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DEGREE OF VERTICES IN VAGUE GRAPHS†

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Abstract. A vague graph is a generalized structure of a fuzzy graph that gives more precision, flexibility and compatibility to a system when compared with systems that are designed using fuzzy graphs. In this paper, we define two new operation on vague graphs namely normal product and tensor product and study about the degree of a vertex in vague graphs which are obtained from two given vague graphs G_1 and G_2 using the operations cartesian product, composition, tensor product and normal product. These operations are highly utilized by computer science, geometry, algebra, number theory and operation research. In addition to the existing operations these properties will also be helpful to study large vague graph as a combination of small, vague graphs and to derive its properties from those of the smaller ones.

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1. Introduction

Graphs and hypergraphs have been applied in a large number of problems including cancer detection, robotics, human cardiac functions, networking and designing. It was Zadeh [25] who introduced fuzzy sets and fuzzy logic into mathematics to deal with problems of uncertainty. As most of the phenomena around us involve much of ambiguity and vagueness, fuzzy logic and fuzzy mathematics have to play a crucial role in modeling real time systems with some level of uncertainty. The most important feature of a fuzzy set is that a fuzzy set A is a class of objects that satisfy a certain (or several) property. Gau and Buehrer [5] proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function

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$t_v(x)$ and a false membership function $f_v(x)$ are used to describe the boundaries of the membership degree. The initial definition given by Kaufmann [6] of a fuzzy graph was based on the fuzzy relation proposed by Zadeh [26]. Later Rosenfeld [15] introduced the fuzzy analogue of several basic graph-theoretic concepts. Mordeson and Nair [7] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Akram et al. [2, 3, 4] introduced vague hypergraphs, certain types of vague graphs and regularity in vague intersection graphs and vague line graphs . Ramakrishna [9] introduced the concept of vague graphs and studied some of their properties. Pal and Rashmanlou [8] studied irregular interval-valued fuzzy graphs. Also, they defined antipodal interval-valued fuzzy graphs [10], balanced interval-valued fuzzy graphs [11], some properties of highly irregular interval-valued fuzzy graphs [12] and a study on bipolar fuzzy graphs [14]. Rashmanlou and Yang Bae Jun investigated complete interval-valued fuzzy graphs [13]. Samanta and Pal defined fuzzy tolerance graphs [16], fuzzy threshold graphs [17], fuzzy planar graphs [18], fuzzy k-competition graphs and p-competition fuzzy graphs [19], irregular bipolar fuzzy graphs [20], fuzzy coloring of fuzzy graphs [21]. In this paper, we defined two new operation on vague graphs namely normal product and tensor product and studied about the degree of a vertex in vague graphs which are obtained from two given vague graphs G_1 and G_2 using the operations cartesian product, composition, tensor product and normal product. For further details, reader may look into [1, 22, 23, 24].

2. Preliminaries

By a graph $G^* = (V, E)$, we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. Formally, given a graph $G^* = (V, E)$, two vertices $x, y \in V$ are said to be neighbors, or adjacent nodes, if $xy \in E$. A fuzzy subset μ on a set X is a map $\mu : X \to [0, 1]$. A fuzzy binary relation on X is a fuzzy subset μ on $X \times X$. A fuzzy graph G is a pair of functions $G = (\sigma, \mu)$ where σ is a fuzzy subset of a non-empty set V and $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on σ, i.e. $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The degree of a vertex u in fuzzy graph G is defined by $d_G(u) = \sum_{u \neq v} \mu(uv) = \sum_{uv \in E} \mu(uv)$. The order of a fuzzy graph G is defined by $O(G) = \sum_{u \in V} \sigma(u)$.

The main objective of this paper is to study of vague graph and this graph is based on the vague set defined below.

Definition 2.1 ([5]). A vague set on an ordinary finite non-empty set X is a pair (t_A, f_A), where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ are true and false membership functions, respectively such that $0 \leq t_A(x) + f_A(x) \leq 1$, for all $x \in X$. Note that $t_A(x)$ is considered as the lower bound for degree of membership of x in A and $f_A(x)$ is the lower bound for negative of membership of x in A. So, the degree of membership of x in the vague set A is characterized by interval $[t_A(x), 1 - f_A(x)]$. Let X and Y be ordinary finite non-empty sets. We call a vague relation to be a vague subset of $X \times Y$, that is an expression R defined by

$$R = \{\langle (x,y),\ t_R(x,y),\ f_R(x,y)\rangle \mid x \in X,\ y \in Y\}$$

where $t_R : X \times Y \to [0,1]$, $f_R : X \times Y \to [0,1]$, which satisfies the condition $0 \leq t_R(x,y) + f_R(x,y) \leq 1$, for all $(x,y) \in X \times Y$.

Definition 2.2 ([9])**.** Let $G^* = (V,E)$ be a crisp graph. A pair $G = (A,B)$ is called a vague graph on a crisp graph $G^* = (V,E)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that

$$t_B(xy) \leq \min(t_A(x), t_A(y)) \textit{ and } f_B(xy) \geq \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

If G is a vague graph, then the order of G is defined and denoted as

$$O(G) = \left(\sum_{u\in V} t_A(u), \sum_{u\in V} f_A(u)\right)$$

and the size of G is

$$S(G) = \left(\sum_{u\neq v u,v\in V} t_B(uv), \sum_{u\neq v u,v\in V} f_B(uv)\right).$$

The open degree of a vertex u in a vague graph $G = (A,B)$ is defined as $d(u) = \big(d^t(u), d^f(u)\big)$ where $d^t(u) = \sum_{\substack{u\neq v\\ u,v\in V}} t_B(uv)$ and $d^f(u) = \sum_{\substack{u\neq v\\ u,v\in V}} f_B(uv)$. If all the vertices have the same open neighborhood degree n, then G is called an n-regular vague graph.

Definition 2.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively.
(1) The cartesian product $G_1 \times G_2$ of G_1 and G_2 is defined as pair $(A_1 \times A_2, B_1 \times B_2)$ such that

$$(i) \begin{cases} (t_{A_1} \times t_{A_2})(u_1,u_2) = \min(t_{A_1}(u_1), t_{A_2}(u_2)) \\ (f_{A_1} \times f_{A_2})(u_1,u_2) = \max(f_{A_1}(u_1), f_{A_2}(u_2)) \end{cases} \text{for all } (u_1,u_2) \in V_1 \times V_2$$

$$(ii) \begin{cases} (t_{B_1} \times t_{B_2})((u,u_2)(u,v_2)) = \min(t_{A_1}(u), t_{B_2}(u_2v_2)) \\ (f_{B_1} \times f_{B_2})((u,u_2)(u,v_2)) = \max(f_{A_1}(u), f_{B_2}(u_2v_2)) \end{cases}$$

for all $u \in V_1$ and $u_2v_2 \in E_2$,

$$(iii) \begin{cases} (t_{B_1} \times t_{B_2})((u_1,z)(v_1,z)) = \min(t_{B_1}(u_1v_1), t_{A_2}(z)) \\ (f_{B_1} \times f_{B_2})((u_1,z)(v_1,z)) = \max(f_{B_1}(u_1v_1), f_{A_2}(z)) \end{cases}$$

for all $z \in V_2$ and $u_1v_1 \in E_1$.

(2) The composition $G_1 \circ G_2$ of G_1 and G_2 is defined as pair $(A_1 \circ A_2, B_1 \circ B_2)$ such that

$$(i)\begin{cases}(t_{A_1}\circ t_{A_2})(u_1,u_2)=\min(t_{A_1}(u_1),t_{A_2}(u_2))\\(f_{A_1}\circ f_{A_2})(u_1,u_2)=\max(f_{A_1}(u_1),f_{A_2}(u_2))\end{cases}\text{for all }(u_1,u_2)\in V_1\times V_2$$

$$(ii)\begin{cases}(t_{B_1}\circ t_{B_2})((u,u_2)(u,v_2))=\min(t_{A_1}(u),t_{B_2}(u_2v_2))\\(f_{B_1}\circ f_{B_2})((u,u_2)(u,v_2))=\max(f_{A_1}(u),f_{B_2}(u_2v_2))\end{cases}$$

for all $u\in V_1$ and $u_2v_2\in E_2$,

$$(iii)\begin{cases}(t_{B_1}\circ t_{B_2})((u_1,z)(v_1,z))=\min(t_{B_1}(u_1v_1),t_{A_2}(z))\\(f_{B_1}\circ f_{B_2})((u_1,z)(v_1,z))=\max(f_{B_1}(u_1v_1),f_{A_2}(z))\end{cases}$$

for all z $\in V_2$ and $u_1v_1\in E_1$,

$$(iv)\begin{cases}(t_{B_1}\circ t_{B_2})((u_1,u_2)(v_1,v_2))=\min(t_{A_2}(u_2),t_{A_2}(v_2),t_{B_1}(u_1v_1))\\(f_{B_1}\circ f_{B_2})((u_1,u_2)(v_1,v_2))=\max(f_{A_2}(u_2),f_{A_2}(v_2),f_{B_1}(u_1v_1))\end{cases}$$

for all $(u_1,u_2)(v_1,v_2)\in E^\circ-E$,

where $E^\circ=E\cup\{(u_1,u_2)(v_1,v_2)\mid u_1v_1\in E_1,\ u_2\neq v_2\}$.

3. Degree of vertices in vague graphs

Operation in fuzzy graph is a great tool to consider large fuzzy graph as a combination of small fuzzy graphs and to derive its properties from those of the smaller ones. Also, they are conveniently used in many combinatorial applications. In various situations they present a suitable construction means. For example in partition theory we deal with complex objects. A typical such object is a fuzzy graph and a fuzzy hypergraph with large chromatic number that we do not know how to compute exactly the chromatic number of these graphs. In such cases, these operations have the main role in solving problems. Hence, in this section, at first we define two new operations on vague graphs namely normal product and tensor product. Then we study about the degree of a vertex in vague graphs which are obtained from two given vague graphs G_1 and G_2 using the operations cartesian product, composition, tensor product and normal product.

Definition 3.1. The normal product of two vague graphs $G_i=(A_i,B_i)$ on $G_i=(V_i,E_i)$, $i=1,2$ is defined as a vague graph $(A_1\bullet A_2,B_1\bullet B_2)$ on $G=(V,E)$ where $V=V_1\times V_2$ and $E=\{((u,u_2)(u,v_2))\mid u\in V_1,u_2v_2\in E_2\}\cup\{((u_1,z)(v_1,z))\mid u_1v_1\in E_1,\ z\in V_2\}\cup\{((u_1,u_2)(v_1,v_2))\mid u_1v_1\in E_1,u_2v_2\in E_2\}$ such that:

$$(i)\begin{cases}(t_{A_1}\bullet t_{A_2})(u_1,u_2)=\min(t_{A_1}(u_1),t_{A_2}(u_2))\\(f_{A_1}\bullet f_{A_2})(u_1,u_2)=\max(f_{A_1}(u_1),f_{A_2}(u_2))\end{cases}\text{for all }(u_1,u_2)\in V_1\times V_2,$$

$$(ii)\begin{cases}(t_{B_1}\bullet t_{B_2})((u,u_2)(u,v_2))=\min(t_{A_1}(u),t_{B_2}(u_2v_2))\\(f_{B_1}\bullet f_{B_2})((u,u_2)(u,v_2))=\max(f_{A_1}(u),f_{B_2}(u_2v_2))\end{cases}$$

for all $u\in V_1$ and $u_2v_2\in E_2$,

$$(iii)\begin{cases}(t_{B_1}\bullet t_{B_2})((u_1,z)(v_1,z))=\min(t_{B_1}(u_1v_1),t_{A_2}(z))\\(f_{B_1}\bullet f_{B_2})((u_1,z)(v_1,z))=\max(f_{B_1}(u_1v_1),f_{A_2}(z))\end{cases}$$

for all $z\in V_2$ and $u_1v_1\in E_1$,

$$(iv)\begin{cases}(t_{B_1}\bullet t_{B_2})((u_1,u_2)(v_1,v_2))=\min(t_{B_1}(u_1v_1),t_{B_2}(u_2v_2))\\(f_{B_1}\bullet f_{B_2})((u_1,u_2)(v_1,v_2))=\max(f_{B_1}(u_1v_1),f_{B_2}(u_2v_2))\end{cases}$$

for all $u_1v_1\in E_1$ and $u_2v_2\in E_2$.

Definition 3.2. The tensor product of two vague graphs $G_i=(A_i,B_i)$ on $G_i=(V_i,E_i)$, $i=1,2$, is defined as a vague graph $(A_1\otimes A_2,B_1\otimes B_2)$ on $G=(V,E)$ where $V=V_1\times V_2$ and $E=\{(u_1,u_2),(v_1,v_2)\mid u_1v_1\in E_1,\ u_2v_2\in E_2\}$ such that

$$(i)\begin{cases}(t_{A_1}\otimes t_{A_2})(u_1,u_2)=\min(t_{A_1}(u_1),t_{A_2}(u_2))\\(f_{A_1}\otimes f_{A_2})(u_1,u_2)=\max(f_{A_1}(u_1),f_{A_2}(u_2))\end{cases}\text{for all } (u_1,u_2)\in V_1\times V_2,$$

$$(ii)\begin{cases}(t_{B_1}\otimes t_{B_2})((u_1,u_2)(v_1,v_2))=\min(t_{B_1}(u_1v_1),t_{B_2}(u_2v_2))\\(f_{B_1}\otimes f_{B_2})((u_1,u_2)(v_1,v_2))=\max(f_{B_1}(u_1v_1),f_{B_2}(u_2v_2))\end{cases}$$

for all $u_1v_1\in E_1$ and $u_2v_2\in E_2$.

Now, we derive degree of a vertex in the cartesian product. By the definition of cartesian product for any vertex $(u_1,u_2)\in V_1\times V_2$,

$$\begin{aligned}d^t_{G_1\times G_2}(u_1,u_2)&=\sum_{(u_1,u_2)(v_1,v_2)\in E}(t_{B_1}\times t_{B_2})((u_1,u_2)(v_1,v_2))\\&=\sum_{u_1=v_1,u_2v_2\in E_2}t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)\\&\quad+\sum_{u_2=v_2,u_1v_1\in E_1}t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\d^f_{G_1\times G_2}(u_1,u_2)&=\sum_{(u_1,u_2)(v_1,v_2)\in E}(f_{B_1}\times f_{B_2})((u_1,u_2)(v_1,v_2))\\&=\sum_{u_1=v_1,u_2v_2\in E_2}f_{A_1}(u_1)\vee f_{B_2}(u_2v_2)\\&\quad+\sum_{u_2=v_2,u_1v_1\in E_1}f_{A_2}(u_2)\vee f_{B_1}(u_1v_1).\end{aligned}$$

Theorem 3.3. *Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two vague graphs. If $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$ and $t_{A_2}\geq t_{B_1}$, $f_{A_2}\leq f_{B_1}$ then*

$$d_{G_1\times G_2}(u_1,u_2)=d_{G_1}(u_1)+d_{G_2}(u_2).$$

Proof. From the definition of a vertex in the cartesian product we have

$$d^t_{G_1\times G_2}(u_1,u_2)=\sum_{u_1=v_1,u_2v_2\in E_2}t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)$$

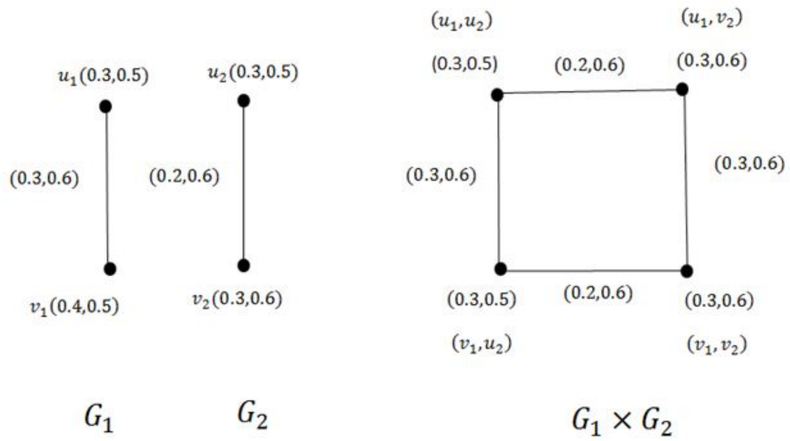

FIGURE 1. Cartesian product of G_1 and G_2

$$\begin{aligned}
&+ \sum_{u_2=v_2, u_1v_1\in E_1} t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
&= \sum_{u_2v_2\in E_2} t_{B_2}(u_2v_2) + \sum_{u_1v_1\in E_1} t_{B_1}(u_1v_1)\\
&= d^t_{G_1}(u_1) + d^t_{G_2}(u_2).
\end{aligned}$$

Also we have

$$\begin{aligned}
d^f_{G_1\times G_2}(u_1,u_2) &= \sum_{u_1=v_1, u_2v_2\in E_2} f_{A_1}(u_1)\vee f_{B_2}(u_2v_2)\\
&\quad + \sum_{u_2=v_2, u_1v_1\in E_1} f_{A_2}(u_2)\vee f_{B_1}(u_1v_1)\\
&= \sum_{u_2v_2\in E_2} f_{B_2}(u_2v_2) + \sum_{u_1v_1\in E_1} f_{B_1}(u_1v_1)\\
&= d^f_{G_1}(u_1) + d^f_{G_2}(u_2).
\end{aligned}$$

Hence, $d_{G_1\times G_2}(u_1,u_2) = d_{G_1}(u_1) + d_{G_2}(u_2)$. □

Example 3.4. Consider the vague graphs G_1, G_2 and $G_1\times G_2$ as follows.

Since $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$, $t_{A_2}\geq t_{B_1}$ and $f_{A_2}\leq f_{B_1}$. By Theorem 3.3, we have

$$\begin{aligned}
d^t_{G_1\times G_2}(u_1,u_2) &= d^t_{G_1}(u_1) + d^t_{G_2}(u_2) = 0.3+0.2 = 0.5,\\
d^f_{G_1\times G_2}(u_1,u_2) &= d^f_{G_1}(u_1) + d^f_{G_2}(u_2) = 0.6+0.6 = 1.2.
\end{aligned}$$

So, $d_{G_1\times G_2}(u_1,u_2)=(0.5,1.2)$.

$$d^t_{G_1\times G_2}(u_1,v_2)=d^t_{G_1}(u_1)+d^t_{G_2}(v_2)=0.3+0.2=0.5,$$
$$d^f_{G_1\times G_2}(u_1,v_2)=d^f_{G_1}(u_1)+d^f_{G_2}(v_2)=0.6+0.6=1.2.$$

Hence, $d_{G_1\times G_2}(u_1,v_2)=(0.5,1.2)$.

Similarly, we can find the degrees of all the vertices in $G_1\times G_2$. This can be verified in the Figure 1.

Now we calculate the degree of a vertex in composition. By the definition of composition for any vertex $(u_1,u_2)\in V_1\times V_2$ we have

$$\begin{aligned}
d^t_{G_1\circ G_2}(u_1,u_2)&=\sum_{(u_1,u_2)(v_1,v_2)\in E}(t_{B_1}\circ t_{B_2})((u_1,u_2)(v_1,v_2))\\
&=\sum_{u_1=v_1,u_2v_2\in E_2}t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)\\
&\quad+\sum_{u_2=v_2,u_1v_1\in E_1}t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
&\quad+\sum_{u_2\neq v_2,u_1v_1\in E_1}t_{A_2}(v_2)\wedge t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
d^f_{G_1\circ G_2}(u_1,u_2)&=\sum_{(u_1,u_2)(v_1,v_2)\in E}(f_{B_1}\circ f_{B_2})((u_1,u_2)(v_1,v_2))\\
&=\sum_{u_1=v_1,u_2v_2\in E_2}f_{A_1}(u_1)\vee f_{B_2}(u_2v_2)\\
&\quad+\sum_{u_2=v_2,u_1v_1\in E_1}f_{A_2}(u_2)\vee f_{B_1}(u_1v_1)\\
&\quad+\sum_{u_2\neq v_2,u_1v_1\in E_1}f_{A_2}(v_2)\vee f_{A_2}(u_2)\vee f_{B_1}(u_1v_1).
\end{aligned}$$

Theorem 3.5. *Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two vague graphs. If $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$, $t_{A_2}\geq t_{B_1}$ and $f_{A_2}\leq f_{B_1}$, then $d_{G_1\circ G_2}(u_1,u_2)=|V_2|d_{G_1}(u_1)+d_{G_2}(u_2)$ for all $(u_1,u_2)\in V_1\times V_2$.*

Proof.

$$\begin{aligned}
d^t_{G_1\circ G_2}(u_1,u_2)=&\sum_{u_1=v_1,u_2v_2\in E_2}t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)\\
&+\sum_{u_2=v_2,u_1v_1\in E_1}t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
&+\sum_{u_2\neq v_2,u_1v_1\in E_1}t_{A_2}(v_2)\wedge t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
=&\sum_{u_2v_2\in E_2}t_{B_2}(u_2v_2)+\sum_{u_2=v_2,u_1v_1\in E_1}t_{B_1}(u_1v_1)
\end{aligned}$$

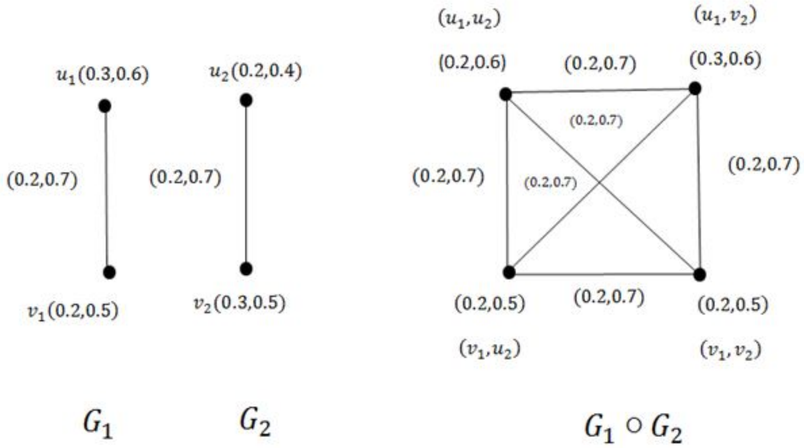

FIGURE 2. Composition of G_1 and G_2

$$
\begin{aligned}
&+\sum_{u_2\neq v_2,u_1v_1\in E_1} t_{B_1}(u_1v_1) \ \ (\text{Since } t_{A_1}\geq t_{B_2} \ \textit{and} \ t_{A_2}\geq t_{B_1})\\
&= d^t_{G_2}(u_2)+|V_2|\sum_{u_1v_1\in E_1} t_{B_1}(u_1v_1)\\
&= d^t_{G_2}(u_2)+|V_2|d^t_{G_1}(u_1).
\end{aligned}
$$

Similarly we can show that

$$d^f_{G_1\circ G_2}(u_1,u_2)=d^f_{G_2}(u_2)+|V_2|d^f_{G_1}(u_1).$$

Hence, $d_{G_1\circ G_2}(u_1,u_2)=d_{G_2}(u_2)+|V_2|d_{G_1}(u_1)$. □

Example 3.6. Consider the vague graphs G_1, G_2 and $G_1\circ G_2$ as follows.

Here, $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$, $t_{A_2}\geq t_{B_1}$ and $f_{A_2}\leq f_{B_1}$. By Theorem 3.5, we have

$$
\begin{aligned}
d^t_{G_1\circ G_2}(u_1,u_2)&=d^t_{G_2}(u_2)+|V_2|d^t_{G_1}(u_1)=0.2+2(0.2)=0.6,\\
d^f_{G_1\circ G_2}(u_1,u_2)&=d^f_{G_2}(u_2)+|V_2|d^f_{G_1}(u_1)=0.7+2(0.7)=2.1.
\end{aligned}
$$

Therefore, $d_{G_1\circ G_2}(u_1,u_2)=(0.6,2.1)$.

$$
\begin{aligned}
d^t_{G_1\circ G_2}(u_1,v_2)&=d^t_{G_2}(v_2)+|V_2|d^t_{G_1}(u_1)=0.2+2(0.2)=0.6,\\
d^f_{G_1\circ G_2}(u_1,v_2)&=d^f_{G_2}(v_2)+|V_2|d^f_{G_1}(u_1)=0.7+2(0.7)=2.1.
\end{aligned}
$$

So, $d_{G_1\circ G_2}(u_1,v_2)=(0.6,2.1)$.

In the same way, we can find the degree of all the vertices in $G_1\circ G_2$. This can be verified in the Figure 2.

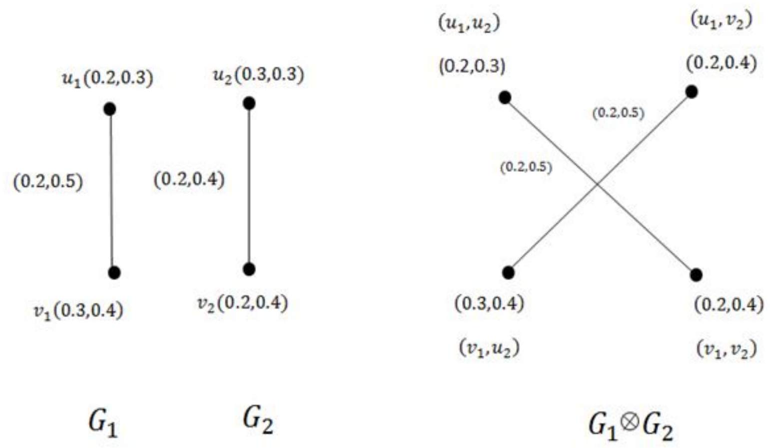

FIGURE 3. Tensor product of G_1 and G_2

Degree of a vertex in the tensor product is as follows.
By definition of tensor product for any $(u_1, u_2) \in V_1 \times V_2$ we have

$$d^t_{G_1 \otimes G_2}(u_1, u_2) = \sum (t_{B_1} \otimes t_{B_2})((u_1, u_2)(v_1, v_2)) = \sum_{u_1 v_1 \in E_1} t_{B_1}(u_1 v_1) \wedge t_{B_2}(u_2 v_2)$$

$$d^f_{G_1 \otimes G_2}(u_1, u_2) = \sum (f_{B_1} \otimes f_{B_2})((u_1, u_2)(v_1, v_2)) = \sum_{u_1 v_1 \in E_1} f_{B_1}(u_1 v_1) \vee f_{B_2}(u_2 v_2).$$

Theorem 3.7. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two vague graphs. If $t_{B_2} \geq t_{B_1}$ and $f_{B_2} \leq f_{B_1}$ then $d_{G_1 \otimes G_2}(u_1, u_2) = d_{G_1}(u_1)$. Also, if $t_{B_1} \geq t_{B_2}$ and $f_{B_1} \leq f_{B_2}$ then $d_{G_1 \otimes \otimes G_2}(u_1, u_2) = d_{G_2}(u_2)$.*

Proof. Let $t_{B_2} \geq t_{B_1}$, $f_{B_2} \leq f_{B_1}$ then we have

$$d^t_{G_1 \otimes G_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1} t_{B_1}(u_1 v_1) \wedge t_{B_2}(u_2 v_2) = \sum t_{B_1}(u_1 v_1) = d^t_{G_1}(u_1),$$

$$d^f_{G_1 \otimes G_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1} f_{B_1}(u_1 v_1) \vee f_{B_2}(u_2 v_2) = \sum f_{B_1}(u_1 v_1) = d^f_{G_1}(u_1).$$

Hence, $d_{G_1 \otimes G_2}(u_1, u_2) = d_{G_1}(u_1)$. Similarly if $t_{B_1} \geq t_{B_2}$ and $f_{B_1} \leq f_{B_2}$, then $d_{G_1 \otimes G_2}(u_1, u_2) = d_{G_2}(u_2)$. □

Example 3.8. In this example we obtain the degree of vertices of $G_1 \otimes G_2$ by Theorem 3.7.

Consider the vague graphs G_1 and G_2 in Figure 3. Here $t_{B_2} \geq t_{B_1}$, $f_{B_2} \leq f_{B_1}$. By Theorem 3.7 we have

$$d^t_{G_1 \otimes G_2}(u_1, u_2) = d^t_{G_1}(u_1) = 0.2, \ d^f_{G_1 \otimes G_2}(u_1, u_2) = d^f_{G_1}(u_1) = 0.5,$$

$$d^t_{G_1\otimes G_2}(v_1,v_2)=d^t_{G_1}(v_1)=0.2,\ d^f_{G_1\otimes G_2}(v_1,v_2)=d^f_{G_1}(v_1)=0.5.$$

So, $d_{G_1\otimes G_2}(u_1,u_2)=(0.2,0.5)$ and $d_{G_1\otimes G_2}(v_1,v_2)=(0.2,0.5)$. Similarly, we can find the degree of all the vertices in $G_1\otimes G_2$. This can be verified in the Figure 3.

Finally, we derive the degree of a vertex in normal product. By the definition of normal product for any $(u_1,u_2)\in V_1\times V_2$ we have

$$\begin{aligned}
d^t_{G_1\bullet G_2}(u_1,u_2) &= \sum_{((u_1,v_1)(u_2,v_2))\in E}(t_{B_1}\bullet t_{B_2})((u_1,v_1)(u_2,v_2))\\
&= \sum_{u_1=v_1,u_2v_2\in E_2} t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)\\
&\quad+\sum_{u_2=v_2,u_1v_1\in E_1} t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
&\quad+\sum_{u_2v_2\in E_2,u_1v_1\in E_1} t_{B_2}(u_2v_2)\wedge t_{B_1}(u_1v_1),\\
d^f_{G_1\bullet G_2}(u_1,u_2) &= \sum_{((u_1,v_1)(u_2,v_2))\in E}(f_{B_1}\bullet f_{B_2})((u_1,v_1)(u_2,v_2))\\
&= \sum_{u_1=v_1,u_2v_2\in E_2} f_{A_1}(u_1)\vee f_{B_2}(u_2v_2)\\
&\quad+\sum_{u_2=v_2,u_1v_1\in E_1} f_{A_2}(u_2)\vee f_{B_1}(u_1v_1)\\
&\quad+\sum_{u_2v_2\in E_2,u_1v_1\in E_1} f_{B_2}(u_2v_2)\vee f_{B_1}(u_1v_1).
\end{aligned}$$

Theorem 3.9. *Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two vague graphs. If $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$, $t_{A_2}\geq t_{B_1}$, $f_{A_2}\leq f_{B_1}$, $t_{B_1}\leq t_{B_2}$ and $f_{B_1}\geq f_{B_2}$ then $d_{G_1\bullet G_2}(u_1,u_2)=|V_2|d_{G_1}(u_1)+d_{G_2}(u_2)$.*

Proof.

$$\begin{aligned}
d^t_{G_1\bullet G_2}(u_1,u_2) &= \sum_{((u_1,v_1)(u_2,v_2))\in E}(t_{B_1}\bullet t_{B_2})((u_1,v_1)(u_2,v_2))\\
&= \sum_{u_1=v_1,u_2v_2\in E_2} t_{A_1}(u_1)\wedge t_{B_2}(u_2v_2)\\
&\quad+\sum_{u_2=v_2,u_1v_1\in E_1} t_{A_2}(u_2)\wedge t_{B_1}(u_1v_1)\\
&\quad+\sum_{u_2v_2\in E_2,u_1v_1\in E_1} t_{B_2}(u_2v_2)\wedge t_{B_1}(u_1v_1)\\
&= \sum_{u_2v_2\in E_2} t_{B_2}(u_2v_2)+\sum_{u_2=v_2,u_1v_1\in E_1} t_{B_1}(u_1v_1)
\end{aligned}$$

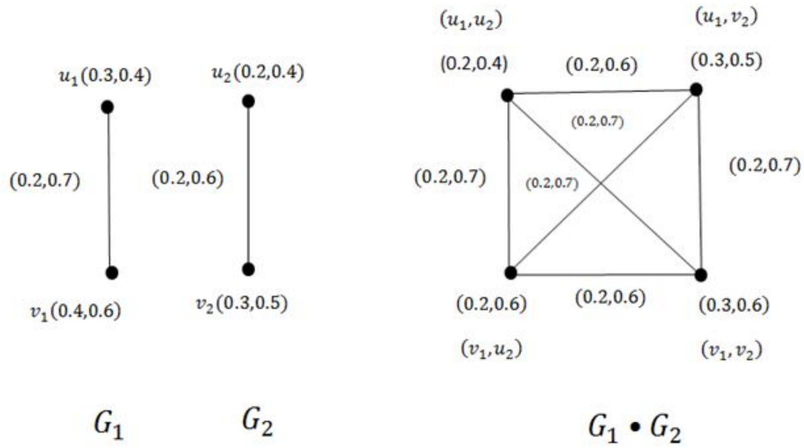

FIGURE 4. Normal product of G_1 and G_2

$$+\sum_{u_1v_1\in E_1} t_{B_1}(u_1v_1),\ (\text{Since } t_{A_1}\geq t_{B_2},\ t_{A_2}\geq t_{B_1},\ t_{B_1}\leq t_{B_2})$$
$$= d^t_{G_2}(u_2) + |V_2| d^t_{G_1}(u_1).$$

In the same way we can show that

$$d^f_{G_1\bullet G_2}(u_1,u_2) = d^f_{G_2}(u_2) + |V_2| d^f_{G_1}(u_1).$$

Hence, $d_{G_1\bullet G_2}(u_1,u_2) = |V_2| d_{G_1}(u_1) + d_{G_2}(u_2)$. $\square$

Example 3.10. In this example we obtain the degree of vertices of $G_1 \bullet G_2$ by Theorem 3.9.

Consider the vague graphs G_1 and G_2 in Figure 4. Here $t_{A_1}\geq t_{B_2}$, $f_{A_1}\leq f_{B_2}$, $t_{A_2}\geq t_{B_1}$, $f_{A_2}\leq f_{B_1}$, $t_{B_1}\leq t_{B_2}$ and $f_{B_1}\geq f_{B_2}$. So, by Theorem 3.9 we have

$$\begin{aligned} d^t_{G_1\bullet G_2}(u_1,u_2) &= d^t_{G_2}(u_2) + |V_2| d^t_{G_1}(u_1) = 0.2 + 2(0.2) = 0.6,\\ d^f_{G_1\bullet G_2}(u_1,u_2) &= d^f_{G_2}(u_2) + |V_2| d^f_{G_1}(u_1) = 0.6 + 2(0.7) = 2. \end{aligned}$$

Therefore, $d_{G_1\bullet G_2}(u_1,u_2) = (0.6, 2)$.

$$\begin{aligned} d^t_{G_1\bullet G_2}(u_1,v_2) &= d^t_{G_2}(v_2) + |V_2| d^t_{G_1}(u_1) = 0.2 + 2(0.2) = 0.6,\\ d^f_{G_1\bullet G_2}(u_1,v_2) &= d^f_{G_2}(v_2) + |V_2| d^f_{G_1}(u_1) = 0.6 + 2(0.7) = 2. \end{aligned}$$

So, $d_{G_1\bullet G_2}(u_1,v_2) = (0.6, 2)$.

Similarly, we can find the degree of all the vertices in $G_1 \bullet G_2$. This can be verified in the Figure 4.

4. Conclusion

Graph theory has several interesting applications in system analysis, operations research, computer applications, and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy systems. It is known that fuzzy graph theory has numerous applications in modern science and engineering, neural networks, expert systems, medical diagnosis, town planning and control theory. In this paper, we have found the degree of vertices in $G_1 \times G_2$, $G_1 \circ G_2$, $G_1 \otimes G_2$ and $G_1 \bullet G_2$ in terms of the degree of vertices in G_1 and G_2 under some conditions and illustrated them through examples. This will be helpful when the graphs are very large and it can help us in studying various properties of cartesian product, composition, tensor product and normal product of two vague graphs.

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