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ASYMPTOTIC-NUMERICAL METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS OF MIXED-TYPE

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Abstract. A computational method for solving singularly perturbed boundary value problem of differential equation with shift arguments of mixed type is presented. When shift arguments are sufficiently small ($o(\varepsilon)$), most of the existing method in the literature used Taylor's expansion to approximate the shift term. This procedure may lead to a bad approximation when the delay argument is of $O(\varepsilon)$. The main idea for this work is to deal with constant shift arguments, which are independent of ε. In the present method, we construct the formally asymptotic solution of the problem using the method of composite expansion. The reduced problem is solved numerically by using operator compact implicit method, and the second problem is solved analytically. Error estimate is derived by using the maximum norm. Numerical examples are provided to support the theoretical results and to show the efficiency of the proposed method.

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1. Introduction

Many phenomena in real life and science may be modeled mathematically by delay differential or differential difference equations (DDEs). Equations of this type arise widely in scientific fields such as biology, medicine, ecology and physics, in which the time evolution depends not only on present states but also on states at or near a given time in the past [3]. If we restrict the class of DDEs to a class in which the highest derivative is multiplied by a small parameter, then we get a class of singularly perturbed differential difference equations (SPDDEs). These equations are used to model a large variety of practical phenomena, for instance, variational problems in control theory [4], description of the so-called

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human pupil-light reflex [10], evolutionary biology [19] and a variety of model for physiological processes or diseases [11].

In this paper, we consider the following SPDDE with negative as well as positive shifts on the interval $\Omega = (0, \ell)$:

$$
\begin{aligned}
&L_\varepsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) + c(x)u(x-\delta) + d(x)u(x+\delta) = f(x),\\
&\quad u(x) = \phi(x), \qquad -\delta \leqslant x \leqslant 0,\\
&\quad u(x) = \psi(x), \qquad \ell \leqslant x \leqslant \ell + \delta,
\end{aligned}
\tag{1}
$$

where $0 < \varepsilon \ll 1$ is the singular perturbation parameter and $\delta(\ell < 2\delta)$ is the shift parameter. We assume that the functions a, b, c, d, f, ϕ and ψ are sufficiently smooth such that

$$
\begin{aligned}
&a(x) \geq \alpha > 0, \quad b(x) > 0, \quad x \in \bar{\Omega},\\
&b(x) - c(x) - d(x) > 0, \quad c(x), d(x) \geq 0 \quad \forall x \in \bar{\Omega}.
\end{aligned}
$$

Under the above assumptions, the solution u of the problem (1) exhibits a single boundary layer at the left end of the interval $\bar{\Omega} = [0, \ell]$. When $a(x) < 0$, the solution exhibits boundary layer at the right end of the interval $\bar{\Omega}$. If $a(x) = 0$, then the solution can exhibit layer or oscillatory behavior depending on the sign of reaction coefficient. Here, we deal with the two cases $a(x) > 0$ and $a(x) < 0$.

It is well known that the standard discretization methods for solving singular perturbation problems are not useful and fail to give accurate results when the perturbation parameter ε tends to zero. This motivates the need for other methods to solve this type of equations, whose accuracy does not depend on the perturbation parameter. For more details, one may refer to [2, 12].

The study of boundary value problems for such equations was initiated by Lange and Miura [8, 9], where the authors provided asymptotic approach to SPDDEs with small shifts in the case of layer behavior and rapid oscillations. In recent years, there has been growing interest to develop numerical methods for SPDDEs. In this area, we mention [5, 6, 7, 13, 17]. Most of these works focused mainly on the problems with very small shifts (of the order of ε). Kadalbajoo and Sharma [6] considered the problem where delay and advance parameter are of $o(\varepsilon)$ and a convection coefficient is absent. Such problems have oscillatory behavior or boundary layers at both ends of the interval. Kadalbajoo et al. [5], Mohapatra and Natesan [13] and Salama and Al-Amery [17] presented numerical methods to solve SPDDEs with small shifts of mixed type when convection coefficient is non-vanishing. Recently, Subburayan and Ramanujam [18] have applied the initial value technique for singularly perturbed convection-diffusion problems with negative shift.

In this paper, we construct an asymptotic-numerical method to solve SPDDEs with shifts of mixed type, these shifts are fixed and do not depend on ε. In this method, an asymptotic expansion of the solution of this problem is constructed using the basic ideas of the method of matched asymptotic expansion [14, 15].

Then, the initial-value problem (the reduced equation) is approximated by operator compact implicit (OCI) method [16], and the second problem, which retains the order of the original problem, is solved analytically. We show that the present method is useful for obtaining the numerical solution of the considered problem in both cases when boundary layer is at the left end as well as at the right end of the interval.

The remainder of the paper is organized as: In Section 2, a maximum principle and some important properties of the exact solution and its derivatives are established. An asymptotic expansion approximation is constructed in Section 3. The proposed numerical method is described in Section 4. Error estimate is derived in Section 5. In Section 6, numerical examples are presented, which validate the theoretical results. Finally, conclusion and discussion are indicated in Section 7.

2. The continuous problem

The problem (1) is equivalent to

$$L_\varepsilon u \equiv \begin{cases} \varepsilon u'' + a(x)u' - b(x)u + d(x)u(x+\delta) = f(x) - c(x)\phi(x-\delta), & x \in \Omega_1, \\ \varepsilon u'' + a(x)u' - b(x)u = f(x) - c(x)\phi(x-\delta) - d(x)\psi(x+\delta), & x \in \Omega_2, \\ \varepsilon u'' + a(x)u' - b(x)u + c(x)u(x-\delta) = f(x) - d(x)\psi(x+\delta), & x \in \Omega_3, \end{cases}$$
$$u(0) = \phi(0), u(\ell) = \psi(\ell).$$

where $\Omega_1 = (0, \ell-\delta]$, $\Omega_2 = [\ell-\delta, \delta]$ and $\Omega_3 = [\delta, \ell)$. Throughout this paper, C is a generic positive constant independent of ε and discretization parameter h, and we use the simple notation for the discrete maximum norm $\|g\| = \max_{1\leqslant i\leqslant N} |g_i|$.

Lemma 2.1 (Continuous minimum principle)**.** *Let $\Psi(x)$ be a smooth function satisfying $\Psi(0) \geq 0$ and $\Psi(\ell) \geq 0$. Then $L_\varepsilon\Psi(x) \leqslant 0$, $\forall x \in \Omega$ implies that $\Psi(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Let $x^\star \in \bar{\Omega}$ be such that $\Psi(x^\star) = \min\{\Psi(x), x \in \bar{\Omega}\}$ and $\Psi(x^\star) < 0$. Clearly $x^\star \neq 0, x^\star \neq \ell$ and also $\Psi'(x^\star) = 0$ and $\Psi''(x^\star) \geq 0$.
We have the following:

(i) $x^\star \in \Omega_1$

$$\begin{aligned} L_\varepsilon\Psi(x^\star) =& \varepsilon\Psi''(x^\star) + a(x^\star)\Psi'(x^\star) - b(x^\star)\Psi(x^\star) + d(x^\star)\Psi(x^\star+\delta) \\ \geq& \varepsilon\Psi''(x^\star) + a(x^\star)\Psi'(x^\star) - b(x^\star)\Psi(x^\star) + d(x^\star)\Psi(x^\star) \\ >& 0, \end{aligned}$$

(ii) $x^\star \in \Omega_2$

$$L_\varepsilon\Psi(x^\star) = \varepsilon\Psi''(x^\star) + a(x^\star)\Psi'(x^\star) - b(x^\star)\Psi(x^\star) > 0,$$

(iii) $x^\star \in \Omega_3$

$$\begin{aligned} L_\varepsilon \Psi(x^\star) =& \varepsilon\Psi''(x^\star) + a(x^\star)\Psi'(x^\star) - b(x^\star)\Psi(x^\star) + c(x^\star)\Psi(x^\star - \delta) \\ \geq& \varepsilon\Psi''(x^\star) + a(x^\star)\Psi'(x^\star) - b(x^\star)\Psi(x^\star) + c(x^\star)\Psi(x^\star) \\ >& 0, \end{aligned}$$

which contradicts the hypothesis that $L_\varepsilon \Psi(x^\star) \leqslant 0$. Therefore $\Psi(x^\star) \geq 0$. But since $x^\star$ was arbitrary point in $\bar{\Omega}$, so that $\Psi(x) \geq 0, \forall x \in \bar{\Omega}$. □

Lemma 2.2. *Let u be the solution of the problem* (1)*, and let $b(x)-c(x)-d(x) \geq \lambda > 0,\ x \in \Omega$. Then*

$$\|u\| \leqslant \lambda^{-1}\|f\| + C_0 \max\big(\|\phi\| + \|\psi\|\big),$$

where C_0 is a positive constant.

Proof. Consider the barrier function $\Psi^\pm$ as

$$\Psi^\pm(x) = \lambda^{-1}\|f\| + C_0 \max\big(\|\phi\| + \|\psi\|\big) \pm u(x).$$

Then, application of the minimum principle to the above barrier function, one can obtain the required result. □

Remark 2.1. Lemma 2.1 implies that the solution of the problem (1) is unique, and the existence of the solution is implied by its uniqueness and the linearity of the considered problem. Further, a bound on the solution is given in Lemma 2.2.

Theorem 2.3. *If a, b, c, d and f are sufficiently smooth functions and $a(x) \geq \alpha > 0,\ \forall x \in \Omega$. Then the derivatives of the solution u of the problem* (1) *satisfy*

$$|u^{(k)}(x)| \leqslant C\Big(1 + \varepsilon^{-k} \exp\big(-\frac{\alpha x}{\varepsilon}\big)\Big), \quad k = 1, 2, \ldots \tag{2}$$

Proof. We rewrite (1) in the form

$$\varepsilon u''(x) + a(x)u'(x) = F(x), \tag{3}$$

where

$$F(x) = f(x) + b(x)u(x) - c(x)u(x-\delta) - d(x)u(x+\delta).$$

Multiplying both sides of (3) by $\frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\int_0^x a(t)dt}$ and integrating over $(0, x)$ we get

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^x a(t)dt} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(t)dt} d\xi. \tag{4}$$

Integrating (4) from 0 to x, we have

$$u(x) = \phi(0) + u'(0)\int_0^x e^{-\frac{1}{\varepsilon}\int_0^\tau a(t)dt} d\tau + \frac{1}{\varepsilon}\int_0^x d\tau \int_0^\tau F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(t)dt} d\xi. \tag{5}$$

Interchanging the order of the double integral, we get

$$u(x) = \phi(0) + u'(0)\int_0^x e^{-\frac{1}{\varepsilon}\int_0^\tau a(t)dt} d\tau + \frac{1}{\varepsilon}\int_0^x F(\xi)d\xi \int_\xi^x e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(t)dt} d\tau. \tag{6}$$

Using the condition $u(\ell)=\psi(\ell)$ in equation (6), we get

$$u'(0)=\frac{\psi(\ell)-\phi(0)-\frac{1}{\varepsilon}\int_0^\ell F(\xi)d\xi\int_\xi^\ell e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(t)dt}d\tau}{\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\tau a(t)dt}d\tau}. \tag{7}$$

Since

$$\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\tau a(t)dt}d\tau \geq \int_0^\ell e^{-\frac{\|a\|\tau}{\varepsilon}}d\tau=\frac{\varepsilon}{\|a\|}\big(1-e^{-\frac{\|a\|\tau}{\varepsilon}}\big)\geq \frac{\varepsilon}{\|a\|}\big(1-e^{-\|a\|\ell}\big)=C_1\,\varepsilon,$$

and a bound on $F(x)$ is obtained by using the bound on u given in Lemma 2.2. Hence, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\ell |F(\xi)|d\xi\int_\xi^\ell e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(t)dt}d\tau &\leqslant \frac{1}{\varepsilon}\int_0^\ell |F(\xi)|d\xi\int_\xi^\ell e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}}d\tau\\ &\leqslant \frac{1}{\varepsilon}\int_0^\ell \Big[\alpha^{-1}\varepsilon\big(1-e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}}\big)\Big]|F(\xi)|d\xi\\ &\leqslant \alpha^{-1}\int_0^\ell |F(\xi)|d\xi \leqslant C_2.\end{aligned}$$

Now, from equation (7) we have

$$\begin{aligned}|u'(0)| &\leqslant \frac{|\psi(\ell)|+|\phi(0)|+\frac{1}{\varepsilon}\int_0^\ell |F(\xi)|d\xi\int_\xi^\ell e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(t)dt}d\tau}{\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\tau a(t)dt}d\tau}\\ &\leqslant \frac{C_1^{-1}\big(|\psi(\ell)|+|\phi(0)|+C_2\big)}{\varepsilon}=\frac{C_3}{\varepsilon}.\end{aligned}$$

Using the above inequality in equation (4), we can conclude that

$$|u'(x)|\leqslant \frac{C_3}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}+C_2\big(1-e^{-\frac{\alpha x}{\varepsilon}}\big).$$

which proves (2) when $k=1$. The estimates for higher order derivatives can be proved by using induction process. $\square$

3. An asymptotic expansion

Since the differential difference equations have special properties, we construct piecewise formal solution in Ω_1, Ω_2 and Ω_3. Moreover, we assume that the solution u satisfies the continuity conditions

$$u((\ell-\delta)^-)=u((\ell-\delta)^+), u'((\ell-\delta)^-)=u'((\ell-\delta)^+), u(\delta^-)=u(\delta^+), u'(\delta^-)=u'(\delta^+),$$

where $u(z^-)$ and $u(z^+)$ denote the left and the right limits of u at $x=z$, respectively.

We represent the solution $u(x)$ in the form

$$u(x)=\begin{cases}\displaystyle\sum_{i=0}^{\infty}\varepsilon^i\left[u_i(x)+v_i(\tau_1)\right], & x\in\Omega_1,\\ \displaystyle\sum_{i=0}^{\infty}\varepsilon^i\left[u_i(x)+v_i(\tau_2)\right], & x\in\Omega_2,\\ \displaystyle\sum_{i=0}^{\infty}\varepsilon^i\left[u_i(x)+v_i(\tau_3)\right], & x\in\Omega_3,\end{cases}\tag{8}$$

where $\tau_1=\frac{x}{\varepsilon}$, $\tau_2=\frac{x-\ell+\delta}{\varepsilon}$ and $\tau_3=\frac{x-\delta}{\varepsilon}$. The coefficients u_i and v_i are called outer and boundary-layer correction coefficients, respectively. We refer the reader to [14, 15] for complete description. From the boundary conditions, we obtain

$$\begin{array}{lll}u_0(0)+v_0(0)=\phi(0), & u_0(\ell)=\psi(\ell), & \\ u_i(0)+v_i(0)=0, & u_i(\ell)=0, & i=1,2,\dots\ .\end{array}\tag{9}$$

Here, we consider the zero order asymptotic expansion approximation to the solution of the original problem (1) which is given by

$$\tilde{u}(x)=u_0(x)+v_0(x),\qquad x\in\Omega,\tag{10}$$

where $u_0(x)$ is the solution of the following reduced problem

$$\begin{cases}a(x)u_0'-b(x)u_0+d(x)u_0(x+\delta)=f(x)-c(x)\phi(x-\delta), x\in\Omega_1,\\ a(x)u_0'-b(x)u_0=f(x)-c(x)\phi(x-\delta)-d(x)\psi(x+\delta), x\in\Omega_2,\\ a(x)u_0'-b(x)u_0+c(x)u_0(x-\delta)=f(x)-d(x)\psi(x+\delta), x\in\Omega_3,\\ u_0(\ell)=\psi(\ell),\end{cases}\tag{11}$$

and $v_0(x)$ is the solution of the boundary value problem

$$\begin{cases}v_0''(\tau_1)+a(0)v_0'(\tau_1)=0, & \tau_1=\frac{x}{\varepsilon}, & x\in\Omega_1,\\ v_0(0)=\phi(0)-u_0(0), & v_0(\infty)=0, & \\ v_0''(\tau_2)+a(\delta)v_0'(\tau_2)=0, & \tau_2=\frac{x-\ell+\delta}{\varepsilon}, & x\in\Omega_2,\\ v_0(0)=u((\ell-\delta)^-)-u_0((\ell-\delta)^+), & v_0(\infty)=0, & \\ v_0''(\tau_3)+a(\delta)v_0'(\tau_3)=0, & \tau_3=\frac{x-\delta}{\varepsilon}, & x\in\Omega_3,\\ v_0(0)=u(\delta^-)-u_0(\delta^+), & v_0(\infty)=0, & \end{cases}\tag{12}$$

which can be solved analytically as

$$v_0(x)=\begin{cases}\left[\phi(0)-u_0(0)\right]\exp\left(\frac{-a(0)x}{\varepsilon}\right), & x\in\Omega_1,\\ \left[u((\ell-\delta)^-)-u_0((\ell-\delta)^+)\right]\exp\left(\frac{-a(\ell-\delta)(x-\ell+\delta)}{\varepsilon}\right), & x\in\Omega_2,\\ \left[u(\delta^-)-u_0(\delta^+)\right]\exp\left(\frac{-a(\delta)(x-\delta)}{\varepsilon}\right), & x\in\Omega_2\ .\end{cases}\tag{13}$$

Remark 3.1. In the case of right boundary layer, the stretching variables $\tau_1=\frac{\ell-\delta-x}{\varepsilon}$ $\tau_2=\frac{\delta-x}{\varepsilon}$ and $\tau_3=\frac{\ell-x}{\varepsilon}$ are used in the intervals $[0,\ell-\delta]$, $[\ell-\delta,\delta]$ and $[\delta,\ell]$, respectively, and one can follow the same procedure used for the case of left boundary layer.

4. Numerical method

In this section, we derive operator compact implicit (OCI) method for the initial value problem (11). Hence the numerical solution of the original problem (1) can be obtained by combining this solution with the solution (13).

4.1. Operator compact implicit method. In the following, we derive OCI method of order four for solving the initial value problem (11). To achieve that, we rewrite it in the form

$$Lu_0(x) = u_0'(x) - \tilde{b}(x)u_0(x) = \tilde{F}(x), \tag{14}$$

where

$$\tilde{b}(x) = \frac{b(x)}{a(x)}, \quad \tilde{F}(x) = \frac{f(x) - c(x)u_0(x-\delta) - d(x)u_0(x+\delta)}{a(x)}.$$

We define the present methods in the form

$$L_N U_{0_j} \equiv \frac{1}{h} R(U_{0_{j-\frac{1}{2}}}) = Q(\tilde{F}_{j-\frac{1}{2}}), \qquad U_{0_N} = \psi(\ell), \tag{15}$$

where the operators R and Q are given by

$$R(U_{0_{j-\frac{1}{2}}}) = r_j^0 U_{0_j} - r_j^1 U_{0_{j-1}} \quad \text{and} \quad Q(\tilde{F}_{j-\frac{1}{2}}) = q_j^0 \tilde{F}_j + q_j^{\frac{1}{2}} \tilde{F}_{j-\frac{1}{2}} + q_j^1 \tilde{F}_{j-1}. \tag{16}$$

It is worthwhile to mention that U_0 and $L_N U_0$ are approximations of u_0 and Lu_0, respectively. The coefficients $r_j^{0,1}$ and $q_j^{0,\frac{1}{2},1}$ are to be functions of $\tilde{b}_j, \tilde{b}_{j-\frac{1}{2}}, \tilde{b}_{j-1}$ and h as we show later. Note that one of the coefficients in (16) must be taken as a multiplicative normalizing factor, so that we consider the following condition

$$q_j^{\frac{1}{2}} = \text{positive constant}, \quad \text{as} \quad h \to 0, \quad 1 \leqslant j \leqslant N. \tag{17}$$

For simplicity of notation we use $\tilde{b}_j = \tilde{b}(x_j)$ and we also omit the subscript j in the quantities $q_j^{0,\frac{i}{2},1}$ when convenient.
The local truncation error $\tau_{j-\frac{1}{2}}$ of (15) at $x_{j-\frac{1}{2}}$ is defined in the form

$$\tau_{j-\frac{1}{2}} = L_N u_0(x_{j-\frac{1}{2}}) - Q\big(Lu_0(x_{j-\frac{1}{2}})\big). \tag{18}$$

For u_0 sufficiently smooth and using Taylor expansion, $\tau_{j-\frac{1}{2}}$ can be written in the form

$$\tau_{j-\frac{1}{2}} = T^0_{j-\frac{1}{2}} u_0(x_{j-\frac{1}{2}}) + T^1_{j-\frac{1}{2}} u_0'(x_{j-\frac{1}{2}}) + \cdots + T^5_{j-\frac{1}{2}} u_0^{(5)}(x_{j-\frac{1}{2}}) + O(h^5),$$

where

$$\begin{aligned}
T^0_{j-\frac{1}{2}} &= \frac{1}{h}\Big[r^0_j - r^1_j + h\big(q^0_j\tilde{b}_j + q^{\frac{1}{2}}_j\tilde{b}_{j-\frac{1}{2}} + q^1_j\tilde{b}_{j-1}\big)\Big],\\
T^1_{j-\frac{1}{2}} &= \frac{1}{2}\Big[r^0_j + r^1_j - 2(q^0_j + q^{\frac{1}{2}}_j + q^1_j) + h\big(q^0_j\tilde{b}_j - q^1_j\tilde{b}_{j-1}\big)\Big],\\
T^k_{j-\frac{1}{2}} &= \frac{h^{k-1}}{2^k k!}\Big[r^0_j + (-1)^{k-1}r^1_j - 2k\sum_{i=0}^{2}(-1)^k(1-i)^{k-1}q^{\frac{i}{2}}_j \\
&\qquad - h\sum_{i=0}^{2}(-1)^k(1-i)^k q^{\frac{i}{2}}_j\tilde{b}_{j-\frac{i}{2}}\Big], \qquad k = 2,3,4,5.
\end{aligned} \tag{19}$$

The sufficient condition for the order of the local truncation error $\tau_{j-\frac{1}{2}}$ is given in the following lemma.

Lemma 4.1. *For the implicit method* (15) *and* (16)*, a sufficient condition for* $\tau_{j-\frac{1}{2}} = O(h^4)$ *is that* $T^k_{j-\frac{1}{2}} = O(h^4)$ *for* $k = 0,1,2,3$*. Furthermore, if* (15) *and* (16) *are normalized according to* (17)*. Then* $\tau_{j-\frac{1}{2}}$ *has formal order no larger than* 4.

Proof. The proof of the this lemma is similar to that of Theorem 1.1 in [16] . □

Here, we construct the method (15) and (16) by the following conditions

$$T^0_{j-\frac{1}{2}} = T^1_{j-\frac{1}{2}} = 0, \tag{20}$$

$$T^k_{j-\frac{1}{2}} = O(h^4), \quad k = 2,3, \quad j = 1,2,\dots,N. \tag{21}$$

Using (19) and the above conditions, we obtain

$$\begin{aligned}
T^2_{j-\frac{1}{2}} &= \frac{h}{8}\Big(4q^1_j - 4q^0_j + h\,q^{\frac{1}{2}}_j\tilde{b}_{j-\frac{1}{2}}\Big),\\
T^3_{j-\frac{1}{2}} &= \frac{h^2}{24}\Big(-2q^1_j - 2q^0_j + q^{\frac{1}{2}}_j\Big), \quad j = 1,2,\cdots,N.
\end{aligned} \tag{22}$$

Now, we define $q^{0,\frac{1}{2},1}_j$ as polynomials in h at each mesh point $x_{j-\frac{1}{2}}$, $j = 1,2,\dots,N$, and for simplifying the notations, the index j in $q^{0,\frac{1}{2},1}_j$ will be dropped.

$$q^{\frac{i}{2}} = \sum_{k=0}^{2} q^{\frac{i}{2}}_k h^k, \qquad i = 0,1,2. \tag{23}$$

Substituting (23) into (22) and imposing (21), we get

$$q^1_0 = 2, \quad q^{\frac{1}{2}}_0 = 8, \quad q^0_0 = 2, \quad q^1_1 = \tilde{b}_{j-\frac{1}{2}}, \quad q^0_1 = -\tilde{b}_{j-\frac{1}{2}}, \quad q^{\frac{1}{2}}_1 = q^1_2 = q^{\frac{1}{2}}_2 = q^0_2 = 0\,.$$

Then, the method is defined in the form

$$\begin{aligned}
&U_{0_N} = \beta,\\
&U_{0_{j-1}} = \frac{1}{r^1_j}\Big[r^0_j U_{0_j} - h\Big(q^0_j\tilde{F}_j + q^{\frac{1}{2}}_j\tilde{F}_{j-\frac{1}{2}} + q^1_j\tilde{F}_{j-1}\Big)\Big],\ j = N, N-1,\cdots,1,
\end{aligned} \tag{24}$$

where

$$
\begin{aligned}
q_j^0 &= 2 - h\tilde{b}_{j-\frac{1}{2}},\\
q_j^{\frac{1}{2}} &= 8,\\
q_j^1 &= 2 + h\tilde{b}_{j-\frac{1}{2}},\\
r_j^0 &= 12 - 4h\tilde{b}_{j-\frac{1}{2}} + h\tilde{b}_j(-2 + h\tilde{b}_{j-\frac{1}{2}}),\\
r_j^1 &= 12 + 4h\tilde{b}_{j-\frac{1}{2}} + h\tilde{b}_{j-1}(2 + h\tilde{b}_{j-\frac{1}{2}}).
\end{aligned}
\tag{25}
$$

4.2. A numerical solution for problem (11)**.** In order to obtain the numerical solution for the problem (11), we apply the OCI method (15) and (16) on a specially designed mesh. The presence of shift parameter makes the problem (11) difficult. To overcome this difficulty, the mesh is designed in such a way that the terms containing shifts lie at the mesh points after the discretization. In what follows, we consider the following uniform mesh on Ω:

$$\Omega_N = \{x_j = jh,\ j = 0, 1, \ldots, N\}, \qquad h = \frac{\ell}{N},$$

and we suppose that $N_0 = \dfrac{\delta}{\ell}N$, where N_0 is a positive integer, i.e., $x_{N_0} = \delta$. Thus, the OCI method (15) and (16) for the initial value problem (11) is defined as follows

$$
\begin{aligned}
U_{0_{N_0}}^{(n)} &= \beta^{(n-1)},\\
U_{0_{j-1}}^{(n)} &= \frac{1}{r_j^1}\Big[r_j^0 U_{0_j}^{(n)} - h\big(q_j^0\tilde{F}_j + q_j^{\frac{1}{2}}\tilde{F}_{j-\frac{1}{2}} + q_j^1\tilde{F}_{j-1}\big)\Big],\ j = N_0(-1)1,\\
U_{0_N}^{(n)} &= \psi(\ell),\\
U_{0_{j-1}}^{(n)} &= \frac{1}{r_j^1}\Big[r_j^0 U_{0_j}^{(n)} - h\big(q_j^0\tilde{F}_j + q_j^{\frac{1}{2}}\tilde{F}_{j-\frac{1}{2}} + q_j^1\tilde{F}_{j-1}\big)\Big],\ j = N(-1)N_0 + 1,
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
\tilde{F}_j &= \frac{f(x_j) - c(x_j)\tilde{U}_1^{(n)}(x_j) - d(x_j)\tilde{U}_2^{(n)}(x_j)}{a(x_j)},\\
\tilde{U}_1^{(n)}(x) &= \begin{cases} \phi(x-\delta),\ x \in [x_{j-1}, x_j],\ j = 1(1)N_0,\\ P_0(x)U_{0_{j-N_0-2}}^{(n)} + P_1(x)U_{0_{j-N_0-1}}^{(n)} + P_2(x)U_{0_{j-N_0}}^{(n)} + P_3(x)U_{0_{j-N_0+1}}^{(n)},\\ \qquad x \in [x_{j-1}, x_j], \qquad j = N_0 + 1(1)N, \end{cases}\\
\tilde{U}_2^{(n)}(x) &= \begin{cases} P_0(x)U_{0_{j+N_0-2}}^{(n-1)} + P_1(x)U_{0_{j+N_0-1}}^{(n-1)} + P_2(x)U_{0_{j+N_0}}^{(n-1)} + P_3(x)U_{0_{j+N_0+1}}^{(n-1)},\\ \qquad x \in [x_{j-1}, x_j], \qquad j = 1(1)N - N_0,\\ \psi(x+\delta),\ x \in [x_{j-1}, x_j],\ j = N - N_0 + 1(1)N, \end{cases}
\end{aligned}
\tag{27}
$$

and

$$\begin{aligned}
P_0(x) &= \frac{x-x_j}{6h}\Big(1-\frac{(x-x_j)^2}{h^2}\Big),\\
P_1(x) &= \frac{x_j-x}{h}\Big(1+\frac{x-x_j}{2h}\Big)\Big(1-\frac{x-x_j}{h}\Big),\\
P_2(x) &= \Big(1+\frac{x-x_j}{2h}\Big)\Big(1-\frac{(x-x_j)^2}{h^2}\Big),\\
P_3(x) &= \frac{x-x_j}{3h}\Big(1+\frac{x-x_j}{2h}\Big)\Big(1+\frac{x-x_j}{h}\Big).
\end{aligned} \tag{28}$$

$\beta^{(n)}$ is the value of (26) for $U_{0_{N_0}}$. The above process is to be repeated with suitable initial value $\beta^{(0)}$ until the profiles stabilize in both regions. For computational purposes, the iterative process stops at the nth iteration if any one of the following conditions is satisfied:

$$|\beta^{(n+1)}-\beta^{(n)}|<\xi, \qquad \text{or} \qquad \|U_0^{(n+1)}-U_0^{(n)}\|<\xi,$$

where ξ is a given tolerance.

4.3. A numerical solution for problem (1)**.** A numerical solution for the original problem (1) is given by

$$U_j=\begin{cases} U_{0_j}+(\phi(0)-U_{0_0})\exp\Big(\dfrac{-a(0)x_j}{\varepsilon}\Big), & 0\leqslant j\leqslant N-N_0,\\ U_{0_j}+(U_{0_{N-N_0}}^{(n-1)}-U_{0_{N-N_0}}^{(n)})\exp\Big(\dfrac{-a(x_{N-N_0})x_{j-N+N_0}}{\varepsilon}\Big), & N-N_0<j\leqslant N_0,\\ U_{0_j}+(U_{0_{N_0}}^{(n-1)}-U_{0_{N_0}}^{(n)})\exp\Big(\dfrac{-a(x_{N_0})x_{j-N_0}}{\varepsilon}\Big), & N_0<j\leqslant N.\end{cases} \tag{29}$$

5. Error estimate

Lemma 5.1. *Let a, b, c and f be sufficiently smooth functions. Then the zero-order asymptotic approximation $\tilde{u}$ given in* (10) *of the solution u for the problem* (1) *satisfies the following*

$$\|u-\tilde{u}\|\leqslant C\varepsilon, \tag{30}$$

where C is a positive constant independent of ε.

Proof. Following the method of proof that is done in [2] on the intervals $[0,\ell-\delta], [\ell-\delta,\delta]$ and $[\delta,\ell]$ yields the desired estimate. □

Denote by $\sigma(x)$ the error of the piecewise cubic interpolation for $u(x)$ defined as

$$\sigma(x_j-\theta h)=u(x_j-\theta h)-\sum_{k=0}^{3}P_k(\theta)u(x_{j+k-2}), \quad \theta\in[0,1], \quad 1\leqslant j\leqslant N.$$

Define

$$\|\sigma\|=\max_{0\leqslant x\leqslant \ell}|\sigma(x)|, \quad M=\max_{1\leqslant j\leqslant N}\sum_{k=0}^{2}|q_j^{\frac{k}{2}}|, \quad \tilde{P}=\sup\Big\{\sum_{k=0}^{3}|P_k(\theta)|,\ \theta\in[0,1]\Big\}.$$

Theorem 5.2. *Let a, b, c, d and f be sufficiently smooth functions and satisfy the condition*

$$b(x)/a(x) \geq b^* > K\tilde{P}, \quad K = \max\big\{\|c/a\|, \|d/a\|\big\}. \tag{31}$$

Then the numerical solution U_0 given in (26) *to the solution u_0 of the problem* (11) *exists and the following error estimate holds:*

$$\big\|E\big\| \leqslant \frac{D}{M}\|\tau\| + (DK+1)\|\sigma\|, \quad \textit{where} \quad D = \frac{\tilde{P}}{b^* - K\tilde{P}}. \tag{32}$$

In particular, the method (26)-(28) *is convergent and the order of convergence is four.*

Proof. Let $e(x_j) = u_0(x_j) - U_{0_j}$. Then, for $0 \leqslant j \leqslant N - N_0$ and using (26), we get

$$\big\|E\big\| \leqslant \max_{1\leqslant j\leqslant N_0} \Big(\Big|\frac{r_j^0}{r_j^1}\Big||u_0(x_j) - U_{0_j}| + h\frac{M\|d/a\|}{|r_j^1|}|u_0(x_j^*) - U_{0_j}^*|\Big) + \frac{h}{|r_j^1|}\|\tau\|.$$

It is clear that, if $x_j^* = x_j + \delta \geq \ell$, then $|u_0(x_j^*) - U_{0_j}^*| = 0$. Otherwise, $x_j^* \in (x_{i-1}, x_i]$ for some $N_0 + 1 \leqslant i \leqslant N$. Thus, $x_j^* = x_i - \theta h, 0 \leqslant \theta < 1$. Using the cubic interpolation, we get

$$\begin{aligned}|u_0(x_j^*) - U_{0_j}^*| \leqslant&\big|u_0(x_j^*) - \sum_{k=0}^{3} P_k(\theta) u_{0_{i+k-2}}\big| + \big|\sum_{k=0}^{3} P_k(\theta)(u_{0_{i+k-2}} - U_{0_{i+k-2}})\big| \\ \leqslant&\tilde{P}\big\|E\big\| + \|\sigma\|.\end{aligned}$$

Thus

$$\big\|E\big\| \leqslant \Big(\Big|\frac{r_j^0}{r_j^1}\Big| + \frac{sh}{|r_j^1|}\Big)\big\|E\big\| + \frac{h}{|r_j^1|}\big(M\|d/a\|\|\sigma\| + \|\tau\|\big), \qquad s = M\tilde{P}\|d/a\|.$$

Hence

$$\big\|E\big\| \leqslant \frac{h}{|r_j^1| - |r_j^0| - sh}\big(M\|d/a\|\|\sigma\| + \|\tau\|\big).$$

Since $|r_j^1| - |r_j^0| \geq hb^*M > 0$, we have $\dfrac{h}{|r_j^1| - |r_j^0| - sh} \leqslant \dfrac{1}{b^*M - s}$. Therefore, we get

$$\big\|E\big\| \leqslant \frac{1}{b^* - \tilde{P}\|d/a\|}\bigg(\|d/a\|\|\sigma\| + \frac{1}{M}\|\tau\|\bigg). \tag{33}$$

Similarly, we have

$$\begin{aligned}&\big\|E\big\| \leqslant \frac{1}{b^*M}\|\tau\|, && N - N_0 < j \leqslant N_0, \\ &\big\|E\big\| \leqslant \frac{1}{b^* - \tilde{P}\|c/a\|}\bigg(\|c/a\|\|\sigma\| + \frac{1}{M}\|\tau\|\bigg), && N_0 \leqslant j \leqslant N.\end{aligned} \tag{34}$$

From the condition (31) and using (33) and (34), we have

$$\|E\| \leqslant \frac{1}{b^* - K\tilde{P}}\left(K\|\sigma\| + \frac{1}{M}\|\tau\|\right), \qquad 0 \leqslant j \leqslant N. \tag{35}$$

To obtain a bound on $e(x)$, we observe that

$$|e(x_j - \theta h)| \leqslant \tilde{P}\|E\| + \|\sigma\|. \tag{36}$$

Let $D = \dfrac{\tilde{P}}{b^* - K\tilde{P}}$, then the above inequality together with (35) complete the proof. □

Remark 5.1. Note that, in the error estimate (32), $\|\tau\| = O(h^4)$ and $\|\sigma\| = O(h^4)$. Therefore, it follows that

$$\|u_0 - U_0\| \leqslant Ch^4. \tag{37}$$

Theorem 5.3. *If u is the solution of the original problem* (1)*, and U is the numerical solution given in* (29)*. Then we have*

$$\|u - U\| \leqslant C(\varepsilon + h^4),$$

where C is a positive constant independent of ε and h.

Proof. Let $\tilde{u} = u_0 + v_0$ be the asymptotic approximation to the solution of problem (1). Then using Lemma 5.1, we have

$$\|u - \tilde{u}\| \leqslant C\varepsilon. \tag{38}$$

On the other hand, for $0 \leqslant j \leqslant N - N_0$, we have

$$U_j = U_{0_j} + V_{0_j},$$

where U_{0_j} is the numerical solution of problem (11) obtained by the OCI method (26)-(28), and V_{0_j} defined as

$$V_{0_j} = (\phi(0) - U_{0_0})e^{-\frac{a(0)x_j}{\varepsilon}}.$$

Hence, using triangle inequality, we get

$$\begin{aligned} |\tilde{u}(x_j) - U_j| &= |u_0 + v_0 - (U_0 + V_0)| \leqslant |(u_0 - U_0) + (U_{0_0} - u_0(0))e^{-\frac{a(0)x_j}{\varepsilon}}| \\ &\leqslant Ch^4|1 - e^{-\frac{a(0)x_j}{\varepsilon}}| \qquad\qquad \text{(using (37))} \\ &\leqslant Ch^4. \end{aligned} \tag{39}$$

Similarly, one can conclude that

$$\begin{aligned} &|\tilde{u}(x_j) - U_j| \leqslant Ch^4, && N - N_0 < j \leqslant N_0, \\ &|\tilde{u}(x_j) - U_j| \leqslant Ch^4, && N_0 \leqslant j \leqslant N. \end{aligned} \tag{40}$$

Combining (38) with (39) and (40), we obtaine

$$\|u - U\| \leqslant \|u - \tilde{u}\| + \|\tilde{u} - U\| \leqslant C(\varepsilon + h^4).$$

□

6. Numerical results

In this section, we present several numerical examples to illustrate the applicability and efficiency of the proposed method, we consider the boundary value problem of singularly perturbed differential difference equations with the left end and the right end boundary layer.

For given value of N, the maximum absolute error is calculated as follows

$$E^N = \max_{0\leqslant j\leqslant N} |u(x_j) - U_j|,$$

where u is the exact solution and U is the numerical solution. In the case of the exact solution is not known, the maximum absolute error and the corresponding rate of convergence are evaluated using the double mesh principle [2]

$$\tilde{E}^N = \max_{0\leqslant j\leqslant N} |U_j^N - U_j^{2N}| \quad \text{and} \quad \tilde{r}^N = \log_2\Big(\frac{\tilde{E}^N}{\tilde{E}^{2N}}\Big).$$

Example 1 ([1])**.** Consider the constant coefficient problem with left boundary layer

$$\varepsilon u''(x) + 128u'(x) + 0.25u(x-1) = 0.25(x-1), \quad x \in \Big(0, \frac{3}{2}\Big),$$

subject to the interval and boundary conditions

$$u(x) = x, \quad -1 \leqslant x \leqslant 0, \qquad u\Big(\frac{3}{2}\Big) = 2.$$

The exact solution is given by

$$u(x) = \begin{cases} c_1\Big(\exp\big(\frac{-128x}{\varepsilon}\big) - 1\Big), & x \in [0,1], \\ c_2 + c_3\exp\big(\frac{-128x}{\varepsilon}\big) + \frac{(x-1)^2}{2^{10}} - \frac{\varepsilon(x-1)^{16}}{2} & \\ \quad +c_1\frac{(x-1)}{2^9}\Big(1+\exp\big(\frac{-128(x-1)}{\varepsilon}\big)\Big), & x \in [1, \frac{3}{2}], \end{cases}$$

where

$$c_1 = \frac{\big(\big(\varepsilon^2 - 64\varepsilon - 2^{24} + 2^{11}\big)\exp\big(\frac{64}{\varepsilon}\big) - \varepsilon^2\big)\exp\big(\frac{128}{\varepsilon}\big)}{256\big(\big(2^{15} - 32 + \varepsilon\big)\exp\big(\frac{192}{\varepsilon}\big) - (32+\varepsilon)\exp\big(\frac{128}{\varepsilon}\big) - 2^{15}\big)},$$

$$c_2 = \frac{\varepsilon^2}{2^{23}} - \Big(1 + \frac{\varepsilon}{2^{15}}\Big)c_1,$$

$$c_3 = \Big(1 + \frac{\varepsilon}{2^{15}}\exp\Big(\frac{128}{\varepsilon}\Big)\Big)c_1 - \frac{\varepsilon^2}{2^{23}}\exp\Big(\frac{128}{\varepsilon}\Big).$$

Example 2 ([18])**.** Consider the constant coefficient problem with right boundary layer

$$\varepsilon u''(x) - 3u'(x) + u(x-1) = 0, \qquad x \in (0,2),$$

subject to the interval and boundary conditions

$$u(x) = 1, \quad -1 \leqslant x \leqslant 0, \qquad u(2) = 2.$$

TABLE 1. The maximum absolute errors E^N for Example 1.

N	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
150	5.351E−07	5.351E−9	5.351E−11	5.298E−13
300	5.351E−07	5.351E−9	5.351E−11	5.338E−13
600	5.351E−07	5.351E−9	5.351E−11	5.351E−13
1200	5.351E−07	5.351E−9	5.351E−11	5.360E−13
2400	5.351E−07	5.351E−9	5.351E−11	5.325E−13

The exact solution is given by

$$u(x) = \begin{cases} 1 + c_1\Big(\exp\big(\frac{3x}{\varepsilon}\big) - 1\Big) + \frac{x}{3}, & x \in [0,1], \\ c_2 + \frac{x}{3} + \frac{(x-1)^2}{18} + \frac{\varepsilon x}{27} - \frac{c_1 x}{3}\big(1 + \exp\big(\frac{3(x-1)}{\varepsilon}\big)\big) & \\ \quad + \exp\big(\frac{3(x-2)}{\varepsilon}\big)\big[\frac{23}{18} - \frac{2\varepsilon}{27} - c_2 + \frac{2c_1}{3}\big(1 + \exp\big(\frac{3}{\varepsilon}\big)\big)\big], & x \in [1,2], \end{cases}$$

where

$$c_1 = \frac{\exp\Big(\frac{-6}{\varepsilon}\Big)\Big(\frac{4\varepsilon}{9} - \frac{\varepsilon^2}{27} - 3\Big)}{3 - 4\exp\big(\frac{-6}{\varepsilon}\big) + \frac{2\varepsilon}{3}\big(\exp\big(\frac{-3}{\varepsilon}\big) - \exp\big(\frac{-6}{\varepsilon}\big)\big)},$$

$$c_2 = \left[\frac{1 - \frac{23}{18}\exp\big(\frac{-3}{\varepsilon}\big) + \frac{2\varepsilon}{27}\exp\big(\frac{-3}{\varepsilon}\big) - \frac{\varepsilon}{27}}{1 - \exp\big(\frac{-3}{\varepsilon}\big)} + \frac{c_1\exp\big(\frac{3}{\varepsilon}\big)\big(1 - \exp\big(\frac{-3}{\varepsilon}\big) - \frac{2}{3}\exp\big(\frac{-6}{\varepsilon}\big)\big)}{1 - \exp\big(\frac{-3}{\varepsilon}\big)}\right].$$

Example 3 ([18])**.** Consider the variable coefficient problem with right boundary layer

$$-\varepsilon u''(x) + (x+10)u'(x) - u(x-1) = x, \qquad x \in (0,2),$$

subject to the interval and boundary conditions

$$u(x) = x, \quad -1 \leqslant x \leqslant 0, \qquad u(2) = 2.$$

The exact solution is not known.

Example 4 ([5])**.** Consider the variable coefficient problem with left boundary layer

$$\varepsilon u''(x) + (1+x^2)u'(x) - (x^3+3)u(x) - (1+x^2)u(x-\delta) + (2+x)u(x+\delta) = 1, \quad x \in (0,1),$$

with the interval conditions

$$u(x) = 0, \quad -\delta \leqslant x \leqslant 0, \qquad u(x) = 1, \quad 1 \leqslant x \leqslant 1+\delta.$$

The exact solution is not known.

TABLE 2. The maximum absolute errors E^N for Example 2.

N	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
100	3.620E−04	3.630E−06	3.630E−08	3.630E−10
200	3.620E−04	3.667E−06	3.667E−08	3.667E−10
400	3.620E−04	3.685E−06	3.685E−08	3.685E−10
800	3.620E−04	3.694E−06	3.694E−08	3.694E−10
1600	3.621E−04	3.699E−06	3.699E−08	3.698E−10

TABLE 3. The values of $\tilde{E}^N$ and $\tilde{r}^N$ for Example 3.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$
10^{-1}	8.25E−12	5.58E−13	3.63E−14	2.36E−15
	3.89	3.94	3.94	
10^{-2}	8.43E−12	5.78E−13	3.77E−14	2.44E−15
	3.87	3.94	3.95	
10^{-5}	8.43E−12	5.78E−13	3.77E−14	2.44E−15
	3.87	3.94	3.95	
10^{-10}	8.43E−12	5.78E−13	3.77E−14	2.44E−15
	3.87	3.94	3.95	
10^{-15}	8.43E−12	5.78E−13	3.77E−14	2.44E−15
	3.87	3.94	3.95	
10^{-20}	8.43E−12	5.78E−13	3.77E−14	2.44E−15
	3.87	3.94	3.95	

7. Conclusion and discussion

Boundary value problem for second order singularly perturbed differential difference equations of mixed type having a boundary layer at one end (left or right) point is considered. To obtain an approximate solution for such type of problems, a mixed asymptotic-numerical method is proposed. In this method, the outer solution which corresponds to the reduced problem is solved numerically using OCI method while the inner solution is obtained analytically by making use of suitable stretching variable. Both cases, when the boundary layer occurs in the left and in the right side of interval are studied. The proposed method is analyzed for convergence, and the error estimate is also discussed.

The maximum absolute errors E_N for different values of ε and N for Examples 1 and 2 are represented in Tables 1 and 2, respectively. The maximum absolute errors $\tilde{E}_N$ and the corresponding rates of convergence $\tilde{r}_N$ for different values of

Table 4. The values of $\tilde{E}^N$ and $\tilde{r}^N$ for Example 4 when $\delta = 0.5$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	3.06E−09	1.93E−10	1.21E−11	7.59E−13	4.71E−14
	3.99	4.00	4.00	4.01	
10^{-2}	3.06E−09	1.94E−10	1.25E−11	7.92E−13	4.90E−14
	3.98	3.96	3.98	4.02	
10^{-5}	3.06E−09	1.94E−10	1.25E−11	7.92E−13	4.90E−14
	3.98	3.96	3.98	4.02	
10^{-10}	3.06E−09	1.94E−10	1.25E−11	7.92E−13	4.90E−14
	3.98	3.96	3.98	4.02	
10^{-15}	3.06E−09	1.94E−10	1.25E−11	7.92E−13	4.90E−14
	3.98	3.96	3.98	4.02	
10^{-20}	3.06E−09	1.94E−10	1.25E−11	7.92E−13	4.90E−14
	3.98	3.96	3.98	4.02	

Table 5. Comparison of maximum absolute errors for Example 1 with [1].

Method	N_0	$\varepsilon = 2^{-2}$	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-12}$	$\varepsilon = 2^{-16}$
Method in [1]	128	1.437E−05	2.341E−05	2.655E−05	2.674E−05	2.675E−05
Present method		1.339E−05	3.345E−06	2.090E−07	1.307E−08	8.165E−10
Method in [1]	512	1.096E−06	3.594E−06	6.480E−06	6.676E−06	6.688E−06
Present method		1.336E−05	3.345E−06	2.090E−07	1.306E−08	8.165E−10

Table 6. Comparison of maximum absolute errors for Example 3 for $\varepsilon = 2^{-6}$ with [18].

Method	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Method in [18]	2.647E−03	8.394E−04	2.583E−04	8.025E−05	2.432E−05
Present method	8.432E−12	5.780E−13	3.765E−14	2.442E−15	5.829E−16

ε for Examples 3 and 4 are shown in Tables 3 and 4, respectively. The numerical results in these tables show that the proposed method is ε-uniformly convergent. Also, from Tables 1 and 2 the maximum absolute error between the numerical solution and exact solution stabilizes as $\varepsilon \to 0$ for each N. While the maximum absolute error $\tilde{E}_N$ stabilizes and decreases rapidly with increasing N for each value of ε as indicated in Tables 3 and 4. Also, it can be observed that the rate of convergence is independent of the value of ε and δ. Comparisons in maximum absolute errors for problems in Examples 1 and 3 with the existing methods are presented in Tables 5 and 6. It is clear that the present method is robust with respect to the perturbation parameter and that our results are much better than the previous ones. The method presented here is very simple to implement, and with a little modification can be extended to high order SPDDEs and other types of differential difference equations like problems with discontinuous source terms and evolution equations.

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References

1. G.M. Amiraliyev and E. Cimen, *Numerical method for a singularly perturbed convection-diffusion problem with delay*, Appl. Math. Comput. **216** (2010), 2351-2359.
2. E.R. Doolan, J.J.H. Miller and W.H.A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dublin, 1980.
3. R.D. Driver, *Ordinary and delay differential equations*, Springer, New York, 1977.
4. V.Y. Glizer, *Asymptotic analysis and solution of a finite-horizon H_∞ control problem for singularly-perturbed linear systems with small state delay*, J. Optim. Theory Appl. **117** (2003), 295-325.
5. M.K. Kadalbajoo, K.C. Patidar and K.K. Sharma, *ϵ-uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs*, Appl. Math. Comput. **182** (2006), 119-139.
6. M.K. Kadalbajoo and K.K. Sharma, *Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations with small shifts of mixed type*, J. Optim. Theory Appl. **115** (2002), 145-163.
7. M.K. Kadalbajoo and K.K. Sharma, *ϵ-uniform fitted mesh method for singularly perturbed differential difference equations: mixed type of shifts with layer behavior*, Int. J. Comput. Math. **81** (2004), 49-62.
8. C.G. Lange and R.M. Miura, *Singular perturbation analysis of boundary value problems of differential-difference equations*, SIAM J. Appl. Math. **42** (1982), 502-531.
9. C.G. Lange and R.M. Miura, *Singular perturbation analysis of boundary-value problems for differential difference equations, V. Small shifts with layer behavior*, SIAM J. Appl. Math. **54** (1994), 249-272.
10. A. Longtin and J. Milton, *Complex oscillations in the human pupil light reflex with mixed and delayed feedback*, Math. Biosci. **90** (1988), 183-199.
11. M.C. Mackey and L. Glass, *Oscillations and chaos in physiological control systems*, Science **197** (1977), 287-289.
12. J.J.H. Miller, E. O'Riordan and G.I. Shishkin, *Fitted numerical methods for singular perturbation problems*, World Scientific, Singapore, 1996.
13. J. Mohapatra and S. Natesan, *Uniformly convergent numerical method for singularly perturbed differential-difference equation using grid equidistribution*, Int. J. Numer. Meth. Biomed. Engng. **27** (2011), 1427-1445.
14. A.H. Nayfeh, *Introduction to perturbation methods*, John Wiley and Sons, New York, 1981.
15. R.E.Jr. O'Malley, *Singular-perturbation methods for ordinary differential equations*, Springer Verlag, New York, 1990.
16. A.A. Salama, *The operator compact implicit methods for the numerical treatment of ordinary differential and integro-differential equations*, Ph.D. Thesis, Assiut University, Assiut, 1992.

17. A.A. Salama and D.G. Al-Amery, *High-order method for singularly perturbed differential-difference equations with small shifts*, Int. J. Pure Appl. Math. **8** (2013), 273-295.
18. V. Subburayan and N. Ramanujam, *An initial value technique for singularly perturbed convection-diffusion problems with a negative shift*, J. Optim Theory Appl. **158** (2013), 234-250.
19. M. Wazewska-Czyzewska and A. Lasota, *Mathematical models of the red cell system*, Mat. Stosow **6** (1976), 25-40.

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