-1} F_{k+1}^2 + b^2 \sum_{k=1}^{n-1} F_k^2 - 2ab \sum_{k=1}^{n-1} F_k F_{k+1}} \\ &= \sqrt{1 + a^2\left(\frac{F_n F_{n+1}}{p} - 1\right) + b^2\left(\frac{F_n F_{n-1}}{p}\right) - 2ab\left(\frac{F_n^2 + F_{n-1}F_{n+1} - 1}{2p}\right)} \end{aligned}$$

$$c_1(D) = \max_j \sqrt{\sum_i (d_{ij})^2} = \sqrt{\sum_{k=0}^{n-1} U_k^2} = \sqrt{\frac{U_{n-1}U_n - ab}{p} + a^2}$$

By theorem (1.3), we obtain the desired result

$$\|\mathsf{A}\|_2 \leq \Bigg[\left(\frac{U_{n-1}U_n - ab}{p} + a^2\right) \times \left\{1 + a^2\left(\frac{F_n F_{n+1}}{p} - 1\right) + b^2\left(\frac{F_n F_{n-1}}{p}\right) - 2ab\left(\frac{F_n^2 + F_{n-1}F_{n+1} - 1}{2p}\right)\right\}\Bigg]^{1/2},$$

□

Remark 3.5. Norms of Toeplitz matrix with Fibonacci and Lucas numbers have been calculated by Akbulak and Bozkurt [1]. Theorem (3.4) is a generalization of their paper.

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