

An Analytical Solution for Regular Progressive Water Waves

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Abstract

In order to provide simple and accurate wave theory in design of offshore structure, an analytical approximation is introduced in this paper. The solution is limited to flat bottom having a constant water depth. Water is considered as inviscid, incompressible and irrotational. The solution satisfies the continuity equation, bottom boundary condition and non-linear kinematic free surface boundary condition exactly. Error for dynamic condition is quite small. The solution is suitable in description of breaking waves. The solution is presented with closed form and dispersion relation is also presented with closed form.

In the last century, there have been two main approaches to the nonlinear problems. One of these is perturbation method. Stokes wave and Cnoidal wave are based on the method. The other is numerical method. Dean's stream function theory is based on the method. In this paper, power series method was considered. The power series method can be applied to certain nonlinear differential equations (initial value problems). The series coefficients are specified by a nonlinear recurrence inherited from the differential equation. Because the non-linear wave problem is a boundary value problem, the power series method cannot be applied to the problem in general. But finite number of coefficients is necessary to describe the wave profile, truncated power series is enough. Therefore the power series method can be applied to the problem. In this case, the series coefficients are specified by a set of equations instead of recurrence. By using the set of equations, the nonlinear wave problem has been solved in this paper.

Keywords: Nonlinear progressive water waves, Breaking waves, Breaking limit, Non-linear free surface boundary condition, The Stokes criterion, The power series method, The Fourier series method, Variational method.

1. Introduction

Structure design is to determine structural configuration, material and dimension to satisfy structural integrity. Hence those are the final output of the structural design. In order to verify structural integrity, structural analysis is definitely necessary. But the final output is used as input data to structural analysis. The output of structural analysis such as stress, strain and displacement etc. is merely reference data in structural design. Because of the reasons, structural design is basically a non-linear process. There are two basic methods to overcome the non-linearity. One of these uses conservative scheme in which design loads considered are greater than actual loads and structural strength and stiffness considered are less than actual strength and stiffness. Therefore integrity of actual structure can be guaranteed when we use the scheme in structure analysis and the result satisfies structural

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integrity requirements. One reason for using this scheme is the simplicity in structure analysis and easy determination of the final output. The complexity of actual structure and loads acting on the structure can be simplified and idealized with conservative scheme. The other is to use analytical solutions instead of numerical solution. Structural designers prefer analytical solution rather than numerical solution because it is possible to determine the final output without stress, displacement and strain when analytical solutions are available in structural design.

The most governing design load is wave force in offshore structure. Numerous water wave theories have been developed which are applicable to different environments dependent upon the specific environmental parameters, e.g., water depth, wave height and wave period. Airy wave theory, Stokes wave theory, Cnoidal wave theory, Dean's numerical stream function are commonly used in description of water wave (Chakrabarti, 1987). These wave theories are limited to a flat bottom having a constant uniform water depth and assume that the waves are periodic and uniform. Additionally, water is assumed incompressible and irrotational. In spite of these assumptions, the complete solution has not been solved. Airy wave theory and Stokes wave theory satisfy the continuity equation and the bottom boundary condition exactly. Cnoidal wave theory satisfies the bottom boundary condition only. However, the nonlinear boundary conditions at the free surface are not satisfied by any of these theories except for the kinematic condition met by the Dean's stream function theory. Hence the stream function theory has a broader range of validity than the other theory (Chakrabarti, 1987; DNV, 2007). The stream function theory is a purely numerical method (DNV, 2007). Hence the stream function theory is seldom used in the design of offshore structure. Because the worst loads and load combinations are always considered in structure analysis, only breaking waves are of interest in structure analysis. Airy wave theory and second order Stokes wave theory are not valid in description of breaking waves. But because these theories are good accord with conservative scheme due to its simplicity and analytical solution, these theories are used in even broader range than their validity range in actual project. Weakness of conservative scheme is to increase structure weight. When structure and its environment are further simplified, structure weight is also increased. In some case, final design weight may be greater than the weight considered in structure analysis. In this case structure design has totally failed. In order to avoid this problem and to provide a wave theory to be good accord with conservative scheme, an analytical solution was presented in this paper.

The linear combination of velocity potential developed in Airy wave theory was considered in this paper. By transformation of variables in which variables were normalized with wave length, the non-linear kinematic boundary condition was transferred from the partial differential equation to an ordinary differential equation. Wave profile was calculated from the ordinary differential equation. Hence the solution satisfies the continuity equation, the bottom boundary condition and kinematic free surface boundary condition exactly.

The dynamic free surface boundary condition also was solved and wave profile was calculated from the dynamic condition. The combination coefficients in velocity potential and the dispersion relation are unknown parameters, which were determined to minimize deviation of the two wave profiles. In the wave profile obtained from the kinematic condition, the combination coefficients are presented with linear terms but in the wave profile obtained from the dynamic condition, the combination coefficients are presented with quadratic terms. Owing to convergence condition of infinite series, the combination coefficients are very small except the first coefficient ($b_1 = 1$). Hence any products of these two coefficients are very small. If we neglect any products of two coefficients $b_n b_m$ for $n, m > 1$, the quadratic terms can be linearized with $(\sum_{n=1}^{\infty} b_n)^2 \approx 1 + 2 \sum_{n=2}^{\infty} b_n$. Hence current solution is an approximation but the error is quite small. Dispersion relation was determined to give the same slope of two wave profiles at point B given in Fig. 1. Two wave profiles can be expanded with two power series at a point "B" given in Fig. 1. All coefficients of two power series are Fourier series. Using least square method to minimize the deviation of two wave profile, we have a set of equations to determine the combination coefficients. In order to verify the result, a wave was calculated and compared with Stokes' fifth-order wave.

2. Velocity Potential

Typical wave profile and coordinate system considered in this analysis was given in Fig. 1 in which H is

wave height, A_c is wave crest height, A_t is wave trough depth, T is wave period and L is wave length, t is time, x is horizontal coordinate and y is vertical coordinate, h is water depth. Flat bottom is assumed in this study. $\eta(x, t)$ is wave profile and η_0 is wave elevation at phase -90° and 90° with respect to the coordinate system in Fig. 1. The linear (Airy) theory gives symmetric profiles about the still-water line. Hence $\eta_0 = 0$ in the linear theory. The nonlinear theories give an asymmetric crest and trough form with crests higher than the depth of the trough. Hence $\eta_0 \neq 0$ in the nonlinear theories. Point A is wave crest, point B is wave elevation at phase -90° and 90° . And point C is wave trough. Velocity potential to satisfy continuity equation and bottom boundary condition can be presented as follows

$$\phi = \sum_{n=1}^N B_n \cosh n\alpha \sin n\beta \quad (1)$$

where B_n are unknown constants and N is the required order of Fourier series. In Eq. (1), the reference phase $\beta = 0$ is considered for the wave such that at $\beta = 0$, the wave profile becomes equal to the wave crest. $\beta = kx - \omega t$ and $\alpha = k(y + h)$ where k is the wave number, defined as $k = 2\pi/L$ and ω is angular frequency, defined as $\omega = 2\pi/T$. L is wave length and T is wave period and π is the ratio of the circumference of a circle to its diameter.

3. An Analytical Solution to Kinematic Boundary Condition

Kinematic boundary condition on free surface is presented as follows

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \left\{ \frac{\partial \phi}{\partial x} \right\} \quad \text{on } y = \eta(x, t) \quad (2)$$

Substituting Eq. (1) into Eq. (2), we have

$$k \sum_{n=1}^N n B_n \sinh n\gamma \sin n\beta = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \left\{ k \sum_{n=1}^N n B_n \cosh n\gamma \cos n\beta \right\} \quad (3)$$

where non-dimensional wave profile is defined as follows

$$\gamma = k(\eta + h) \quad (4)$$

Knowing γ , wave profile can be easily calculated as follows

$$\eta = \frac{\gamma}{k} - h \quad (5)$$

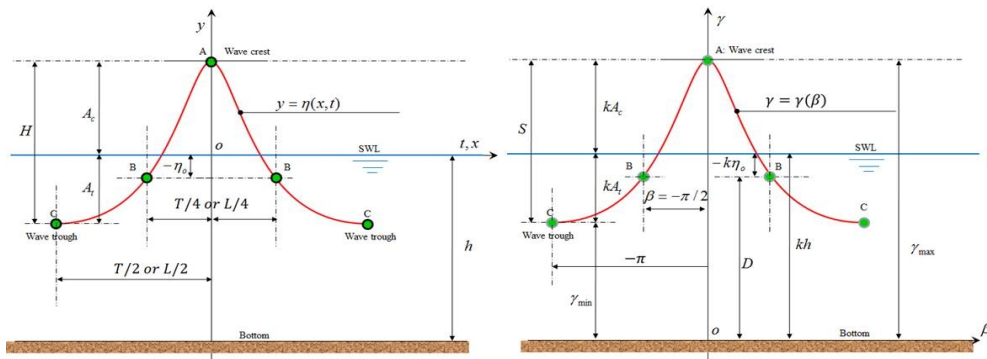


Fig. 1. Typical wave profile and coordinate system considered in this analysis and non-dimensional wave profile.

By using chain rule and Eq. (4), we have $\partial\eta/\partial t = -\omega/k \, d\gamma/d\beta$ and $\partial\eta/\partial x = d\gamma/d\beta$. Substituting the relations into Eq. (3) and then exchanging the first term of the right hand side and the term of left hand side, we have

$$\frac{\omega}{k} \frac{d\gamma}{d\beta} = k \sum_{n=1}^N B_n \frac{d\{\sinh n\gamma \cos n\beta\}}{d\beta} \quad (6)$$

By integrating Eq. (6) with respect to β and applying the condition $\gamma(\pm\pi/2) = D$, as defined in Fig. 1. Then we have

$$\gamma - D = \bar{S} \sum_{n=1}^N b_n \left\{ \sinh n\gamma \cos n\beta - \sinh nD \cos\left(\frac{n\pi}{2}\right) \right\} \quad (7)$$

where $b_n = B_n/B_1$. Hence $b_1 = 1$.

$$D = k(h + \eta_0) \quad (8)$$

$$\bar{S} = \frac{B_1 k^2}{\omega} \quad (9)$$

η_0 is determined with the following continuity equation.

$$h = \frac{1}{2\pi k} \int_{-\pi}^{\pi} \gamma d\beta \quad (10)$$

Eq. (7) is an analytical solution to the kinematic boundary condition. Because Eq. (7) is an implicit function with respect to wave profile γ , wave profile cannot be computed directly. Hence using Newton's method, wave profile in Eq. (7) can be calculated as follows.

$$\gamma = \lim_{i \rightarrow \infty} \gamma_{i+1} \quad (11)$$

Where

$$\gamma_{i+1} = \gamma_i - \frac{f_k(\gamma_i)}{f'_k(\gamma_i)} \quad (12)$$

$$f_k(\gamma) = -(\gamma - D) + \bar{S} \sum_{n=1}^N b_n \left\{ \sinh(n\gamma) \cos(n\beta) - \sinh nD \cos\left(\frac{n\pi}{2}\right) \right\} \quad (13)$$

$$f'_k(\gamma) = \frac{\partial f}{\partial \gamma} = -1 + \bar{S} \sum_{n=1}^N n b_n \cosh(n\gamma) \cos(n\beta) \quad (14)$$

The first step solution in the Newton's method can be calculated with the truncated power series expansion of Eq.

(7) which gives a polynomial equation with regards to $(\gamma - D)$. Power series of $\sinh(n\gamma)$ around D can be expressed as follows

$$\sinh(n\gamma) = \sum_{m=0}^M \frac{n^m E(nD; m)}{m!} (\gamma - D)^m \quad (15)$$

Where

$$E(nD; m) = \begin{cases} \sinh(nD) & \text{for } m : \text{even} \\ \cosh(nD) & \text{for } m : \text{odd} \end{cases} \quad (16)$$

Note that the notation “;” in $E(nD; m)$ is introduced to distinguish independent variable nD from parameter m . The above function was defined in order to express $\sinh(n\gamma)$ and $\cosh(n\gamma)$ by the same power series form as follows

$$\cosh(n\gamma) = \sum_{m=0}^M \frac{n^m E(nD; m+1)}{m!} (\gamma - D)^m \quad (17)$$

where M is the required order of power series. Obviously M is infinite. Because breaking wave steepness is 0.14 in deep water wave (Chakrabarti, 1987), wave steepness is always less than equal to 0.14. Hence we have $M \max(\gamma - D) = kA_c < kH < 0.88$. In addition, $0 \leq |b_n| \leq \sinh \gamma / (n \sinh n\gamma)$. Finite number of M is enough to present Eq. (7), which leads truncated power series expansion as shown in Eq. (15) and Eq. (17). Using Eq. (15), we have

$$\bar{S} \sum_{n=1}^N b_n \{ \sinh n\gamma \cos n\beta \} = \sum_{m=0}^M \sigma_m(\beta) (\gamma - D)^m \quad (18)$$

Where

$$\sigma_m(\beta) = \sum_{n=1}^N G_n(\beta; m) b_n \quad (19)$$

$$G_n(\beta; m) = \frac{n^m E(nD; m)}{m!} \bar{S} \cos(n\beta) \quad (20)$$

Substituting Eq. (18) into Eq. (7), we have a polynomial equation as follows.

$$\sum_{m=2}^M \sigma_m(\beta) (\gamma - D)^m + \{ \sigma_1(\beta) - 1 \} (\gamma - D) + \left\{ \sigma_0(\beta) - \sigma_0\left(\frac{\pi}{2}\right) \right\} = 0 \quad (21)$$

Noting that Eq. (19) is a truncated Fourier series, we can know coefficients of each term in Eq. (21) are expressed as Fourier series with unknown constants b_n (except $b_1 = 1$) which will be determined to satisfy dynamic boundary condition later. For $M = 2$, Eq. (21) is a quadratic equation with respect to $(\gamma - D)$. Then we have the first step solution for the Newton's method as follows

$$\gamma_1 = D + \frac{-\{\sigma_1(\beta) - 1\} - \sqrt{\{\sigma_1(\beta) - 1\}^2 - 4\sigma_2(\beta)\{\sigma_o(\beta) - \sigma_o\left(\frac{\pi}{2}\right)\}}}{2\sigma_2(\beta)} \quad (22)$$

The other of two roots in Eq. (21) for $M = 2$ is a trivial solution because non-dimensional wave profile defined in Eq. (4) is always positive $\gamma \geq 0$. Eq. (22) is an accurate approximation for Eq. (7). Therefore when we use the first step solution given in Eq. (22), Newton's method in Eq.(11) will be converged rapidly. The root includes height ranging up to near breaking. When wave height reaches the limitation, the particle velocity at the crest of the wave reaches the celerity which is Stokes' breaking-wave criterion (Chakrabarti, 1987). Eq. (7) satisfies the Stokes' criterion.

5. An Approximation to Dynamic Boundary Condition

Dynamic boundary condition on free surface is presented as follows

$$\frac{1}{g} \frac{\partial \phi}{\partial t} + \frac{1}{2g} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right\} + \eta = Q(t) \quad \text{on } y = \eta(x, t) \quad (23)$$

where g is acceleration due to gravity and $Q(t)$ is the Bernoulli's constant which can be determined with the condition $\eta = \eta_o$ at phase $\beta = \pm\pi/2$. Substituting Eq. (1) into Eq. (23), and then multiplying wave number to the result, non-dimensional form of wave profile can be calculated. If we neglect any products of two coefficients $b_n b_m$ for $n, m > 1$, then Eq. (23) can be presented as follows.

$$\begin{aligned} \gamma - D = \Omega \bar{S} \sum_{n=1}^N n b_n \cosh(n\gamma) \cos(n\beta) - \frac{\Omega \bar{S}^2}{4} \{ \cosh(2\gamma) + \cos(2\beta) \} \\ - \frac{\Omega \bar{S}^2}{2} \sum_{n=2}^N n b_n \cosh(n+1)\gamma \cos(n-1)\beta \\ - \frac{\Omega \bar{S}^2}{2} \sum_{n=2}^N n b_n \cosh(n-1)\gamma \cos(n+1)\beta + \tau_o \left(\frac{\pi}{2} \right) - \frac{\Omega \bar{S}^2}{2} \end{aligned} \quad (24)$$

Where

$$\Omega = \frac{\omega^2}{gk} \quad (25)$$

By using the power series of $\cosh(n\gamma)$ given in Eq. (17), Eq. (24) can be rewritten as follows

$$\sum_{m=2}^M \tau_m(\beta) (\gamma - D)^m + \{\tau_1(\beta) - 1\} (\gamma - D) + \left\{ \tau_o(\beta) - \tau_o\left(\frac{\pi}{2}\right) \right\} - \frac{\Omega \bar{S}^2}{4} \{ \cos 2\beta + 1 \} = 0 \quad (26)$$

Where

$$\tau_m(\beta) = F_0(\beta; m) + F_1(\beta; m) + \sum_{n=2}^N F_n(\beta; m) b_n \quad \text{for } m \geq 0 \quad (27)$$

$$F_o(\beta; m) = -\frac{\Omega \bar{S}^2}{4} \frac{2^m}{m!} E(2D; m+1) \quad (28)$$

$$F_1(\beta; m) = \frac{\Omega \bar{S}}{m!} E(D; m+1) \cos \beta \quad (29)$$

$$F_n(\beta; m) = \Omega \bar{S} \frac{n^{m+1}}{m!} E(nD; m+1) \cos(n\beta) - \frac{\Omega \bar{S}^2}{2} \left\{ \frac{n(n-1)^m}{m!} E((n-1)D; m+1) \cos(n+1)\beta + \frac{n(n+1)^m}{m!} E((n+1)D; m+1) \cos(n-1)\beta \right\} \text{ for } n \geq 2 \quad (30)$$

5. The Dispersion Relation

Slope of two wave profiles is matched at phase $\pm\pi/2$. By using the method and neglecting any products of two coefficients $b_n b_m$ for $n, m > 1$ in order to avoid coupling to combination coefficients, dispersion relation can be determined as follows.

$$\Omega = \frac{\tanh D}{1 - \bar{S}^2 \sinh^2 D} \text{ for } \bar{S} < \frac{1}{\sinh D} \quad (31)$$

Note that the dispersion relation in linear theory is $\Omega = \tanh D$ and $D = kh$. We have wave height limitation $\bar{S} < 1/\sinh D$ because of $\Omega > 0$.

6. Combination Coefficient

Eq. (26) is good accord with Eq. (21) when $D \geq 2$, $b_n = 0$ for $n \geq 2$ and $b_1 = 1$. Hence the method considered in this chapter is applied when $D < 2$. Because Eq. (26) is approximation, Eq. (26) is not equal to Eq. (21). Hence variational approach (least square method) is suitable to determine combination coefficients(Dym, 1973), b_n . The deviation of two terms in the same order of Eq. (21) and Eq. (26) is defined as follows

$$\varepsilon_0(\beta) = \sigma_M(\beta) \tau_o(\beta) - \sigma_0(\beta) \tau_M(\beta) - \sigma_M(\beta) \tau_o\left(\frac{\pi}{2}\right) + \sigma_o\left(\frac{\pi}{2}\right) \tau_M(\beta) - \frac{\Omega \bar{S}^2}{4} (\cos 2\beta + 1) \sigma_M(\beta) \quad (32)$$

$$\varepsilon_1(\beta) = \sigma_M(\beta) \tau_1(\beta) - \sigma_1(\beta) \tau_M(\beta) - \sigma_M(\beta) + \tau_M(\beta) \quad (33)$$

$$\varepsilon_m(\beta) = \sigma_M(\beta) \tau_m(\beta) - \tau_M(\beta) \sigma_m(\beta) \text{ for } 2 \leq m \leq M-1 \quad (34)$$

If we neglect any products of two coefficients $b_n b_m$ for $n, m > 1$, then we have

$$\sigma_j(\beta_1) \tau_m(\beta_2) = W_1(\beta_1, \beta_2; j, m) + \sum_{n=2}^3 W_n(\beta_1, \beta_2; j, m) b_n \text{ for } 0 \leq j, m \leq M \quad (35)$$

Where

$$W_1(\beta_1, \beta_2; j, m) = G_1(\beta_1; j) \{F_0(\beta_2; m) + F_1(\beta_2; m)\} \quad (36)$$

$$W_n(\beta_1, \beta_2; j, m) = G_1(\beta_1; j) F_n(\beta_2; m) + G_n(\beta_1; j) \{F_0(\beta_2; m) + F_1(\beta_2; m)\} \quad (37)$$

$$\varepsilon_m(\beta) = R_{m,1}(\beta) + \sum_{n=2}^N R_{m,n}(\beta) b_n \quad \text{for } m \geq 0 \quad (38)$$

Substituting Eq. (35) into Eq. (32)-(34), Eq. (32)-(34) can be rewritten as follows
Where

$$R_{m,n}(\beta) = \sum_{j=0}^J \bar{R}_{m,n,j} \cos(j\beta) \quad (39)$$

$$R_{0,n}(\beta) = W_n(\beta; M, 0) - W_n(\beta; 0, M) - W_n\left(\beta, \frac{\pi}{2}; M, 0\right) + W_n\left(\frac{\pi}{2}, \beta; 0, M\right) - \frac{\Omega \bar{S}^2}{2} \{\cos 2\beta + 1\} G_n(\beta; M) \quad \text{for } n \geq 1 \quad (40)$$

$$R_{1,1}(\beta) = W_1(\beta; M, 1) - W_1(\beta; 1, M) - G_1(\beta; M) + F_0(\beta; M) + F_1(\beta; M) \quad (41)$$

$$R_{1,n}(\beta) = W_n(\beta; M, 1) - W_n(\beta; 1, M) - G_n(\beta; M) + F_n(\beta; M) \quad \text{for } n \geq 2 \quad (42)$$

$$R_{m,n}(\beta) = W_n(\beta; M, m) - W_n(\beta; m, M) \quad \text{for } n \geq 3 \text{ and } m \geq 2 \quad (43)$$

Where $W_n(\beta; j, m) = W_n(\beta, \beta; j, m)$. $R_{m,n}(\beta)$ are truncated Fourier series and $\bar{R}_{m,n,j}$ are Fourier series coefficients in which commas was introduced to distinguish clearly 'm' and 'n' when they are written with Arabian number like $R_{1,2}(\beta)$. For $M = 2$ and $N = 3$, Fourier series coefficients $\bar{R}_{m,n,j}$ were shown in APPENDIX. Using least square method to minimize the deviation of two wave profile, we have a set of equations to determine the combination coefficients as follows.

$$\sum_{m=0}^{M-1} \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} \varepsilon_m \frac{\partial \varepsilon_m}{\partial b_q} d\beta \right\} = 0 \quad \text{for } q > 2 \quad (44)$$

Differentiating Eq. (38), we have

$$\frac{\partial \varepsilon_m}{\partial b_q} = R_{m,q}(\beta) \quad (45)$$

Substituting Eq. (38) and Eq. (45) into Eq. (44), we have a set of equations as follows

$$[K_{q,n}] \{b_n\} = \{Y_q\} \quad \text{for } n, q \geq 2 \quad (46)$$

Where

$$K_{q,n} = \sum_{m=0}^{M-1} \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} R_{m,n}(\beta) R_{m,q}(\beta) d\beta \right\} \quad \text{for } q, n \geq 2 \quad (47)$$

And

$$Y_q = -\sum_{m=0}^{M-1} \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} R_{m,1}(\beta) R_{m,q}(\beta) d\beta \right\} \text{ for } q, n \geq 2 \quad (48)$$

Where $[K_{q,n}]$ is N-1 by N-1 matrix and $\{b_n\}$ and $\{Y_q\}$ are N-1 dimensional column vectors. Using the orthogonality of cosine function, the matrix and the column vector can be easily calculated. Substituting Eq. (39) into Eq. (47) and Eq. (48), and using the orthogonality of cosine function we have

$$K_{q,n} = \sum_{m=0}^{M-1} \left\{ 2\bar{R}_{m,n,0}\bar{R}_{m,q,0} + \sum_{j=1}^J \bar{R}_{m,n,j}\bar{R}_{m,q,j} \right\} \text{ for } q, n \geq 2 \quad (49)$$

$$Y_q = -\sum_{m=0}^{M-1} \left\{ 2\bar{R}_{m,1,0}\bar{R}_{m,q,0} + \sum_{j=1}^J \bar{R}_{m,1,j}\bar{R}_{m,q,j} \right\} \text{ for } q, n \geq 2 \quad (50)$$

7. Wave Height Condition

The remained condition is wave height condition given as follows

$$S = \gamma_{max} - \gamma_{min} \quad (51)$$

\bar{S} is determined with the above equation where $\gamma_{max} = \gamma(0)$ and $\gamma_{min} = \gamma(\pi)$.

9. Results and Conclusion

Analytical solution of Eq. (10) and Eq. (51) were not found in this study. Instead of analytical solution, numerical method was considered to solve Eq. (10). Given wave profile, Eq. (10) can be easily solved with well-known numerical method. Wave height also can be easily calculated from Eq. (51). In addition, all variables are normalized with wave length to be determined from the dispersion relation. The dispersion relation is a function of \bar{S} and D . Because of the above reasons, the solution is obtained in an iterative way with initial estimation of \bar{S} and D . Because $0 < D < k_0 h$ and $0 \leq \bar{S} \leq \bar{S}_b$, we can easily determine \bar{S} and D . k_0 is the wave number for linear wave and \bar{S}_b is wave breaking limit which is less than $1/\sinh D$. Because the Newton's method diverges at breaking limit, we can easily know \bar{S}_b . In order to verify the result, a wave near breaking limitation was calculated and compared with Stokes' 5th order wave. The result was shown in Fig. 2. Error was calculated with the method proposed by Dean(Chakrabarti, S.K, 1987). Error of this study in kinematic condition is zero and error of this study in dynamic condition is 6.7%. Hence total error of this study is 6.7%. But error of Stokes' 5th order in kinematic condition is 56.6% and error of Stokes' 5th order in dynamic condition is 19.4%. Hence total error of Stokes' 5th order is 76%.

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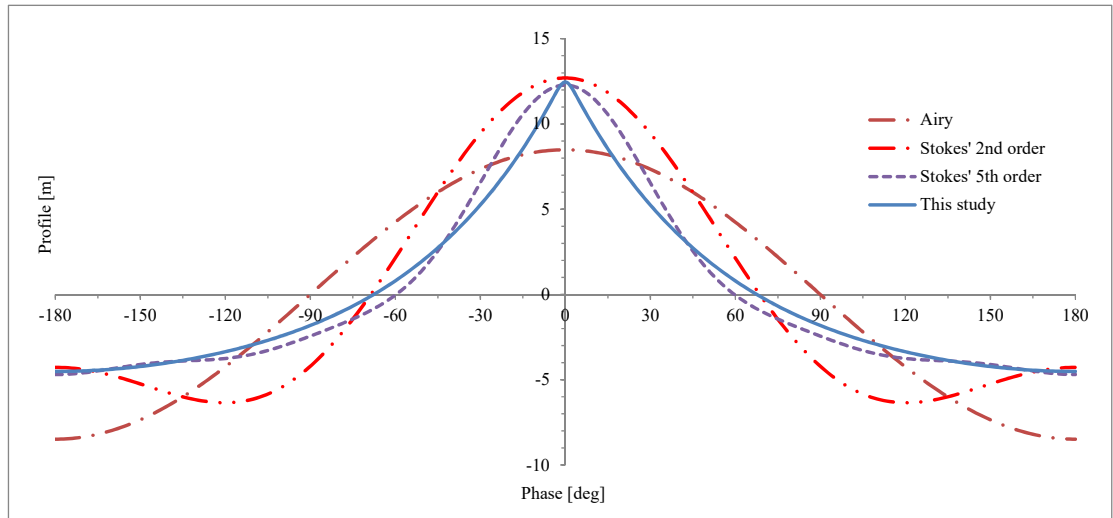


Fig. 2. Wave profile (water depth: 21.714m, wave height: 16.974m, Period: 10sec, wave length 125m by Eq.(31))

Appendix. Fourier Series Coefficient For $M = 2$ and $N=3$,

$\bar{R}_{0,1,1} = \frac{1}{2} \Omega \bar{S}^2 \sinh D \left\{ \cosh 2D - \frac{3}{8} \right\}$	$\bar{R}_{0,1,2} = -\frac{1}{16} (\Omega \bar{S}^2 \sinh D)$
$\bar{R}_{0,2,0} = \Omega \bar{S}^2 \left\{ 2 \sinh D \cosh 3D - \frac{1}{4} \sinh 2D + \frac{1}{2} \sinh 2D \cosh 2D \right\}$	$\bar{R}_{0,2,1} = \Omega \bar{S}^2 \left\{ \frac{1}{4} \sinh 2D \cosh D + \frac{7}{2} \sinh D \cosh 2D \right\}$
$\bar{R}_{0,2,2} = \Omega \bar{S}^2 \left\{ 2 \sinh D \cosh 3D + \frac{1}{2} \sinh 2D \cosh 2D - \frac{1}{2} \sinh 2D \right\}$	$\bar{R}_{0,2,3} = -\Omega \bar{S}^2 \left\{ \frac{3}{2} \sinh D \cosh 2D - \frac{3}{4} \sinh 2D \cosh D \right\}$
$\bar{R}_{0,3,0} = -\frac{1}{4} \Omega \bar{S}^2 \sinh 2D$	$\bar{R}_{0,3,1} = \Omega \bar{S}^2 \left\{ \frac{3}{4} \sinh D \cosh 2D + \frac{39}{8} \sinh D \cosh 4D \right\}$
$\bar{R}_{0,3,2} = \frac{1}{4} \Omega \bar{S}^2 \{-6 \sinh D \cosh 3D + 8 \sinh 3D \cosh D\}$	$\bar{R}_{0,3,3} = \Omega \bar{S}^2 \left\{ \frac{9}{8} \sinh D \cosh 2D + \frac{45}{8} \sinh D \cosh 4D \right\}$
$\bar{R}_{0,3,4} = \frac{1}{2} \Omega \bar{S}^2 \{-3 \sinh D \cosh 3D + 4 \sinh 3D \cosh D\}$	$\bar{R}_{0,3,5} = \Omega \bar{S}^2 \left\{ \frac{9}{8} \sinh D \cosh 2D - \frac{9}{16} \sinh 3D \right\}$
$\bar{R}_{1,1,0} = -\frac{\Omega \bar{S}^2}{4} - \frac{\Omega \bar{S}^2}{2} \cosh 3D$	$\bar{R}_{1,1,1} = -\Omega \bar{S}^2 \left\{ \frac{1}{2} \sinh D \sinh 2D - \frac{1}{2} \cosh D \cosh 2D \right\}$
$\bar{R}_{1,2,0} = -\frac{\Omega \bar{S}^2}{4}$	$\bar{R}_{1,2,1} = -\frac{3\Omega \bar{S}^2}{4} \{\sinh D \sinh 3D - 3 \cosh D \cosh 3D\}$
$\bar{R}_{1,2,2} = \frac{1}{4} \Omega \bar{S}^2 \{8 \sinh D \sinh 2D - 10 \cosh D \cosh 2D - 18 \cosh 3D\}$	$\bar{R}_{1,2,3} = \frac{1}{4} \Omega \bar{S}^2 \{5 + 9 \cosh D \cosh 3D - 3 \sinh D \sinh 3D\}$
$\bar{R}_{1,2,4} = \frac{1}{4} \Omega \bar{S}^2 \{8 \sinh D \sinh 2D - 10 \cosh D \cosh 2D - 2 \cosh D\}$	$\bar{R}_{1,2,5} = \frac{\Omega \bar{S}^2}{4}$

$\bar{R}_{1,z,1} = -\frac{3\Omega\bar{S}^2}{2}\{\sinh D \sinh 4D - 4 \cosh D \cosh 4D\}$	$\bar{R}_{1,z,2} = \frac{1}{2}\Omega\bar{S}^2\{9 \sinh D \sinh 3D - 15 \cosh D \cosh 3D\}$
$\bar{R}_{1,z,3} = \left[\frac{3}{4}\Omega\bar{S}^2\{2 \cosh D \cosh 2D + 8 \cosh D \cosh 4D + 2 \cosh(3D) \cosh 5D\} \right.$	$\bar{R}_{1,z,4} = \frac{1}{4}\Omega\bar{S}^2\{18 \sinh D \sinh 3D - 30 \cosh D \cosh 3D\}$
$\quad - \frac{3}{4}\Omega\bar{S}^2\{\sinh D \sinh 2D + 2 \sinh D \sinh 4D + 3 \sinh 3D\}$	$\bar{R}_{1,z,5} = \frac{1}{4}\Omega\bar{S}^2\{18 \sinh D \sinh 3D - 30 \cosh D \cosh 3D\}$
$\quad - 3\Omega\bar{S}^2 \cosh 2D \}$	$\bar{R}_{1,z,6} = -\frac{6\Omega\bar{S}^2}{8}\{\sinh D \sinh 2D - 2 \cosh D \cosh 2D\}$

All the other coefficients are zero.